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On golden semisymmetric metric F-connections

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Abstract: In this paper, we construct a golden semisymmetric metric F-connection on a locally decomposable golden Riemannian manifold and investigate some properties of its curvature, conharmonic curvature, Weyl projective curvature, and torsion tensors. Moreover, we define the transposed connection of this connection and study its curvature properties.

Key words: Golden Riemannian structure, semisymmetric metric F-connection, Tachibana operator, transposed connection

1. Introduction

Hayden introduced the idea of a metric connection with torsion on a Riemannian manifold [4]. A linear connection $\widetilde{\nabla}$ on a Riemannian manifold (M, g) is called a metric connection with torsion with respect to g if $\widetilde{\nabla} g = 0$ and its torsion tensor is nonzero. In [15], Yano defined a semisymmetric metric connection on a Riemannian manifold. A semisymmetric metric connection is a metric connection whose torsion tensor is in the form: $S(X,Y) = p(Y)X - p(X)Y$, where p is a 1-form. Chaki and Konar [1] obtained the expression for the curvature tensor of a Riemannian manifold that admits a semisymmetric metric connection with vanishing curvature and recurrent torsion tensors.

A locally decomposable Riemannian manifold can be defined as a triplet (M, g, F), which consists of a manifold M endowed with an almost product structure F and a Riemannian metric g such that $g(FX, Y) = g(X, FY)$ and $\nabla F = 0$ for all vector fields X and Y on M, where ∇ is the Levi-Civita connection of g. For a locally decomposable Riemannian manifold (M, g, F), Prvanovic defined a product semisymmetric F-connection that is a generalization of semisymmetric metric connections and proved that if this has a zero curvature tensor then the Riemannian space is product conformally flat [11]. A more general product semisymmetric F-connection is also studied: this is not a metric connection but is a product-recurrent metric connection. For further references, we refer to [7–10].

The notion of golden Riemannian structure was introduced in [2] as a $(1,1)$-tensor field F and a Riemannian metric g on a manifold M that satisfy $F^2 = F + I$ and $g(FX, FY) = g(FX, Y) + g(X, Y)$ for all vector fields X and Y on M. In [3], the second author and collaborators proved that a necessary and sufficient condition for the triplet (M, g, F) to be a locally decomposable golden Riemannian manifold is that $\phi_F g = 0$, where ϕ_F is the Tachibana operator applied to g. In this paper, we construct a golden

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semisymmetric metric F-connection on a locally decomposable golden Riemannian manifold. First we study some properties concerning its torsion tensor. Then we calculate curvature, conharmonic curvature, and Weyl projective curvature tensors of this connection and investigate their properties. Finally we define and study the transposed connection of this connection.

2. Preliminaries

Let M be an n-dimensional differentiable manifold of class C^∞ covered by any system of coordinate neighbourhoods (x^h), where here and in the sequel the indices $h, i, j, k, ...$ run over the range $1, 2, ..., n$. Also note that summation over repeated indices is always implied.

Consider a $(1,1)$-tensor F with components $F_i^{\ j}$ and a (p,q)-tensor K with components $K_{i_1 i_2 ... i_q}^{j_1 j_2 ... j_p}$ satisfying the condition

$$\begin{aligned} K_{mi_2...i_q}^{j_1...j_p} F_{i_1}^{\ m} &= K_{i_1 m ... i_q}^{j_1...j_p} F_{i_2}^{\ m} = ... = K_{i_1 i_2 ... m}^{j_1...jp} F_{i_q}^{\ m} = \\ K_{i_1...i_q}^{m j_2 ... j_p} F_m^{\ j_1} &= K_{i_1...i_q}^{j_1 m ... j_p} F_m^{\ j_2} = ... = K_{i_1...i_q}^{j_1 j_2 ... m} F_m^{\ j_p}; \end{aligned}$$

then the tensor K is called a pure tensor with respect to the tensor F.

The Tachibana operator ϕ_F applied to a pure (p,q)-tensor K is given by

$$\begin{aligned} & (\phi_F K)_{k i_1 ... i_q}^{j_1 ... j_p} \\ = \ & F_k^{\ m} \partial_m K_{i_1...i_q}^{j_1...j_p} - \partial_k (K \circ F)_{i_1...i_q}^{j_1...j_p} \\ & + \sum_{\lambda=1}^{q} (\partial_{i_\lambda} F_k^{\ m}) \ K_{i_1...m...i_q}^{j_1...j_p} + \sum_{\mu=1}^{p} \left(\partial_k F_m^{\ j_\mu} - \partial_m F_k^{\ j_\mu} \right) \ K_{i_1...i_q}^{j_1...m...j_p}, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} (K \circ F)_{i_1...i_q}^{j_1...j_p} &= K_{m i_2 ... i_q}^{j_1...j_p} F_{i_1}^{\ m} = ... = K_{i_1 i_2 ... m}^{j_1...jp} F_{i_q}^{\ m} \\ &= K_{i_1...i_q}^{m j_2 ... j_p} F_m^{\ j_1} = ... = K_{i_1...i_q}^{j_1 j_2 ... m} F_m^{\ j_p}. \end{aligned}$$

The operator (2.1) first introduced by Tachibana [14] (for Tachibana operator and its applications, see [13] and [16]). If the pure tensor K satisfies $\phi_F K = 0$, then it is called a ϕ-tensor. If the $(1,1)$-tensor F is a product structure, then a ϕ-tensor is a decomposable tensor [14].

A golden Riemannian manifold is a manifold M equipped with a $(1,1)$-tensor field F with components F_i^j and a Riemannian metric g with components g_{ij} that satisfy the following conditions:

$$F_i^{\ k} F_k^{\ j} = F_i^{\ j} + \delta_i^j \tag{2.2}$$

and

$$F_i^{\ k} g_{kj} = F_j^{\ k} g_{ki} \text{ (or equivalently } F_i^{\ k} F_j^{\ t} g_{kt} = F_i^{\ k} g_{kj} + g_{ij}). \tag{2.3}$$

It follows from (2.3) that $F_{ij} = F_{ji}$. Riemannian golden and almost product structures are related to each other. In fact, the connection between a golden structure F and almost product structure J on M is as follows

[2]:

$$J = \mp\frac{1}{\sqrt{5}}(2F - I) \tag{2.4}$$

or conversely

$$F = \frac{1}{2}(I \mp \sqrt{5}J). \tag{2.5}$$

Furthermore, it is clear that a Riemannian metric g is pure with respect to a golden structure F if and only if the Riemannian metric g is pure with respect to the corresponding almost product structure J. Using the above-mentioned relation, from (2.1) a simple computation gives the following:

$$\phi_F K = \mp\frac{\sqrt{5}}{2}\phi_J K \tag{2.6}$$

for any (p,q)-tensor K. A golden Riemannian manifold (M,g,F) is a locally decomposable golden Riemannian manifold if and only if the Riemannian metric g is a decomposable tensor, i.e. $(\phi_J g)_{kij} = 0$ [3]. We note that the condition $(\phi_J g)_{kij} = 0$ is equivalent to $\nabla_k J_i{}^j = 0$ [13]. Here we use the notation ∇_k to denote the operator of the Riemannian covariant derivation. Throughout this paper, the notation ∇_k will be used for the same purpose.

3. The golden semisymmetric metric F-connection

Let (M,g,F) be a locally decomposable golden Riemannian manifold. We consider an affine connection $\widetilde{\nabla}$ with components $\widetilde{\Gamma}^k_{ij}$ on M. If the affine connection $\widetilde{\nabla}$ satisfies

$$\begin{aligned} i)\ \widetilde{\nabla}_h g_{ij} &= 0 \\ ii)\ \widetilde{\nabla}_h F_i{}^j &= 0, \end{aligned} \tag{3.7}$$

then it is called a metric F-connection. When the torsion tensor $\widetilde{S}^k_{ij}$ of $\widetilde{\nabla}$ is given by

$$\widetilde{S}^k_{ij} = p_j\delta^k_i - p_i\delta^k_j + p_t F_j{}^t F_i{}^k - p_t F_i{}^t F_j{}^k, \tag{3.8}$$

where p_i are local components of an 1-form (covector field), we call the affine connection $\widetilde{\nabla}$ a golden semisymmetric metric F-connection.

Let $\widetilde{\Gamma}^k_{ij}$ be the components of the golden semisymmetric metric F-connection $\widetilde{\nabla}$. If we put

$$\widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + T^k_{ij}, \tag{3.9}$$

where Γ^k_{ij} are the components of the Levi-Civita connection ∇ of g and T^k_{ij} are the components of a $(1,2)$-tensor field T on M, then the torsion tensor $\widetilde{S}^k_{ij}$ of $\widetilde{\nabla}$ is given by

$$\widetilde{S}^k_{ij} = \widetilde{\Gamma}^k_{ij} - \widetilde{\Gamma}^k_{ji} = T^k_{ij} - T^k_{ji}.$$

Since the connection (3.9) satisfies the condition (i) of (3.7), by following the method shown in [4], we get

$$T^k_{ij} = p_j\delta^k_i - p^k g_{ij} + p_t F_j{}^t F_i{}^k - p_t F^{kt} F_{ij},$$

where $p^k = p_i g^{ik}$, $F^{kt} = F_i{}^t g^{ik}$ and $F_{ij} = F_j{}^k g_{ik}$. Thus the connection (3.9) becomes the following form:

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + p_j \delta_i^k - p^k g_{ij} + p_t F_j{}^t F_i{}^k - p_t F^{kt} F_{ij}. \tag{3.10}$$

Moreover, using (3.10) once verifies

$$\widetilde{\nabla}_k F_i{}^j = g_{ki}(p^t F_t{}^j - p_t F^{jt}) = 0.$$

Therefore the components $\widetilde{\Gamma}_{ij}^k$ of the golden semisymmetric metric F-connection $\widetilde{\nabla}$ can be expressed in the form (3.10).

4. Torsion properties of the golden semisymmetric metric F-connection

In this section, we investigate some properties concerning the torsion tensor of the connection (3.10).

