

ON GOOD REDUCTION OF SOME K3 SURFACES RELATED TO ABELIAN SURFACES

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(Received July 1, 2013, revised January 20, 2014)

Abstract. The Néron–Ogg–Šafarevič criterion for abelian varieties tells that the Galois action on the l -adic étale cohomology of an abelian variety over a local field determines whether the variety has good reduction or not. We prove an analogue of this criterion for a certain type of K3 surfaces closely related to abelian surfaces. We also prove its p -adic analogue. This paper includes T. Ito’s unpublished result on Kummer surfaces.

Introduction. We consider the problem of determining whether a variety over a local field have good reduction in terms of the Galois action on the l -adic étale cohomology of the variety.

An ideal situation is the case of abelian variety: the reduction type (good or bad) is completely determined by the Galois action on the (first) l -adic étale cohomology group (the Néron–Ogg–Šafarevič criterion, see Theorem 1.15). Kulikov [11] essentially showed a similar (potential) criterion for K3 surfaces in the category of complex manifolds, not schemes. In the mixed characteristic case, Tetsushi Ito obtained in 2001 a result for some special K3 surfaces, the Kummer surfaces (see Theorem 1.16).

In this paper, we prove analogous results for K3 surfaces admitting Shioda–Inose structures of product type (see Definition 1.12), which are closely related to abelian surfaces.

(Recently we obtained a similar result for far more general classes of K3 surfaces ([15]) by using different methods.)

We state our main theorems. First let us fix the notation. Let K be a local field (by which we mean a complete discrete valuation field with perfect residue field) of characteristic 0 and denote by \mathcal{O}_K its ring of integers, by p the residue characteristic, and by G_K the absolute Galois group of K . A proper smooth variety X over K is said to have *good reduction* over K if there exists a proper smooth scheme \mathcal{X} over \mathcal{O}_K having X as the generic fiber. A G_K -module is said to be *unramified* if the inertia subgroup I_K of G_K acts on it trivially. Our first result is the following:

THEOREM 0.1. *Let K be a local field of characteristic 0 and of residue characteristic $p \neq 2, 3$, and l a prime number different from p . Let Y be a K3 surface over K admitting a Shioda–Inose structure of product type. If $H_{\text{ét}}^2(Y_{\overline{K}}, \mathbb{Q}_l)$ is unramified, then $Y_{K'}$ has good reduction for some finite extension K' of K of ramification index 1, 2, 3, 4 or 6.*

At present we do not know whether a field extension is necessary.

Although we stated our results for K of characteristic 0, they are valid for K of positive characteristic if we replace the phrases “finite extension of K of ramification index N ” in the statements and the proofs by “finite extension of K which is purely inseparable over a finite extension of K of ramification index N ”. We omit the details.

We also prove results concerning p -adic cohomology, for both Kummer surfaces and K3 surfaces with Shioda–Inose structure of product type. (This time we cannot consider positive characteristic case, since the notion of crystalline representation is defined only for characteristic 0.)

THEOREM 0.2. *Let K be a local field with residue characteristic $p \neq 2$, and X a Kummer surface over K . Assume that X has at least one K -rational point. If $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline, then $X_{K'}$ has good reduction for some finite unramified extension K'/K .*

THEOREM 0.3. *Let K be a local field with residue characteristic $p \neq 2, 3$ and Y a K3 surface over K with Shioda–Inose structure of product type. If $H_{\text{ét}}^2(Y_{\overline{K}}, \mathbb{Q}_p)$ is crystalline, then $Y_{K'}$ has good reduction for some finite extension K'/K of ramification index 1, 2, 3, 4 or 6.*

As an immediate corollary of Theorems 0.1 to 0.3 (and Theorem 1.14), we have a criterion for potential good reduction. The word “potential(ly)” means after replacing the base field by a finite extension.

COROLLARY 0.4. *Let X be a K3 surface that belongs to one of the above two types. Then the following properties are equivalent:*

- (1) *The surface X has potential good reduction.*
- (2) *For some prime $l \neq p$, the second l -adic étale cohomology of X is potentially unramified.*
- (3) *For any prime $l \neq p$, the second l -adic étale cohomology of X is potentially unramified.*
- (4) *The second p -adic étale cohomology of X is potentially crystalline.*

There is an application to the reduction of singular K3 surface. Recall that a K3 surface over a field of characteristic 0 is called *singular* if it has the maximum possible geometric Picard number 20. Note that the word singular here does not mean non-smooth.