Theorem 4.1 *Let (M, g, F) be a locally decomposable golden Riemannian manifold endowed with the connection (3.10). The torsion tensor $\widetilde{S}$ of the connection (3.10) is pure.*

Proof By using (2.2) and (3.8), it follows that $\widetilde{S}_{mj}^k F_i{}^m = \widetilde{S}_{im}^k F_j{}^m = \widetilde{S}_{ij}^m F_m{}^k$, i.e. the torsion tensor $\widetilde{S}$ is pure. □

A F-connection is pure if and only if its torsion tensor is pure [13]. Thus we can say that the connection (3.10) is pure with respect to F, i.e., the following condition holds:

$$\widetilde{\Gamma}_{mj}^k F_i{}^m = \widetilde{\Gamma}_{im}^k F_j{}^m = \widetilde{\Gamma}_{ij}^m F_m{}^k.$$

Theorem 4.2 *Let (M, g, F) be a locally decomposable golden Riemannian manifold endowed with the connection (3.10). The torsion tensor $\widetilde{S}$ of the connection (3.10) is a ϕ-tensor if the generator p is a ϕ-tensor.*

Proof Let (M, g, F) be a locally decomposable golden Riemannian manifold and ∇ be its Levi-Civita connection with components Γ_{ij}^h. A zero tensor is pure and therefore we have: a torsion-free F-connection is always pure. Thus the Levi-Civita connection ∇ of g on M is pure with respect to F.

If we apply the Tachibana operator ϕ_F to the torsion tensor $\widetilde{S}$ of the connection (3.10), we get

$$\begin{aligned}(\phi_F \widetilde{S})_{kij}{}^l &= F_k{}^m(\partial_m \widetilde{S}_{ij}^l) - \partial_k(\widetilde{S}_{mj}^l F_i{}^m) \\ &= F_k{}^m(\nabla_m \widetilde{S}_{ij}^l + \Gamma_{mi}^s \widetilde{S}_{sj}^l + \Gamma_{mj}^s \widetilde{S}_{is}^l - \Gamma_{ms}^l \widetilde{S}_{ij}^s) \\ &\quad - F_i{}^m(\nabla_k \widetilde{S}_{mj}^l + \Gamma_{km}^s \widetilde{S}_{sj}^l + \Gamma_{kj}^s \widetilde{S}_{ms}^l - \Gamma_{ks}^l \widetilde{S}_{mj}^s).\end{aligned}$$

Using the purity of the torsion tensor $\widetilde{S}$ and Levi-Civita connection ∇, the relation above reduces to

$$(\phi_F \widetilde{S})_{kij}{}^l = F_k{}^m(\nabla_m \widetilde{S}_{ij}^l) - F_i{}^m(\nabla_k \widetilde{S}_{mj}^l). \tag{4.11}$$

For the generator p, we calculate

$$\begin{aligned}(\phi_F p)_{kj} &= F_k{}^m(\partial_m p_j) - \partial_k(F_j{}^m p_m) \\ &= F_k{}^m(\nabla_m p_j + \Gamma_{mj}^s p_s) - F_j{}^m(\nabla_k p_m + \Gamma_{km}^s p_s) \\ &= F_k{}^m(\nabla_m p_j) - F_j{}^m(\nabla_k p_m).\end{aligned}$$

From here, we can say that the generator p is a ϕ-tensor if and only if

$$F_k^m(\nabla_m p_j) = F_j^m(\nabla_k p_m). \tag{4.12}$$

Substituting (3.8) into (4.11), we find

$$\begin{aligned}(\phi_F \widetilde{S})_{kij}{}^l &= [F_k{}^m(\nabla_m p_j) - F_j{}^m(\nabla_k p_m)]\delta_i^l - [F_k{}^m(\nabla_m p_i) - F_i{}^m(\nabla_k p_m)]\delta_j^l \\ &\quad + [F_k{}^m F_j{}^s(\nabla_m p_s) - F_j{}^s(\nabla_k p_s) - \nabla_k p_j]F_i{}^l \\ &\quad - [F_k{}^m F_i{}^s(\nabla_m p_s) - F_i{}^s(\nabla_k p_s) - \nabla_k p_i]F_j{}^l.\end{aligned}$$

Assuming that the generator p is a ϕ-tensor, by virtue of (2.2) the last relation becomes $(\phi_F \widetilde{S})_{kij}{}^l = 0$, i.e. the torsion tensor $\widetilde{S}$ is a ϕ-tensor. This completes the proof. □

In view of (2.6), we can easily say that the ϕ-tensor p is also a decomposable tensor, i.e. $\phi_J p = 0$, where J is the product structure associated with the golden structure F. From on now, we shall consider such a special case of golden semisymmetric metric F-connections whose generator p is a ϕ-tensor (or decomposable tensor), i.e. the following condition always holds:

$$F_k{}^m(\nabla_m p_j) = F_j{}^m(\nabla_k p_m)$$

or equivalently

$$J_k{}^m(\nabla_m p_j) = J_j{}^m(\nabla_k p_m).$$

In this case, (4.11) gives the following condition:

$$F_k{}^m(\nabla_m \widetilde{S}_{ij}^l) = F_i{}^m(\nabla_k \widetilde{S}_{mj}^l) = F_j{}^m(\nabla_k \widetilde{S}_{im}^l). \tag{4.13}$$

Theorem 4.3 *Let (M, g, F) be a locally decomposable golden Riemannian manifold endowed with the connection (3.10). If the torsion tensor $\widetilde{S}$ of the connection (3.10) satisfies the following*

$$\widetilde{\nabla}_k \widetilde{S}_{ij}^l + \widetilde{\nabla}_j \widetilde{S}_{ki}^l + \widetilde{\nabla}_i \widetilde{S}_{jk}^l = 0, \tag{4.14}$$

then, under the condition of $(traceF - 2)^2 - (n-4)(n + traceF - 6) \neq 0$, the generator p is closed, i.e. $dp = 0$. Conversely, if the generator p is closed, then the relation (4.14) is satisfied.

Proof We calculate

$$\begin{aligned}& \widetilde{\nabla}_k \widetilde{S}_{ij}^l + \widetilde{\nabla}_j \widetilde{S}_{ki}^l + \widetilde{\nabla}_i \widetilde{S}_{jk}^l \qquad (4.15)\\ =\; & (\widetilde{\nabla}_k p_j - \widetilde{\nabla}_j p_k)\delta_i^l + (\widetilde{\nabla}_i p_k - \widetilde{\nabla}_k p_i)\delta_j^l \\ & + (\widetilde{\nabla}_i p_j - \widetilde{\nabla}_j p_i)\delta_k^l + [(\widetilde{\nabla}_k p_t)F_j{}^t - (\widetilde{\nabla}_j p_t)F_k{}^t]F_i{}^l \\ & + [(\widetilde{\nabla}_i p_t)F_k{}^t - (\widetilde{\nabla}_k p_t)F_i{}^t]F_j{}^l + [(\widetilde{\nabla}_j p_t)F_i{}^t - (\widetilde{\nabla}_i p_t)F_j{}^t]F_k{}^l \\ =\; & (\widetilde{\nabla}_k p_j - \widetilde{\nabla}_j p_k)\delta_i^l + (\widetilde{\nabla}_i p_k - \widetilde{\nabla}_k p_i)\delta_j^l + (\widetilde{\nabla}_i p_j - \widetilde{\nabla}_j p_i)\delta_k^l \\ & + (\widetilde{\nabla}_k p_t - \widetilde{\nabla}_t p_k)F_j{}^t F_i{}^l + (\widetilde{\nabla}_i p_t - \widetilde{\nabla}_t p_i)F_k{}^t F_j{}^l + (\widetilde{\nabla}_j p_t - \widetilde{\nabla}_t p_j)F_i{}^t F_k{}^l.\end{aligned}$$

It is easy to see that $\widetilde{\nabla}_k p_t - \widetilde{\nabla}_t p_k = \nabla_k p_t - \nabla_t p_k = (dp)_{kt}$. Because of this, (4.15) immediately gives that $dp = 0$ implies $\widetilde{\nabla}_k \widetilde{S}_{ij}^l + \widetilde{\nabla}_j \widetilde{S}_{ki}^l + \widetilde{\nabla}_i \widetilde{S}_{jk}^l = 0$.

If we assume that $\widetilde{\nabla}_k \widetilde{S}_{ij}^l + \widetilde{\nabla}_j \widetilde{S}_{ki}^l + \widetilde{\nabla}_i \widetilde{S}_{jk}^l = 0$, then we get

$$\begin{aligned}
&(\nabla_k p_j - \nabla_j p_k)\delta_i^l + (\nabla_i p_k - \nabla_k p_i)\delta_j^l + (\nabla_i p_j - \nabla_j p_i)\delta_k^l \\
&+(\nabla_k p_t - \nabla_t p_k)F_j{}^t F_i{}^l + (\nabla_i p_t - \nabla_t p_i)F_k{}^t F_j{}^l + (\nabla_j p_t - \nabla_t p_j)F_i{}^t F_k{}^l = 0.
\end{aligned} \tag{4.16}$$

Contracting (4.16) with g_{lm}, we have

$$\begin{aligned}
&(\nabla_k p_j - \nabla_j p_k)g_{im} + (\nabla_i p_k - \nabla_k p_i)g_{jm} + (\nabla_i p_j - \nabla_j p_i)g_{km} \\
&+(\nabla_k p_t - \nabla_t p_k)F_j{}^t F_{im} + (\nabla_i p_t - \nabla_t p_i)F_k{}^t F_{jm} + (\nabla_j p_t - \nabla_t p_j)F_i{}^t F_{km} = 0.
\end{aligned}$$

Transvecting the last relation with g^{im} and F^{im} respectively, it follows that

$$(n-4)\,(\nabla_j p_k - \nabla_k p_j) + (traceF - 2)\,F_k{}^t\,(\nabla_j p_t - \nabla_t p_j) = 0 \tag{4.17}$$

and

$$(traceF - 2)\,(\nabla_j p_k - \nabla_k p_j) + (n + traceF - 6)\,F_k{}^t\,(\nabla_j p_t - \nabla_t p_j) = 0. \tag{4.18}$$