COROLLARY 0.5. *Any singular K3 surface has potential good reduction.*

The structure of this paper is as follows. In Section 1 we give some preliminary results. We prove Theorem 0.1 and Corollary 0.5 in Section 2 and Theorems 0.2, 0.3 in Section 3. As an appendix, we give a proof of Ito’s unpublished result (Theorem 1.16) in Section 4.

Acknowledgments. I am deeply grateful to my advisor Atsushi Shiho for his invaluable support and also for helpful suggestions on Section 3. I also thank Tetsushi Ito for the permission to use his unpublished master’s thesis as an appendix. I would also like to thank Shouhei Ma and Takeshi Saito for their helpful comments.

1. Preliminaries.

1.1. General results. In this subsection, we prove some basic results which will be used later.

LEMMA 1.1. *Let X be a geometrically connected variety over F and assume that X has at least one F -rational point. Then the natural map $\text{Pic } X \rightarrow (\text{Pic } X_{\overline{F}})^{G_F}$ is an isomorphism.*

PROOF. Recall that $\text{Pic } X \cong H_{\text{ét}}^1(X, \mathbb{G}_m)$ and $\text{Pic } X_{\overline{F}} \cong H_{\text{ét}}^1(X_{\overline{F}}, \mathbb{G}_m)$. We use the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(G_F, H_{\text{ét}}^q(X_{\overline{F}}, \mathbb{G}_m)) \Rightarrow H_{\text{ét}}^{p+q}(X, \mathbb{G}_m).$$

Since

$$E^{1,0} = H^1(G_F, H_{\text{ét}}^0(X_{\overline{F}}, \mathbb{G}_m)) = H^1(G_F, \overline{F}^*) = 0,$$

we have an exact sequence

$$0 \rightarrow \text{Pic } X \rightarrow (\text{Pic } X_{\overline{F}})^{G_F} \rightarrow H_{\text{ét}}^2(\text{Spec } F, \mathbb{G}_m) \rightarrow H_{\text{ét}}^2(X, \mathbb{G}_m).$$

Since X has at least one F -rational point by assumption, the morphism $X \rightarrow \text{Spec } F$ has a section $s: \text{Spec } F \rightarrow X$, which induces a splitting $s^*: H_{\text{ét}}^2(X, \mathbb{G}_m) \rightarrow H_{\text{ét}}^2(\text{Spec } F, \mathbb{G}_m)$ of the last map in the above sequence. Hence the map $H_{\text{ét}}^2(\text{Spec } F, \mathbb{G}_m) \rightarrow H_{\text{ét}}^2(X, \mathbb{G}_m)$ is injective. The conclusion follows from this. \square

LEMMA 1.2. *Let S be a scheme, X a scheme over S , and $Z \subset X$ a closed subscheme of X . Assume that X is smooth over S and that the composite $Z \hookrightarrow X \rightarrow S$ is an isomorphism. Then for any S -scheme S' , the canonical morphism $\text{Bl}_{(Z \times_S S')} (X \times_S S') \rightarrow (\text{Bl}_Z X) \times_S S'$ is an isomorphism. (In short, this blow-up is compatible with base change.)*

PROOF. An easy computation shows that the assertion holds if $X = \mathbb{A}_S^d$ and Z is the image of an S -valued point of X . In the general case, since the assertion is local, we may assume that $X \rightarrow S$ factors $f: X \rightarrow X_0 = \mathbb{A}_S^d$ with f étale. Let Z_0 be the scheme-theoretic image of Z under f . It follows that the composite $Z_0 \hookrightarrow X_0 \rightarrow S$ is an isomorphism and that Z is an open and closed subscheme of $Y = X \times_{X_0} Z_0$. Using the assertion for the case $X = \mathbb{A}_S^d$ and the fact that blow-up commutes with flat base change, we obtain, for arbitrary $S' \rightarrow S$,

$$\begin{aligned} \text{Bl}_{Y'} X' &\cong (\text{Bl}_{Z'_0} X'_0) \times_{X'_0} X' \xrightarrow{\sim} (\text{Bl}_{Z_0} X_0) \times_S S' \times_{X'_0} X' \\ &\cong (\text{Bl}_{Z_0} X_0) \times_{X_0} X \times_S S' \cong (\text{Bl}_Y X) \times_S S'. \end{aligned}$$

Here the symbol $'$ means the base change by $S' \rightarrow S$. The assertion follows from this and the fact that $\text{Bl}_Z X$ is isomorphic to $\text{Bl}_Y X$ outside $Y \setminus Z$ and to X outside Z and the corresponding fact for $\text{Bl}_{Z'} X'$. \square

LEMMA 1.3. *Let F be a field of characteristic $\neq 2$ and X be a connected smooth proper variety over F . Let Z be an effective divisor on X over F with no multiple component. Then the class $[Z]$ of Z in $\text{Pic}(X)$ is divisible by 2 if and only if there is a double covering*

$Y \rightarrow X$ whose branch locus is Z . If $\text{Pic}(X)$ has no 2-torsion and F is algebraically closed, then such a covering is unique up to isomorphism.