When we multiply (4.17) by $(n + traceF - 6)$ and (4.18) by $(traceF - 2)$ and subtract the first from the second, then we obtain

$$\left[(traceF - 2)^2 - (n-4)(n + traceF - 6)\right](\nabla_j p_k - \nabla_k p_j) = 0.$$

If $(traceF - 2)^2 - (n-4)(n+traceF-6) \neq 0$, then $(\nabla_j p_k - \nabla_k p_j) = (dp)_{jk} = 0$, i.e. the generator p is closed. □

5. Curvature properties of the golden semisymmetric metric F-connection

The curvature tensor $\widetilde{R}_{ijk}{}^l$ of the connection (3.10) is obtained from the well-known formula

$$\widetilde{R}_{ijk}{}^l = \partial_i \widetilde{\Gamma}_{jk}^l - \partial_j \widetilde{\Gamma}_{ik}^l + \widetilde{\Gamma}_{im}^l \widetilde{\Gamma}_{jk}^m - \widetilde{\Gamma}_{jm}^l \widetilde{\Gamma}_{ik}^m.$$

Then the curvature tensor $\widetilde{R}_{ijk}{}^l$ can be expressed as follows:

$$\begin{aligned}
\widetilde{R}_{ijk}{}^l \;=\;& R_{ijk}{}^l - \delta_i^l \mathcal{A}_{jk} + \delta_j^l \mathcal{A}_{ik} + g_{ik}\mathcal{A}_j{}^l - g_{jk}\mathcal{A}_i{}^l \\
&- F_i{}^l F_k{}^t \mathcal{A}_{jt} + F_j{}^l F_k{}^t \mathcal{A}_{it} + F_{ik}F^{lt}\mathcal{A}_{jt} - F_{jk}F^{lt}\mathcal{A}_{it},
\end{aligned} \tag{5.19}$$

where $R_{ijk}{}^l$ are the components of the Riemann curvature tensor of the Levi-Civita connection ∇ and

$$\mathcal{A}_{jk} = \nabla_j p_k - p_j p_k + \frac{1}{2}p^m p_m g_{kj} - p_m p_t F_k{}^t F_j{}^m + \frac{1}{2}p^m p_t F_m{}^t F_{jk}. \tag{5.20}$$

On lowering the upper index in (5.19), we obtain

$$\widetilde{R}_{ijkl} = R_{ijkl} - g_{il}\mathcal{A}_{jk} + g_{jl}\mathcal{A}_{ik} + g_{ik}\mathcal{A}_{jl} - g_{jk}\mathcal{A}_{il} - F_{il}F_k{}^t\mathcal{A}_{jt} + F_{jl}F_k{}^t\mathcal{A}_{it} + F_{ik}F_l{}^t\mathcal{A}_{jt} - F_{jk}F_l{}^t\mathcal{A}_{it}. \quad (5.21)$$

It is obvious that the curvature tensor satisfies $\widetilde{R}_{ijkl} = -\widetilde{R}_{jikl}$ and $\widetilde{R}_{ijkl} = -\widetilde{R}_{ijlk}$.

In order to use it later, we need the following useful lemma.

Lemma 5.1 *Let (M, g, F) be a locally decomposable golden Riemannian manifold endowed with the connection (3.10). Then the tensor $\mathcal{A}$ given by (5.20) is a ϕ-tensor and thus the following relation holds:*

$$(\nabla_m\mathcal{A}_{ij})\,F_k{}^m = (\nabla_k\mathcal{A}_{mj})\,F_i{}^m = (\nabla_k\mathcal{A}_{im})\,F_j{}^m.$$

Proof The tensor $\mathcal{A}$ is pure with respect to F. In fact

$$F_k{}^t\mathcal{A}_{it} - F_i{}^t\mathcal{A}_{tk} = (\nabla_i p_t)\,F_k{}^t - (\nabla_t p_k)\,F_i{}^t = 0.$$

We calculate

$$\begin{aligned}(\phi_F\mathcal{A})_{kij} &= F_k{}^m\,(\partial_m\mathcal{A}_{ij}) - \partial_k(\mathcal{A}_{mj}F_i{}^m) \\ &= F_k{}^m\left(\nabla_m\mathcal{A}_{ij} + \Gamma^s_{mi}\mathcal{A}_{sj} + \Gamma^s_{mj}\mathcal{A}_{is}\right) \\ &\quad - F_i{}^m(\nabla_k\mathcal{A}_{mj} + \Gamma^s_{km}\mathcal{A}_{sj} + \Gamma^s_{kj}\mathcal{A}_{ms}).\end{aligned}$$

Using the purity of the tensor $\mathcal{A}$ and the Levi-Civita connection ∇, we get

$$(\phi_F\mathcal{A})_{kij} = (\nabla_m\mathcal{A}_{ij})\,F_k{}^m - (\nabla_k\mathcal{A}_{mj})\,F_i{}^m. \quad (5.22)$$

Substituting (5.20) into (5.22), standard calculations give

$$(\phi_F\mathcal{A})_{kij} = (\nabla_m\nabla_i p_j)\,F_k{}^m - (\nabla_k\nabla_m p_j)\,F_i{}^m. \quad (5.23)$$

If we apply the Ricci identity to the generator p, then we have

$$\begin{aligned}(\nabla_m\nabla_i p_j)\,F_k{}^m &= (\nabla_i\nabla_m p_j)\,F_k{}^m - p_s R_{mij}{}^s F_k{}^m \\ &= (\nabla_i\nabla_k p_m)\,F_j{}^m - p_s R_{mij}{}^s F_k{}^m\end{aligned}$$

and

$$\begin{aligned}(\nabla_k\nabla_m p_j)\,F_i{}^m &= (\nabla_k\nabla_i p_m)\,F_j{}^m \\ &= (\nabla_i\nabla_k p_m)\,F_j{}^m - p_s R_{kim}{}^s F_j{}^m.\end{aligned}$$

With the help of the last two equations, from (5.23), it follows that

$$(\phi_F\mathcal{A})_{kij} = -p_s(R_{mij}{}^s F_k{}^m - R_{kim}{}^s F_j{}^m).$$

In a locally decomposable golden Riemannian manifold (M, g, F), the Riemannian curvature tensor R is pure [3]. This immediately gives $(\phi_F\mathcal{A})_{kij} = 0$. Hence, from (5.22) we can write

$$(\nabla_m\mathcal{A}_{ij})\,F_k{}^m = (\nabla_k\mathcal{A}_{mj})\,F_i{}^m = (\nabla_k\mathcal{A}_{im})\,F_j{}^m.$$

□

Theorem 5.2 *Let (M, g, F) be a locally decomposable golden Riemannian manifold endowed with the connection (3.10). The curvature tensor $\widetilde{R}$ of the connection (3.10) is a ϕ-tensor and thus the following relation holds:*

$$(\nabla_m \widetilde{R}_{ijl}{}^t)F_k{}^m = (\nabla_k \widetilde{R}_{mjl}^{\ \ \ t})F_i{}^m = (\nabla_k \widetilde{R}_{iml}^{\ \ \ t})F_j{}^m = (\nabla_k \widetilde{R}_{ijm}^{\ \ \ t})F_l{}^m = (\nabla_k \widetilde{R}_{ijl}{}^m)F_m^{\ t}.$$

Proof Using the purity of the tensor $\mathcal{A}$, standard calculations give

$$\widetilde{R}_{ijk}{}^m F_m^{\ l} = \widetilde{R}_{mjk}^{\ \ \ l} F_i{}^m = \widetilde{R}_{imk}^{\ \ \ l} F_j{}^m = \widetilde{R}_{ijm}^{\ \ \ l} F_k{}^m,$$

i.e. the curvature tensor $\widetilde{R}$ is pure.

Applying the Tachibana operator ϕ_F to the curvature tensor $\widetilde{R}$, we have

$$\begin{aligned} & (\phi_F \widetilde{R})_{kijl}{}^t \\ = \ & F_k{}^m(\partial_m \widetilde{R}_{ijl}{}^t) - \partial_k(\widetilde{R}_{mjl}^{\ \ \ t} F_i{}^m) \\ = \ & F_k{}^m(\nabla_m \widetilde{R}_{ijl}{}^t + \Gamma_{mi}^s \widetilde{R}_{sjl}{}^t + \Gamma_{mj}^s \widetilde{R}_{isl}{}^t + \Gamma_{ml}^s \widetilde{R}_{ijs}{}^t - \Gamma_{ms}^t \widetilde{R}_{ijl}{}^m) \\ & - F_i{}^m(\nabla_k \widetilde{R}_{mjl}^{\ \ \ t} + \Gamma_{km}^s \widetilde{R}_{sjl}{}^t + \Gamma_{kj}^s \widetilde{R}_{msl}^{\ \ \ t} + \Gamma_{kl}^s \widetilde{R}_{mjs}^{\ \ \ t} - \Gamma_{ks}^t \widetilde{R}_{mjl}^{\ \ \ s}) \\ = \ & (\nabla_m \widetilde{R}_{ijl}{}^t)F_k{}^m - (\nabla_k \widetilde{R}_{mjl}^{\ \ \ t})F_i{}^m \end{aligned} \tag{5.24}$$

from which, by (5.19), we find

$$\begin{aligned} (\phi_F \widetilde{R})_{kijl}{}^t &= (\phi_F R)_{kijl}{}^t + [(\nabla_k \mathcal{A}_{jm}) F_l{}^m - (\nabla_m \mathcal{A}_{jl}) F_k{}^m]\delta_i^t \\ &\quad + [(\nabla_m \mathcal{A}_{il}) F_k{}^m - (\nabla_k \mathcal{A}_{im}) F_l{}^m]\delta_j{}^t. \end{aligned}$$

In a locally decomposable golden Riemannian manifold (M, g, F), the Riemannian curvature tensor R is a $\phi-$tensor [3]. Thus, taking into account Lemma 5.1, the preceding relation becomes $(\phi_F \widetilde{R})_{kijl}{}^t = 0$, i.e. the curvature tensor $\widetilde{R}$ is a ϕ-tensor. Thus, by (5.24), we can write

$$(\nabla_m \widetilde{R}_{ijl}^t)F_k{}^m = (\nabla_k \widetilde{R}_{mjl}^t)F_i{}^m = (\nabla_k \widetilde{R}_{iml}^t)F_j{}^m = (\nabla_k \widetilde{R}_{ijm}^t)F_l{}^m.$$

This completes the proof. □

Using (2.6), we obtain, as a result to Theorem 4.2 and 5.2, the following theorem.