PROOF. This is easy. \square

1.2. K3 surfaces. We collect facts concerning K3 surfaces. Recall that K3 surface is a proper smooth minimal surface X with $H^1(X, \mathcal{O}_X) = 0$ and $\Omega_X^2 \cong \mathcal{O}_X$.

LEMMA 1.4. *Let F be an algebraically closed field of characteristic $\neq 2$ and X a K3 surface over F . Then the following properties hold.*

- (1) $\text{Pic}(X)$ is a finitely generated group.
- (2) $\text{Pic}(X)$ is 2-torsion-free.

PROOF. (1) The Picard group $\text{Pic}(X)$ has a scheme structure over F and the connected component $\text{Pic}^0(X)$ of the identity is an abelian variety of dimension $\leq \dim H^1(X, \mathcal{O}_X)$ ([8, n° 236, Proposition 2.10]). Since X is a K3 surface, we have $\dim H^1(X, \mathcal{O}_X) = 0$. Then the assertion follows from the fact that the Néron–Severi group $\text{Pic}(X)/\text{Pic}^0(X)$ is finitely generated for any proper smooth variety X .

(2) For each $n \geq 1$, we have an injection $\text{Pic}(X)/l^n \text{Pic}(X) \hookrightarrow H_{\text{ét}}^2(X, \mu_{l^n})$ by the Kummer sequence

$$0 \rightarrow \mu_{l^n} \rightarrow \mathbb{G}_m \xrightarrow{l^n} \mathbb{G}_m \rightarrow 0.$$

The inverse limit of these injections is also injective. Since $\text{Pic}(X)$ is finitely generated, it follows that $\varprojlim_n \text{Pic}(X)/l^n \text{Pic}(X) = \text{Pic}(X) \otimes \mathbb{Z}_l$. So it suffices to show that $\varprojlim_n H_{\text{ét}}^2(X, \mu_{l^n}) = H_{\text{ét}}^2(X, \mathbb{Z}_l)(1)$ is (2-)torsion free for $l = 2$.

If F is of characteristic 0, since the singular cohomology $H^2(X, \mathbb{Z})$ of a complex K3 surface is torsion-free ([1, Proposition VIII.3.3]), we obtain the assertion by using the comparison theorem. If F is not of characteristic 0, we lift X to characteristic 0 (this is always possible for K3 surfaces by Deligne [5, Corollaire 1.8]) and use the proper base change theorem to reduce to the characteristic 0 case. \square

An automorphism of a K3 surface is said to be *symplectic* if it fixes a non-vanishing holomorphic 2-form. Note that, since the canonical divisor of a K3 surface is trivial, such a 2-form exists and is unique up to constant multiple.

The next lemma is important in studying symplectic involutions. This is a part of a result of Nikulin [17, Section 5] for characteristic 0, and Dolgachev–Keum [6, Theorem 3.3] pointed out that Nikulin’s argument stays valid for arbitrary characteristic $\neq 2$.

LEMMA 1.5. *Let ι be a symplectic involution of a K3 surface X over an algebraically closed field of characteristic $\neq 2$. Then ι fixes exactly eight points and $X/\langle \iota \rangle$ is birational to a K3 surface.*

Next propositions are useful when we want K3 surfaces to have elliptic surface structures.

PROPOSITION 1.6 (Pjateckiĭ–Šapiro–Šafarevič [18, Section 3, Theorem 1]). *Let F be a field of characteristic $\neq 2, 3$. Let X be a K3 surface over F and D a non-trivial nef effective*

divisor on X satisfying $D^2 = 0$. Then the linear system $|D_{\overline{F}}|$ over \overline{F} contains a divisor of the form mC where $m > 0$ and C is an elliptic curve over \overline{F} .

PROPOSITION 1.7. (1) Let X and D be as in Proposition 1.6 and Z a smooth rational curve on X . Assume that D is connected and $Z \cdot D = 1$. Then $|D|$ gives an elliptic fibration $X \rightarrow \mathbb{P}^1$ having Z as the image of a section.

(2) Let D' be another divisor on X satisfying the same condition as D , and assume that $\text{Supp } D \cap \text{Supp } D' = \emptyset$. Then D' is another fiber of the elliptic fibration in (1).