Theorem 5.3 *Let (M, g, F) be a locally decomposable golden Riemannian manifold endowed with the connection (3.10). Both the torsion tensor $\widetilde{S}$ and the curvature tensor $\widetilde{R}$ of the connection (3.10) are decomposable tensors with respect to the product structures associated with the golden structure F.*

Theorem 5.4 *Let (M, g, F) be a locally decomposable golden Riemannian manifold endowed with the connection (3.10). If the curvature tensor $\widetilde{R}$ of the connection (3.10) satisfies the following*

$$\begin{cases} i) \ \widetilde{R}_{ijkl} - \widetilde{R}_{klij} = 0 \\ ii) \ \widetilde{R}_{ijkl} + \widetilde{R}_{kijl} + \widetilde{R}_{jkil} = 0 \end{cases} \tag{5.25}$$

then, under the condition of $(traceF-2)^2-(n-4)(n+traceF-6)\neq 0$, *the generator* p *is closed. Conversely, if the generator* p *is closed, then the relations (5.25) are satisfied.*

Proof *i)* From (5.21), we obtain

$$\begin{aligned}\widetilde{R}_{ijkl}-\widetilde{R}_{klij} &= (\mathcal{A}_{li}-\mathcal{A}_{il})\,g_{jk}+(\mathcal{A}_{kj}-\mathcal{A}_{jk})\,g_{il}+(\mathcal{A}_{ik}-\mathcal{A}_{ki})\,g_{jl}\\ &\quad+(\mathcal{A}_{jl}-\mathcal{A}_{lj})\,g_{ik}+F_{il}\left(F_j{}^t\mathcal{A}_{kt}-F_k{}^t\mathcal{A}_{jt}\right)\\ &\quad+F_{jl}\left(F_k{}^t\mathcal{A}_{it}-F_i{}^t\mathcal{A}_{kt}\right)+F_{ik}\left(F_l{}^t\mathcal{A}_{jt}-F_j{}^t\mathcal{A}_{lt}\right)\\ &\quad-F_{jk}(F_l{}^t\mathcal{A}_{it}-F_i{}^t\mathcal{A}_{lt})\end{aligned}$$

from which

$$\begin{aligned}&\widetilde{R}_{ijkl}-\widetilde{R}_{klij}\\ =\;&(\mathcal{A}_{li}-\mathcal{A}_{il})\,g_{jk}+(\mathcal{A}_{kj}-\mathcal{A}_{jk})\,g_{il}+(\mathcal{A}_{ik}-\mathcal{A}_{ki})\,g_{jl}\\ &+(\mathcal{A}_{jl}-\mathcal{A}_{lj})\,g_{ik}+F_{il}F_j{}^t\,(\mathcal{A}_{kt}-\mathcal{A}_{tk})\\ &+F_{jl}F_k{}^t\,(\mathcal{A}_{it}-\mathcal{A}_{ti})+F_{ik}F_l{}^t\,(\mathcal{A}_{jt}-\mathcal{A}_{tj})\\ &-F_{jk}F_l{}^t\,(\mathcal{A}_{it}-\mathcal{A}_{ti})\,.\end{aligned} \tag{5.26}$$

It immediately follows from (5.26) that $dp=0$ implies $\widetilde{R}_{ijkl}-\widetilde{R}_{klij}=0$.

If we assume that $\widetilde{R}_{ijkl}-\widetilde{R}_{klij}=0$, then (5.26) becomes

$$\begin{aligned}0 &= (\mathcal{A}_{li}-\mathcal{A}_{il})\,g_{jk}+(\mathcal{A}_{kj}-\mathcal{A}_{jk})\,g_{il}+(\mathcal{A}_{ik}-\mathcal{A}_{ki})\,g_{jl}\\ &\quad+(\mathcal{A}_{jl}-\mathcal{A}_{lj})\,g_{ik}+F_{il}F_j{}^t\,(\mathcal{A}_{kt}-\mathcal{A}_{tk})\\ &\quad+F_{jl}F_k{}^t\,(\mathcal{A}_{it}-\mathcal{A}_{ti})+F_{ik}F_l{}^t\,(\mathcal{A}_{jt}-\mathcal{A}_{tj})\\ &\quad-F_{jk}F_l{}^t\,(\mathcal{A}_{it}-\mathcal{A}_{ti})\,.\end{aligned}$$

Transvecting the last relation with g^{il} and F^{il} respectively, we find

$$(n-4)\,(\mathcal{A}_{jk}-\mathcal{A}_{kj})+(traceF-2)\,F_k{}^t\,(\mathcal{A}_{jt}-\mathcal{A}_{tj})=0$$

and

$$(traceF-2)\,(\mathcal{A}_{jk}-\mathcal{A}_{kj})+(n+traceF-6)\,F_k{}^t\,(\mathcal{A}_{jt}-\mathcal{A}_{tj})=0.$$

The common solution of the two equations above gives

$$\left[(traceF-2)^2-(n-4)(n+traceF-6)\right](\mathcal{A}_{jk}-\mathcal{A}_{kj})=0.$$

If $(traceF-2)^2-(n-4)\,(n+tarceF-6)\neq 0$, then the last relation gives

$$\mathcal{A}_{jk}-\mathcal{A}_{kj}=0$$

from which the result follows because of $\mathcal{A}_{jk}-\mathcal{A}_{kj}=\nabla_j p_k-\nabla_j p_k$.

ii) From (5.21) and (3.8), we have

$$\widetilde{R}_{ijkl} + \widetilde{R}_{kijl} + \widetilde{R}_{jkil} = (\widetilde{\nabla}_k \widetilde{S}^h_{ij} + \widetilde{\nabla}_j \widetilde{S}^h_{ki} + \widetilde{\nabla}_i \widetilde{S}^h_{jk}) g_{hl}. \tag{5.27}$$

With the help of (5.27) and Theorem 4.3, the result immediately follows. □

Example 5.5 *The Euclidean space* $\mathbb{R}^n$ *is given by Euclidean metric*

$$\begin{aligned} g &= (g_{\alpha\beta}) = \begin{pmatrix} g_{ij} & g_{\bar{i}j} \\ g_{i\bar{j}} & g_{\bar{i}\bar{j}} \end{pmatrix} \\ &= \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \delta_{\bar{i}\bar{j}} \end{pmatrix}, i,j = 1,...,k, \;\; \bar{i},\bar{j} = k+1,...,n. \end{aligned}$$

In the example, Greek indices take on values 1 *to* n. *Take* J *as the standard product structure on* $\mathbb{R}^n$, *so that we have*

$$\begin{aligned} (J_\alpha{}^\beta) &= \begin{pmatrix} J^j_i & J^j_{\bar{i}} \\ J^{\bar{j}}_i & J^{\bar{j}}_{\bar{i}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \delta^j_{\bar{i}} \\ \delta^{\bar{j}}_i & 0 \end{pmatrix}, \;\; i,j = 1,...,k, \;\; \bar{i},\bar{j} = k+1,...,n. \end{aligned}$$

Then the triplet $(\mathbb{R}^n, g, J)$ *is a locally decomposable Euclidean space. The golden structures* $F_\pm$ *on* $\mathbb{R}^n$ *obtained from* J *are as follows:*

$$F_\pm = \begin{pmatrix} F^j_i & F^j_{\bar{i}} \\ F^{\bar{j}}_i & F^{\bar{j}}_{\bar{i}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\delta^j_i & \pm\frac{\sqrt{5}}{2}\delta^j_{\bar{i}} \\ \pm\frac{\sqrt{5}}{2}\delta^{\bar{j}}_i & \frac{1}{2}\delta^{\bar{j}}_{\bar{i}} \end{pmatrix}.$$

Also note that the triplet $(\mathbb{R}^n, g, F_\pm)$ *is a locally decomposable golden Euclidean space.*

We suppose that p_α *is a gradient,* $p_\alpha = (p_i, p_{\bar{i}}) = (\partial_i f, \partial_{\bar{i}} f)$, f *being a decomposable function. The condition for the function* f *to be locally decomposable is given by [13]*

$$(\phi_J df)_{\sigma\beta} = J^\alpha_\sigma \partial_\alpha \partial_\beta f - \partial_\sigma (J^\alpha_\beta \partial_\alpha f) + (\partial_\beta J^\alpha_\sigma) \partial_\alpha f = 0.$$

Then the components of the golden semisymmetric metric $F_\pm$*-connection in* $(\mathbb{R}^n, g, F_\pm)$ *are the following:*

$$\begin{aligned} \widetilde{\Gamma}^k_{ij} &= \widetilde{\Gamma}^{\bar{k}}_{\bar{i}j} = \widetilde{\Gamma}^{\bar{k}}_{i\bar{j}} = \widetilde{\Gamma}^k_{\bar{i}\bar{j}} \\ &= \frac{5}{4}\left[(\partial_j f)\,\delta^k_i - (\partial_h f)\,\delta^{hk}\delta_{ij}\right] \pm \frac{\sqrt{5}}{4}\left[\left(\partial_{\bar{j}} f\right)\delta^k_i - \left(\partial_{\bar{h}} f\right)\delta^{\overline{hk}}\delta_{ij}\right] \end{aligned}$$

$$\begin{aligned} \widetilde{\Gamma}^k_{\bar{i}j} &= \widetilde{\Gamma}^k_{i\bar{j}} = \widetilde{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}} = \widetilde{\Gamma}^{\bar{k}}_{ij} \\ &= \frac{5}{4}\left[\left(\partial_{\bar{j}} f\right)\delta^k_i - \left(\partial_{\bar{h}} f\right)\delta^{\overline{hk}}\delta_{ij}\right] \pm \frac{\sqrt{5}}{4}\left[(\partial_j f)\,\delta^k_i - (\partial_h f)\,\delta^{hk}\delta_{ij}\right]. \end{aligned}$$

The torsion tensor of the golden semisymmetric metric $F_{\pm}$-connection has the components

$$\begin{aligned}\widetilde{S}_{ij}^{k} &= \widetilde{S}_{\bar{i}j}^{\bar{k}} = \widetilde{S}_{i\bar{j}}^{\bar{k}} = \widetilde{S}_{\bar{i}\bar{j}}^{k} \\ &= \frac{5}{4}\left[(\partial_j f)\,\delta_i^k - (\partial_i f)\,\delta_j^k\right] \pm \frac{\sqrt{5}}{4}\left[\left(\partial_{\bar{j}} f\right)\delta_i^k - (\partial_{\bar{i}} f)\,\delta_j^k\right]\end{aligned}$$

$$\begin{aligned}\widetilde{S}_{\bar{i}j}^{k} &= \widetilde{S}_{i\bar{j}}^{k} = \widetilde{S}_{\bar{i}\bar{j}}^{\bar{k}} = \widetilde{S}_{ij}^{\bar{k}} \\ &= \frac{5}{4}\left[\left(\partial_{\bar{j}} f\right)\delta_i^k - (\partial_{\bar{i}} f)\,\delta_j^k\right] \pm \frac{\sqrt{5}}{4}\left[(\partial_j f)\,\delta_i^k - (\partial_i f)\,\delta_j^k\right]\end{aligned}$$

Simple calculations show that the torsion tensor $\widetilde{S}$ is pure with respect to $F_{\pm}$ and furthermore $(\phi_{F_{\pm}} S)_{\sigma\alpha\beta}^{\gamma} = 0$ or equivalently $(\phi_J S)_{\sigma\alpha\beta}^{\gamma} = 0$, i.e. $\widetilde{S}$ is decomposable.

The components of the curvature tensor $\widetilde{R}$ of the golden semisymmetric metric $F_{\pm}$-connection are the following:

$$\begin{aligned}\widetilde{R}_{ijk}{}^{l} &= \widetilde{R}_{\bar{i}\bar{j}k}{}^{l} = \widetilde{R}_{\bar{i}j\bar{k}}{}^{l} = \widetilde{R}_{i\bar{j}\bar{k}}{}^{l} = \widetilde{R}_{\bar{i}jk}{}^{\bar{l}} \\ &= \widetilde{R}_{i\bar{j}k}{}^{\bar{l}} = \widetilde{R}_{ij\bar{k}}{}^{\bar{l}} = \widetilde{R}_{\bar{i}\bar{j}\bar{k}}{}^{\bar{l}} \\ &= -\delta_i^l\left(\frac{5}{4}\mathcal{A}_{jk} \pm \frac{\sqrt{5}}{4}\mathcal{A}_{\bar{j}k}\right) + \delta_j^l\left(\frac{5}{4}\mathcal{A}_{ik} \pm \frac{\sqrt{5}}{4}\mathcal{A}_{\bar{i}k}\right) \\ &\quad +\delta_{ik}\left(\frac{5}{4}\mathcal{A}_j^l \pm \frac{\sqrt{5}}{4}\mathcal{A}_{\bar{j}}^l\right) - \delta_{jk}\left(\frac{5}{4}\mathcal{A}_i^l \pm \frac{\sqrt{5}}{4}\mathcal{A}_{\bar{i}}^l\right)\end{aligned}$$

$$\begin{aligned}\widetilde{R}_{\bar{i}jk}{}^{l} &= \widetilde{R}_{i\bar{j}k}{}^{l} = \widetilde{R}_{ij\bar{k}}{}^{l} = \widetilde{R}_{ijk}{}^{\bar{l}} \\ &= \widetilde{R}_{\bar{i}\bar{j}k}{}^{\bar{l}} = \widetilde{R}_{\bar{i}j\bar{k}}{}^{\bar{l}} = \widetilde{R}_{i\bar{j}\bar{k}}{}^{\bar{l}} = \widetilde{R}_{\bar{i}\bar{j}\bar{k}}{}^{l} \\ &= -\delta_i^l\left(\frac{5}{4}\mathcal{A}_{\bar{j}k} \pm \frac{\sqrt{5}}{4}\mathcal{A}_{jk}\right) + \delta_j^l\left(\frac{5}{4}\mathcal{A}_{\bar{i}k} \pm \frac{\sqrt{5}}{4}\mathcal{A}_{ik}\right) \\ &\quad +\delta_{ik}\left(\frac{5}{4}\mathcal{A}_{\bar{j}}^l \pm \frac{\sqrt{5}}{4}\mathcal{A}_j^l\right) - \delta_{jk}\left(\frac{5}{4}\mathcal{A}_{\bar{i}}^l \pm \frac{\sqrt{5}}{4}\mathcal{A}_i^l\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}_{jk} &= \mathcal{A}_{\bar{j}\bar{k}} = \partial_k\partial_j f - \frac{5}{4}\left[\left(\partial_{\bar{k}} f\right)\left(\partial_{\bar{j}} f\right) + (\partial_k f)\,(\partial_j f)\right] \\ &\quad \mp\frac{\sqrt{5}}{4}\left[(\partial_k f)\left(\partial_{\bar{j}} f\right) + \left(\partial_{\bar{k}} f\right)(\partial_j f)\right] \\ &\quad +\frac{5}{8}\delta^{hm}\delta_{jk}\left[\left(\partial_{\bar{h}} f\right)(\partial_{\bar{m}} f) + (\partial_h f)\,(\partial_m f)\right] \\ &\quad \pm\frac{\sqrt{5}}{8}\delta^{hm}\delta_{jk}\left[\left(\partial_{\bar{h}} f\right)(\partial_m f) + (\partial_h f)\,(\partial_{\bar{m}} f)\right]\end{aligned}$$

$$
\begin{aligned}
\mathcal{A}_{\bar{j}k} &= \mathcal{A}_{j\bar{k}} = \partial_{\bar{k}}\partial_j f - \frac{5}{4}\left[(\partial_k f)\left(\partial_{\bar{j}} f\right) + (\partial_{\bar{k}} f)(\partial_j f)\right] \\
&\mp \frac{\sqrt{5}}{4}\left[(\partial_{\bar{k}} f)\left(\partial_{\bar{j}} f\right) + (\partial_k f)(\partial_j f)\right] \\
&\pm \frac{\sqrt{5}}{8}\delta^{hm}\delta_{jk}\left[(\partial_{\bar{h}} f)(\partial_{\bar{m}} f) + (\partial_h f)(\partial_m f)\right] \\
&+ \frac{5}{8}\delta^{hm}\delta_{jk}\left[(\partial_{\bar{h}} f)(\partial_m f) + (\partial_h f)(\partial_{\bar{m}} f)\right]
\end{aligned}
$$

and $\mathcal{A}_\sigma^\beta = g^{\alpha\beta}\mathcal{A}_{\sigma\alpha}$. *One checks that* $(\phi_{F_\pm}\mathcal{A})_{\sigma\alpha\beta} = 0$. *Using this, one verifies that the curvature tensor* $\widetilde{R}$ *is pure with respect to* $F_\pm$ *and furthermore* $(\phi_{F_\pm}\widetilde{R})_{\sigma\alpha\beta\gamma}^{\ \ \ \ \ \eta} = 0$ *or equivalently* $(\phi_J\widetilde{R})_{\sigma\alpha\beta\gamma}^{\ \ \ \ \ \eta} = 0$, *i.e.* $\widetilde{R}$ *is decomposable.*

The components of the curvature $(0,4)$*-tensor* $\widetilde{R}$ *are the following:*

$$
\begin{aligned}
\widetilde{R}_{ijkl} &= \widetilde{R}_{\bar{i}\bar{j}kl} = \widetilde{R}_{\bar{i}j\bar{k}l} = \widetilde{R}_{i\bar{j}\bar{k}l} = \widetilde{R}_{\bar{i}jk\bar{l}} \\
&= \widetilde{R}_{i\bar{j}k\bar{l}} = \widetilde{R}_{ij\bar{k}\bar{l}} = \widetilde{R}_{\bar{i}\bar{j}\bar{k}\bar{l}} \\
&= -\delta_{il}\left(\frac{5}{4}\mathcal{A}_{jk} \pm \frac{\sqrt{5}}{4}\mathcal{A}_{\bar{j}k}\right) + \delta_{jl}\left(\frac{5}{4}\mathcal{A}_{ik} \pm \frac{\sqrt{5}}{4}\mathcal{A}_{\bar{i}k}\right) \\
&+\delta_{ik}\left(\frac{5}{4}\mathcal{A}_{jl} \pm \frac{\sqrt{5}}{4}\mathcal{A}_{\bar{j}l}\right) - \delta_{jk}\left(\frac{5}{4}\mathcal{A}_{il} \pm \frac{\sqrt{5}}{4}\mathcal{A}_{\bar{i}l}\right)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{R}_{\bar{i}jkl} &= \widetilde{R}_{i\bar{j}kl} = \widetilde{R}_{ij\bar{k}l} = \widetilde{R}_{ijk\bar{l}} \\
&= \widetilde{R}_{\bar{i}\bar{j}k\bar{l}} = \widetilde{R}_{\bar{i}j\bar{k}\bar{l}} = \widetilde{R}_{i\bar{j}\bar{k}\bar{l}} = \widetilde{R}_{\bar{i}\bar{j}\bar{k}l} \\
&= -\delta_{il}\left(\frac{5}{4}\mathcal{A}_{\bar{j}k} \pm \frac{\sqrt{5}}{4}\mathcal{A}_{jk}\right) + \delta_{jl}\left(\frac{5}{4}\mathcal{A}_{\bar{i}k} \pm \frac{\sqrt{5}}{4}\mathcal{A}_{ik}\right) \\
&+\delta_{ik}\left(\frac{5}{4}\mathcal{A}_{\bar{j}l} \pm \frac{\sqrt{5}}{4}\mathcal{A}_{jl}\right) - \delta_{jk}\left(\frac{5}{4}\mathcal{A}_{\bar{i}l} \pm \frac{\sqrt{5}}{4}\mathcal{A}_{il}\right)
\end{aligned}
$$