PROOF. (1) We may assume $F = \overline{F}$. Let m and C be as in Proposition 1.6. Then m divides $mC \cdot Z = D \cdot Z = 1$ and hence we have $m = 1$.

By the same argument as in [18, Section 3], we have $\dim_F |D| = 2$ and hence a morphism $\Phi: X \rightarrow \mathbb{P}^1$. Then Φ is an elliptic fibration since Φ has a geometric fiber which is an elliptic curve by Proposition 1.6. By construction D is a fiber. Since $Z \cdot D = 1$, the composite $Z \hookrightarrow X \rightarrow \mathbb{P}^1$ is an isomorphism, hence its inverse is a section.

(2) Since each component of D' is disjoint from D , it is mapped to a point by Φ . Since D' is connected, every component goes to the same point p . Such a divisor has self-intersection 0 if and only if it is a rational multiple of the whole fiber $\Phi^{-1}(p)$ (this follows from an elementary computation, or see [24, Proposition III.8.2]). Comparing the intersection numbers with Z , it follows that D' coincides with $\Phi^{-1}(p)$. \square

In the following two lemmas, we consider surfaces over a local field K . We denote by l a prime different from the residue characteristic of K .

LEMMA 1.8 ([9, Lemmas 2.1 and 2.4]). Let X be a $K3$ surface or an abelian surface over K . Assume that $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified. Then the following holds.

- (1) Let $C \subset X_{\overline{K}}$ be a smooth rational curve. Then C is defined over a finite unramified extension of K .
- (2) Assume that X has a K -rational point. Let $X' \rightarrow X_{\overline{K}}$ be a double covering ramified along $\bigcup_i C_i \subset X_{\overline{K}}$ where each C_i is a smooth rational curve. Then $X' \rightarrow X_{\overline{K}}$ is defined over a finite unramified extension of K .

PROOF. (1) Recall that there exists the cycle map $\text{cl}: Z^1(X_{\overline{K}}) \rightarrow H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l)$ (1) which is compatible with the Galois action and the intersection pairing. Take any $\sigma \in I_K = G_{K^{\text{un}}}$. By the unramifiedness assumption, σ acts trivially on the image of cl . Therefore we have $C \cdot \sigma(C) = C \cdot C$. By the adjunction formula, this value is equal to -2 . Since distinct curves cannot have a negative intersection number, we have $\sigma(C) = C$. Since this holds for any $\sigma \in G_{K^{\text{un}}}$, it follows that C is defined over K^{un} and hence over an extension of desired type.

(2) The divisor $\bigcup C_i$ is defined over K^{un} since each C_i is defined over K^{un} by (1). By Lemma 1.3 we can take $Y \in \text{Pic } X_{\overline{K}}$ such that $2Y = [\bigcup C_i]$. Then since $\text{Pic } X_{\overline{K}}$ has no 2-torsion (Lemma 1.4), Y is $G_{K^{\text{un}}}$ -invariant. Since X has a K -rational point, Y is in $\text{Pic } X_{K^{\text{un}}}$ by Lemma 1.1. This shows, by Lemma 1.3 again, that $X' \rightarrow X_{\overline{K}}$ is defined over K^{un} and hence over an extension of desired type.

(Note that we would need further inseparable extensions if K is of positive characteristic.) \square

REMARK 1.9. By a similar argument, we have the following: for a K3 surface X over a field F and a field L containing \overline{F} , any smooth rational curve C on $X_{\overline{L}}$ is defined over \overline{F} .

LEMMA 1.10. *Let A be an abelian surface over K such that $H_{\text{ét}}^2(A_{\overline{K}}, \mathbb{Q}_l)$ is unramified. If $A_{\overline{K}}$ is isomorphic to the product of two elliptic curves, then so is $A_{K'}$ for some finite unramified extension K' of K .*

PROOF. Take a decomposition $A_{\overline{K}} = C_1 \times C_2$. We identify C_1 with the closed subscheme $C_1 \times \{0\}$ of A . By a similar argument as in the proof of Lemma 1.8 (1), it follows that $C_1 \cdot \sigma(C_1) = C_1 \cdot C_1 = 0$ for any $\sigma \in I_K = G_{K^{\text{un}}}$. The origin of A is in C_1 , and since it is a K -rational point, it is also in $\sigma(C_1)$. It follows that $\sigma(C_1) = C_1$ since otherwise $C_1 \cdot \sigma(C_1)$ should be ≥ 1 . This means that C_1 is defined over K^{un} and hence over a finite subextension K' of K^{un} . The same holds for C_2 . Then the addition map $C_1 \times C_2 \rightarrow A_{K'}$ is an isomorphism. \square

In this paper we consider two specific classes of K3 surfaces: (1) Kummer surfaces and (2) K3 surfaces which admit Shioda–Inose structures of product type.