It is a straightforward verification that the conditions

$$
\begin{cases}
\widetilde{R}_{\sigma\alpha\beta\gamma} = -\widetilde{R}_{\alpha\sigma\beta\gamma}, \\
\widetilde{R}_{\sigma\alpha\beta\gamma} = -\widetilde{R}_{\sigma\alpha\gamma\beta},
\end{cases}
$$

$$
\begin{cases}
\widetilde{R}_{\sigma\alpha\beta\gamma} = \widetilde{R}_{\beta\gamma\sigma\alpha}, \\
\widetilde{R}_{\sigma\alpha\beta\gamma} + \widetilde{R}_{\alpha\beta\sigma\gamma} + \widetilde{R}_{\beta\sigma\alpha\gamma} = 0,
\end{cases}
\quad \text{for } n > 4
$$

are fulfilled.

Contracting (5.19) with respect to i and l, we obtain

$$
\begin{aligned}
\widetilde{R}_{jk} &= R_{jk} - (n-4)\mathcal{A}_{jk} - trace\mathcal{A}g_{jk} \\
&- (traceF - 2)F_k^{\ t}\mathcal{A}_{jt} - F_{jk}F^{lt}\mathcal{A}_{lt}
\end{aligned}
\tag{5.28}
$$

for Ricci tensors of the connection (3.10) ($\widetilde{R}_{jk}$) and the Levi-Civita connection (R_{jk}) respectively. Contracting the preceding relation with g^{jk}, for the scalar curvatures $\overline{\tau}$ and τ of these connections, we have

$$\overline{\tau} = \tau - 2\,(n-2)\,trace\mathcal{A} - 2\,(traceF - 1)\,F^{lt}\mathcal{A}_{lt}.$$

Actually, we can get

$$\widetilde{R}_{jk} - \widetilde{R}_{kj} = (n-4)\,(\mathcal{A}_{kj} - \mathcal{A}_{jk}) + (traceF - 2)\,F_j{}^t\,(\mathcal{A}_{kt} - \mathcal{A}_{tk})\,.$$

From this, it is easy to see that if the generator p is closed, then $\widetilde{R}_{jk} = \widetilde{R}_{kj}$.

The conharmonic curvature tensor with respect to the connection (3.10) is given by

$$\widetilde{V}_{ijkl} = \widetilde{R}_{ijkl} - \frac{1}{n-2}\left[\widetilde{R}_{jk}g_{il} - \widetilde{R}_{ik}g_{jl} - \widetilde{R}_{jl}g_{ik} + \widetilde{R}_{il}g_{jk}\right]. \tag{5.29}$$

Putting the values of $\widetilde{R}_{ijkl}$ and $\widetilde{R}_{ik}$ from (5.21) and (5.28) respectively in (5.29), we have

$$\begin{aligned} &\widetilde{V}_{ijkl} \\ = \ & V_{ijkl} - F_{il}F_k{}^t\mathcal{A}_{jt} + F_{jl}F_k{}^t\mathcal{A}_{it} + F_{ik}F_l{}^t\mathcal{A}_{jt} \\ & -F_{jk}F_l{}^t\mathcal{A}_{it} - \frac{1}{n-2}[(2\mathcal{A}_{jk} - g_{jk}(trace\mathcal{A}) \\ & +F_{jk}F^{mt}\mathcal{A}_{mt} + (2 - traceF)F_k{}^t\mathcal{A}_{jt})g_{il} \\ & -(2\mathcal{A}_{ik} - g_{ik}(trace\mathcal{A}) + F_{ik}F^{mt}\mathcal{A}_{mt} + (2 - traceF)F_k{}^t\mathcal{A}_{it})g_{jl} \\ & -(2\mathcal{A}_{jl} - g_{jl}(trace\mathcal{A}) + F_{jl}F^{mt}\mathcal{A}_{mt} + (2 - traceF)F_l{}^t\mathcal{A}_{jt})g_{ik} \\ & +(2\mathcal{A}_{il} - g_{il}(trace\mathcal{A}) + F_{il}F^{mt}\mathcal{A}_{mt} + (2 - traceF)F_l{}^t\mathcal{A}_{it})g_{jk}], \end{aligned} \tag{5.30}$$

where V_{ijkl} is the conharmonic curvature tensor with respect to the Levi-Civita connection. Since in a Riemannian manifold $V_{ijkl} = -V_{jikl}$ and $V_{ijkl} = -V_{ijlk}$, we can easily find $\widetilde{V}_{ijkl} = -\widetilde{V}_{jikl}$ and $\widetilde{V}_{ijkl} = -\widetilde{V}_{ijlk}$.

Theorem 5.6 *Let (M, g, F) be a locally decomposable golden Riemannian manifold endowed with the connection (3.10). If the conharmonic curvature tensors with respect to the connection (3.10) and the Levi-Civita connection coincide, then the 1-form p satisfies*

$$\nabla_l p^l + \lambda_1\left(\nabla_t p^l\right)F_l{}^t + \lambda_2 p_l p^l + \lambda_3 p_t p^l F_l{}^t = 0,$$

where $\lambda_1 = \frac{traceF-1}{n-2}$, $\lambda_2 = \frac{(traceF-1)(traceF-2)+(n-2)(n-4)}{2(n-2)}$ *and*

$\lambda_3 = \frac{(traceF-2)(n-2)+(traceF-1)(traceF+n-6)}{2(n-2)}$.

Proof Let us assume that $\widetilde{V}_{ijkl} = V_{ijkl}$. Then from (5.30) we have

$$\begin{aligned} & -F_{il}F_k^{\ t}\mathcal{A}_{jt} + F_{jl}F_k^{\ t}\mathcal{A}_{it} + F_{ik}F_l^{\ t}\mathcal{A}_{jt} \\ & -F_{jk}F_l^{\ t}\mathcal{A}_{it} - \frac{1}{n-2}[(2\mathcal{A}_{jk} - g_{jk}(trace\mathcal{A}) \\ & +F_{jk}F^{mt}\mathcal{A}_{mt} + (2-traceF)F_k^{\ t}\mathcal{A}_{jt})g_{il} \\ & -(2\mathcal{A}_{ik} - g_{ik}(trace\mathcal{A}) + F_{ik}F^{mt}\mathcal{A}_{mt} + (2-traceF)F_k^{\ t}\mathcal{A}_{it})g_{jl} \\ & -(2\mathcal{A}_{jl} - g_{jl}(trace\mathcal{A}) + F_{jl}F^{mt}\mathcal{A}_{mt} + (2-traceF)F_l^{\ t}\mathcal{A}_{jt})g_{ik} \\ & +(2\mathcal{A}_{il} - g_{il}(trace\mathcal{A}) + F_{il}F^{mt}\mathcal{A}_{mt} + (2-traceF)F_l^{\ t}\mathcal{A}_{it})g_{jk}] \\ = & \ 0. \end{aligned} \tag{5.31}$$

Contracting (5.31) with g^{il}, we find

$$trace\mathcal{A} + \frac{traceF-1}{n-2}F^{lt}\mathcal{A}_{tl} = 0,$$

from which it follows that

$$\nabla_l p^l + \lambda_1 \left(\nabla_t p^l\right) F_l^{\ t} + \lambda_2 p_l p^l + \lambda_3 p_t p^l F_l^{\ t} = 0,$$

where $\lambda_1 = \frac{traceF-1}{n-2}$, $\lambda_2 = \frac{(traceF-1)(traceF-2)+(n-2)(n-4)}{2(n-2)}$ and

$\lambda_3 = \frac{(traceF-2)(n-2)+(traceF-1)(traceF+n-6)}{2(n-2)}$. □

Theorem 5.7 *Let (M, g, F) be a locally decomposable golden Riemannian manifold endowed with the connection (3.10). If the conharmonic curvature tensor with respect to the connection (3.10) vanishes, then the 1-form p satisfies*

$$\nabla_l p^l + \lambda_1 \left(\nabla_t p^l\right) F_l^{\ t} + \lambda_2 p_l p^l + \lambda_3 p_t p^l F_l^{\ t} + \lambda_4 \tau = 0,$$

where $\lambda_1 = \frac{traceF-1}{n-2}$, $\lambda_2 = \frac{(traceF-1)(traceF-2)+(n-2)(n-4)}{2(n-2)}$,

$\lambda_3 = \frac{(traceF-2)(n-2)+(traceF-1)(traceF+n-6)}{2(n-2)}$, $\lambda_4 = \frac{1}{2(2-n)}$ *and τ is the scalar curvature of (M, g).*

Proof Let us assume that $\widetilde{V}_{ijkl} = 0$. Then from (5.30) we have

$$\begin{aligned} & V_{ijkl} - F_{il}F_k^{\ t}\mathcal{A}_{jt} + F_{jl}F_k^{\ t}\mathcal{A}_{it} + F_{ik}F_l^{\ t}\mathcal{A}_{jt} \\ & -F_{jk}F_l^{\ t}\mathcal{A}_{it} - \frac{1}{n-2}[(2\mathcal{A}_{jk} - g_{jk}(trace\mathcal{A}) \\ & +F_{jk}F^{mt}\mathcal{A}_{mt} + (2-traceF)F_k^{\ t}\mathcal{A}_{jt})g_{il} \\ & -(2\mathcal{A}_{ik} - g_{ik}(trace\mathcal{A}) + F_{ik}F^{mt}\mathcal{A}_{mt} + (2-traceF)F_k^{\ t}\mathcal{A}_{it})g_{jl} \\ & -(2\mathcal{A}_{jl} - g_{jl}(trace\mathcal{A}) + F_{jl}F^{mt}\mathcal{A}_{mt} + (2-traceF)F_l^{\ t}\mathcal{A}_{jt})g_{ik} \\ & +(2\mathcal{A}_{il} - g_{il}(trace\mathcal{A}) + F_{il}F^{mt}\mathcal{A}_{mt} + (2-traceF)F_l^{\ t}\mathcal{A}_{it})g_{jk}] \\ = & \ 0. \end{aligned}$$

When we contract the last equation by g^{il}, using the condition $V_{ijkl}g^{il} = V_{ljk}{}^{l} = -\frac{\tau}{n-2}g_{jk}$, we obtain

$$\begin{aligned} trace\mathcal{A} + \frac{traceF-1}{n-2}F^{lt}\mathcal{A}_{tl} + \frac{1}{2(2-n)}\tau &= 0 \\ \nabla_l p^l + \lambda_1\left(\nabla_t p^l\right)F_l{}^t + \lambda_2 p_l p^l + \lambda_3 p_t p^l F_l{}^t + \lambda_4\tau &= 0, \end{aligned}$$

where $\lambda_1 = \frac{traceF-1}{n-2}$, $\lambda_2 = \frac{(traceF-1)(traceF-2)+(n-2)(n-4)}{2(n-2)}$,

$\lambda_3 = \frac{(traceF-2)(n-2)+(traceF-1)(traceF+n-6)}{2(n-2)}$, $\lambda_4 = \frac{1}{2(2-n)}$ and τ is the scalar curvature of (M,g). This completes the proof. □

The Weyl projective curvature tensor with respect to the connection (3.10) is given by

$$\widetilde{P}_{ijkl} = \widetilde{R}_{ijkl} - \frac{1}{n-1}\left[\widetilde{R}_{jk}g_{il} - \widetilde{R}_{ik}g_{jl}\right]. \tag{5.32}$$

Substituting the values of $\widetilde{R}_{ijkl}$ and $\widetilde{R}_{ik}$ from (5.21) and (5.28) respectively into (5.32) we get

$$\begin{aligned} \widetilde{P}_{ijkl} = {} & P_{ijkl} + g_{ik}\mathcal{A}_{jl} - g_{jk}\mathcal{A}_{il} \\ & -F_{il}F_k{}^t\mathcal{A}_{jt} + F_{jl}F_k{}^t\mathcal{A}_{it} + F_{ik}F_l{}^t\mathcal{A}_{jt} - F_{jk}F_l{}^t\mathcal{A}_{it} \\ & -\frac{1}{n-1}[(3\mathcal{A}_{jk} - g_{jk}(trace\mathcal{A}) + F_{jk}F^{mt}\mathcal{A}_{mt} \\ & +(2-traceF)F_k{}^t\mathcal{A}_{jt})g_{il} - (3\mathcal{A}_{ik} - g_{ik}\mathcal{A}_m^m + \\ & F_{ik}F^{mt}\mathcal{A}_{mt} + (2-traceF)F_k{}^t\mathcal{A}_{it})g_{jl}], \end{aligned} \tag{5.33}$$

where P_{ijkl} is the Weyl projective curvature tensor with respect to the Levi-Civita connection. Interchanging i and j in (5.33), and then adding it to (5.33), we obtain

$$\widetilde{P}_{ijkl} + \widetilde{P}_{jikl} = P_{ijkl} + P_{jikl}.$$

Since in a Riemannian manifold $P_{ijkl} + P_{jikl} = 0$, we have $\widetilde{P}_{ijkl} = -\widetilde{P}_{jikl}$.

Theorem 5.8 *Let (M,g,F) be a locally decomposable golden Riemannian manifold endowed with the connection (3.10). If the Weyl projective curvature tensor with respect to the connection (3.10) vanishes, then the 1-form p is closed, under the condition of $(traceF-2)^2 - (n-4)(n+traceF-6) \neq 0$.*

Proof Let us assume that $\widetilde{P}_{ijkl} = 0$. Then from (5.33) it follows that

$$\begin{aligned} 0 = {} & P_{ijkl} + g_{ik}\mathcal{A}_{jl} - g_{jk}\mathcal{A}_{il} \\ & -F_{il}F_k{}^t\mathcal{A}_{jt} + F_{jl}F_k{}^t\mathcal{A}_{it} + F_{ik}F_l{}^t\mathcal{A}_{jt} - F_{jk}F_l{}^t\mathcal{A}_{it} \\ & -\frac{1}{n-1}[(3\mathcal{A}_{jk} - g_{jk}(trace\mathcal{A}) + F_{jk}F^{mt}\mathcal{A}_{mt} \\ & +(2-traceF)F_k{}^t\mathcal{A}_{jt})g_{il} - (3\mathcal{A}_{ik} - g_{ik}\mathcal{A}_m^m + \\ & F_{ik}F^{mt}\mathcal{A}_{mt} + (2-traceF)F_k{}^t\mathcal{A}_{it})g_{jl}]. \end{aligned}$$

Contracting the previous equation with g^{kl}, using $P_{ijk}{}^{k}=0$ we get

$$(n-4)\left(\nabla_j p_k-\nabla_k p_j\right)+(traceF-2)\,F_k{}^t\left(\nabla_j p_t-\nabla_t p_j\right)=0. \tag{5.34}$$

Transvecting (5.34) by $F_k{}^i$, we find

$$\left[(traceF-2)^2-(n-4)(n+traceF-6)\right]\left(\nabla_j p_k-\nabla_k p_j\right)=0$$

from which the result immediately follows, under the condition of $(traceF-2)^2-(n-4)(n+traceF-6)\neq 0$. □

6. The transposed connection of the golden semisymmetric metric F-connection

The transpose of a connection ∇' on M is defined by

$${}^t\nabla'_X Y=\nabla'_Y X+[X,Y] \tag{6.35}$$

for all vector fields X and Y on M. These type connections have been studied by some author (see [5, 6, 10]). Taking into account $S'(X,Y)=\nabla'_X Y-\nabla'_Y X-[X,Y]$, (6.35) can be rewritten in the form

$${}^t\nabla'_X Y=\nabla'_X Y-S'(X,Y).$$

The torsion tensor ${}^tS'$ of the transposed connection ${}^t\nabla'$ is ${}^tS'=-S'$. Denoting by ${}^t\widetilde{\nabla}$ the transposed connection of the connection (3.10), the relations (3.8) and (3.10) lead to

$${}^t\widetilde{\Gamma}^k_{ij}=\Gamma^k_{ij}+p_i\delta_j{}^k-p^k g_{ij}+p_tF_i{}^tF_j{}^k-p^tF_t{}^kF_{ij}. \tag{6.36}$$

In view of Theorem 5.3, the torsion tensor ${}^tS'$ of the transposed connection (6.36) is a decomposable tensor.

Covariant differentiating the golden structure F and the Riemannian metric g with respect to the transposed connection ${}^t\widetilde{\nabla}$ we find

$${}^t\widetilde{\nabla}_k F^j_i=0$$

and

$${}^t\widetilde{\nabla}_k g_{ij}=p_ig_{jk}+p_jg_{ik}-2p_kg_{ij}-2p_tF_t{}^kF_{ij}+p_tF_j{}^tF_{ki}+p_tF_i{}^tF_{kj}\neq 0.$$

Hence we can say that the transposed connection (6.36) is a golden semisymmetric nonmetric F-connection.

Now we shall prove the following lemma.

Lemma 6.1 *Let (M,g,F) be a locally decomposable golden Riemannian manifold endowed with the connection (3.10) and $\widetilde{S}$ be the torsion tensor of the connection (3.10). Then for all vector fields X,Y and Z on M*

$$\underset{X,Y,Z}{\sigma}\widetilde{S}\left(\widetilde{S}(X,Y)\,Z\right)=0,$$

where σ is the cyclic sum by three arguments.

Proof In local coordinates, we can write

$$\underset{X,Y,Z}{\sigma}\widetilde{S}\left(\widetilde{S}\left(X,Y\right)Z\right)_{ijk}^{l}=\widetilde{S}_{ij}^{m}\widetilde{S}_{mk}^{l}+\widetilde{S}_{ki}^{m}\widetilde{S}_{mj}^{l}+\widetilde{S}_{jk}^{m}\widetilde{S}_{mi}^{l}$$

from which, using (3.8), standard calculations directly give the result. □

For the curvature tensor ${}^{t}\widetilde{R}$ of the transposed connection (6.36), we have

$$\begin{aligned}{}^{t}\widetilde{R}_{ijk}{}^{l} &= \partial_i{}^{t}\widetilde{\Gamma}_{jk}^{l}-\partial_j{}^{t}\widetilde{\Gamma}_{ik}^{l}+{}^{t}\widetilde{\Gamma}_{im}^{l}{}^{t}\widetilde{\Gamma}_{jk}^{m}-{}^{t}\widetilde{\Gamma}_{jm}^{l}{}^{t}\widetilde{\Gamma}_{ik}^{m}\\ &= \widetilde{R}_{ijk}{}^{l}-\widetilde{\nabla}_i\widetilde{S}_{jk}^{l}-\widetilde{\nabla}_j\widetilde{S}_{ki}^{l}-(\widetilde{S}_{ij}^{m}\widetilde{S}_{mk}^{l}+\widetilde{S}_{ki}^{m}\widetilde{S}_{mj}^{l}+\widetilde{S}_{jk}^{m}\widetilde{S}_{mi}^{l}).\end{aligned}$$

In view of Lemma 6.1, we obtain

$${}^{t}\widetilde{R}_{ijk}{}^{l}=\widetilde{R}_{ijk}{}^{l}-\widetilde{\nabla}_i\widetilde{S}_{jk}^{l}-\widetilde{\nabla}_j\widetilde{S}_{ki}^{l}. \tag{6.37}$$

It follows immediately by (6.37) that

$${}^{t}\widetilde{R}_{jik}{}^{l}=\widetilde{R}_{jik}{}^{l}-\widetilde{\nabla}_j\widetilde{S}_{ik}^{l}-\widetilde{\nabla}_i\widetilde{S}_{kj}^{l}=-\widetilde{R}_{ijk}{}^{l}+\widetilde{\nabla}_i\widetilde{S}_{jk}^{l}+\widetilde{\nabla}_j\widetilde{S}_{ki}^{l}=-{}^{t}\widetilde{R}_{ijk}{}^{l}.$$

Theorem 6.2 *Let (M,g,F) be a locally decomposable golden Riemannian manifold endowed with the transposed connection (6.36). The curvature tensor ${}^{t}\widetilde{R}$ of the transposed connection (6.36) is a ϕ-tensor.*

Proof From (6.37), we get

$$\begin{aligned}{}^{t}\widetilde{R}_{ijk}{}^{m}F_m^{\ l} &= \widetilde{R}_{ijk}{}^{m}F_m^{\ l}-(\widetilde{\nabla}_i\widetilde{S}_{jk}^{m})F_m^{\ l}-(\widetilde{\nabla}_j\widetilde{S}_{ki}^{m})F_m^{\ l},\\ {}^{t}\widetilde{R}_{mjk}{}^{l}F_i^{\ m} &= \widetilde{R}_{mjk}{}^{l}F_i^{\ m}-(\widetilde{\nabla}_m\widetilde{S}_{jk}^{l})F_i^{\ m}-(\widetilde{\nabla}_j\widetilde{S}_{km}^{l})F_i^{\ m},\\ {}^{t}\widetilde{R}_{imk}{}^{l}F_j^{\ m} &= \widetilde{R}_{imk}{}^{l}F_j^{\ m}-(\widetilde{\nabla}_i\widetilde{S}_{mk}^{l})F_j^{\ m}-(\widetilde{\nabla}_m\widetilde{S}_{ki}^{l})F_j^{\ m},\\ {}^{t}\widetilde{R}_{ijm}{}^{l}F_k^{\ m} &= \widetilde{R}_{ijm}{}^{l}F_k^{\ m}-(\widetilde{\nabla}_i\widetilde{S}_{jm}^{l})F_k^{\ m}-(\widetilde{\nabla}_j\widetilde{S}_{mi}^{l})F_k^{\ m}.\end{aligned}$$

Due to $(\widetilde{\nabla}_m\widetilde{S}_{ij}^{l})F_k^{\ m}-(\widetilde{\nabla}_k\widetilde{S}_{mj}^{l})F_i^{\ m}=(\nabla_m\widetilde{S}_{ij}^{l})F_k^{\ m}-(\nabla_k\widetilde{S}_{mj}^{l})F_i^{\ m}=0$, using the purity of the curvature tensor $\widetilde{R}$ and the torsion tensor $\widetilde{S}$ of the connection (3.10) we have

$${}^{t}\widetilde{R}_{ijk}{}^{m}F_m^{\ l}={}^{t}\widetilde{R}_{mjk}{}^{l}F_i^{\ m}={}^{t}\widetilde{R}_{imk}{}^{l}F_j^{\ m}={}^{t}\widetilde{R}_{ijm}{}^{l}F_k^{\ m},$$

i.e. the curvature tensor ${}^{t}\widetilde{R}$ is pure.

When we apply the Tachibana operator ϕ_F to the curvature tensor ${}^{t}\widetilde{R}$, we obtain

$$\begin{aligned}(\phi_F{}^{t}\widetilde{R})_{kijl}{}^{s} &= F_k^{\ m}(\partial_m{}^{t}\widetilde{R}_{ijl}{}^{s})-\partial_k({}^{t}\widetilde{R}_{ijl}{}^{m}F_m^{\ s})\\ &= F_k^{\ m}(\nabla_m{}^{t}\widetilde{R}_{ijl}{}^{s}+\Gamma_{mi}^{n}{}^{t}\widetilde{R}_{njl}{}^{s}+\Gamma_{mj}^{n}{}^{t}\widetilde{R}_{inl}{}^{s}+\Gamma_{ml}^{n}{}^{t}\widetilde{R}_{ijn}{}^{s}-\Gamma_{mn}^{s}{}^{t}\widetilde{R}_{ijl}{}^{n})\\ &\quad -F_m^{\ s}(\nabla_k{}^{t}\widetilde{R}_{ijl}{}^{m}+\Gamma_{ki}^{n}{}^{t}\widetilde{R}_{njl}{}^{m}+\Gamma_{kj}^{n}{}^{t}\widetilde{R}_{inl}{}^{m}+\Gamma_{kl}^{n}{}^{t}\widetilde{R}_{ijn}{}^{m}-\Gamma_{kl}^{n}{}^{t}\widetilde{R}_{ijn}{}^{m})\\ &= (\nabla_m{}^{t}\widetilde{R}_{ijl}{}^{s})F_k^{\ m}-(\nabla_k{}^{t}\widetilde{R}_{ijl}{}^{m})F_m^{\ s}.\end{aligned}$$

Substituting (6.37) into the above relation, we obtain

$$(\phi_F {}^t\widetilde{R})_{kijl}{}^s = (\nabla_m \widetilde{R}_{ijl}{}^s) F_k{}^m - (\nabla_k \widetilde{R}_{ijl}{}^m) F_m{}^s - (\nabla_m \widetilde{\nabla}_i \widetilde{S}^s_{jl}) F_k{}^m$$
$$-(\nabla_m \widetilde{\nabla}_j \widetilde{S}^s_{li}) F_k{}^m + (\nabla_k \widetilde{\nabla}_i \widetilde{S}^m_{jl}) F_m{}^s + (\nabla_k \widetilde{\nabla}_j \widetilde{S}^m_{li}) F_m{}^s. \tag{6.38}$$

Substituting (3.8) into (6.38), in view of Theorem 5.2, we get

$$(\phi_F {}^t\widetilde{R})_{kijl}{}^s = [(\nabla_m \widetilde{\nabla}_i p_j) F_k{}^m - (\nabla_k \widetilde{\nabla}_i p_m) F_j^m] \delta_l^s$$
$$-[(\nabla_m \widetilde{\nabla}_i p_l) F_k{}^m - (\nabla_k \widetilde{\nabla}_i p_m) F_l{}^m] \delta_j^s$$
$$+[(\nabla_m \widetilde{\nabla}_j p_i) F_k{}^m - (\nabla_k \widetilde{\nabla}_j p_m) F_i{}^m] \delta_l^s$$
$$+[(\nabla_m \widetilde{\nabla}_j p_l) F_k{}^m - (\nabla_k \widetilde{\nabla}_j p_m) F_l{}^m] \delta_i^s. \tag{6.39}$$

It is easy to see that $(\nabla_m \widetilde{\nabla}_i p_j) F_k{}^m - (\nabla_k \widetilde{\nabla}_i p_m) F_j{}^m = (\nabla_m \nabla_i p_j) F_k{}^m - (\nabla_k \nabla_i p_m) F_j{}^m$. Thus (6.39) can be rewritten in the form

$$(\phi_F {}^t\widetilde{R})_{kijl}{}^s = [(\nabla_m \nabla_i p_j) F_k{}^m - (\nabla_k \nabla_i p_m) F_j{}^m] \delta_l{}^s$$
$$-[(\nabla_m \nabla_i p_l) F_k{}^m - (\nabla_k \nabla_i p_m) F_l{}^m] \delta_j{}^s$$
$$+[(\nabla_m \nabla_j p_i) F_k{}^m - (\nabla_k \nabla_j p_m) F_i{}^m] \delta_l{}^s$$
$$+[(\nabla_m \nabla_j p_l) F_k{}^m - (\nabla_k \nabla_j p_m) F_l{}^m] \delta_i{}^s,$$

from which the result follows from the fact that $(\nabla_m \nabla_i p_j) F_k{}^m = (\nabla_k \nabla_i p_m) F_j{}^m$ (see the proof of lemma 5.1). □

As a direct consequence of (2.6) and Theorem 6.2, we state the following theorem.

Theorem 6.3 *Let (M, g, F) be a locally decomposable golden Riemannian manifold endowed with the transposed connection (6.36). The curvature tensor ${}^t\widetilde{R}$ of the transposed connection (6.36) is a decomposable tensor with respect to the product structures associated with the golden structure F.*

By (5.27) and (6.37), we obtain

$$ {}^t\widetilde{R}_{ijk}{}^l + {}^t\widetilde{R}_{jki}{}^l + {}^t\widetilde{R}_{kij}{}^l = \widetilde{R}_{ijk}{}^l + \widetilde{R}_{jki}{}^l + \widetilde{R}_{kij}{}^l$$
$$-2(\widetilde{\nabla}_k \widetilde{S}^l_{ij} + \widetilde{\nabla}_j \widetilde{S}^l_{ki} + \widetilde{\nabla}_i \widetilde{S}^l_{jk})$$
$$= -(\widetilde{R}_{ijk}{}^l + \widetilde{R}_{jki}{}^l + \widetilde{R}_{kij}{}^l),$$

which in view of Theorem 5.4 gives the following theorem.

Theorem 6.4 *Let (M, g, F) be a locally decomposable golden Riemannian manifold endowed with the transposed connection (6.36). If the curvature tensor ${}^t\widetilde{R}$ of the connection (6.36) satisfies the following:*

$$ {}^t\widetilde{R}_{ijk}{}^l + {}^t\widetilde{R}_{jki}{}^l + {}^t\widetilde{R}_{kij}{}^l = 0 \tag{6.40}$$

then, under the condition of $(traceF - 2)^2 - (n-4)(n + traceF - 6) \neq 0$, the generator p is closed. Conversely, if the generator p is closed, then the relation (6.40) is satisfied.

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