DEFINITION 1.11. A K3 surface X over a field F of characteristic $\neq 2$ is a *Kummer surface* if, for some abelian surface A over \overline{F} , $X_{\overline{F}}$ is isomorphic to the minimal desingularization $\text{Km } A$ of the quotient surface $A/\langle -1 \rangle$ of A by the multiplication-by- (-1) map.

DEFINITION 1.12. We say that a K3 surface Y over a field F admits a *Shioda–Inose structure of product type* if $Y_{\overline{F}}$ admits an elliptic fibration $\Phi: Y_{\overline{F}} \rightarrow \mathbb{P}_{\overline{F}}^1$ which admits a section and two (singular) fibers of type II^* (in Kodaira’s notation).

REMARK 1.13. The usual notion of Shioda–Inose structure is as follows: a K3 surface Y over \mathbb{C} admits a *Shioda–Inose structure* if there exists a (necessarily symplectic) involution ι of Y such that the minimal desingularization X of the quotient surface $Y/\langle \iota \rangle$ is the Kummer surface of an abelian surface A and that the quotient maps induce a Hodge isometry $T_Y \cong T_A$, where T denotes the transcendental lattice of a complex surface.

A K3 surface Y over \mathbb{C} admits a Shioda–Inose structure of product type in the sense of Definition 1.12 if and only if it admits a Shioda–Inose structure in this sense with the corresponding abelian surface A being the product of two elliptic curves. For a proof of this assertion, see Shioda–Inose [23, Theorem 3]. We prefer Definition 1.12 since it is valid for an arbitrary base field.

One may ask when or how often a K3 surface admits a Shioda–Inose structure, and when it is of product type.

Naively thinking, since the K3 surfaces which admit Shioda–Inose structures (resp. those of product type) are in one-to-one correspondence to the abelian surfaces (resp. product abelian surfaces), they form a 3-dimensional (resp. 2-dimensional) moduli.

Another answer (for surfaces over \mathbb{C}) is the following criterion in terms of transcendental lattice: a K3 surface X over \mathbb{C} admits a Shioda–Inose structure if and only if there exists a primitive embedding $T_X \hookrightarrow U^3$ (Morrison [16, Theorem 6.3]), and it is of product type if and only if there exists a primitive embedding $T_X \hookrightarrow U^2$ ([16, Theorem 6.3] combined with [14, Corollary 3.5]). Here U denotes the hyperbolic plane (the lattice of rank 2 generated by e_1, e_2 with $e_i \cdot e_j = 1 - \delta_{ij}$). In particular, if X admits a Shioda–Inose structure (resp. of product type) then its Picard number is at least 17 (resp. at least 18).

1.3. Known criteria for good reduction. We recall the relation between cohomology and reduction of varieties over local fields, and the criteria for good reduction of abelian varieties. In this subsection K is a local field.

For general varieties, we have the following necessary condition for good reduction.

THEOREM 1.14. *Let X be a variety over K which has good reduction. Then the following properties hold.*

- (1) (consequence of the smooth base change theorem [21, Exposé XVI]) *For any prime $l \neq p$, the l -adic étale cohomology group $H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l)$ of X is unramified for any i .*
- (2) (consequence of the crystalline conjecture (Faltings [7, Theorems 5.3 and 5.6] and Tsuji [25, Theorem 0.2])) *The p -adic étale cohomology group $H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_p)$ of X is crystalline for any i .*

For abelian varieties, this condition is also sufficient.

THEOREM 1.15. *Let X be an abelian variety over K . Then the condition that X has good reduction is equivalent to each of the following.*

- (1) (Néron–Ogg–Šafarevič criterion, Serre–Tate [20, Theorem 1]) *For some (any) prime $l \neq p$, the first (all) l -adic étale cohomology of X is unramified.*
- (2) (Coleman–Iovita [4, Theorem 4.7]) *The first (all) p -adic étale cohomology of X is crystalline.*

The next result of Ito is an analogue of the above criterion for Kummer surfaces. In Ito’s paper it was assumed that $\text{char } K = 0$, but it is valid for positive characteristic in the sense we discussed in the introduction. Since his paper is unpublished, we include the proof of this theorem as an appendix under his permission.

THEOREM 1.16 (Ito [9, Corollary 4.3]). *Let K be a local field with residue characteristic $p \neq 2$ and l be a prime number different from p . Let X be a Kummer surface over K . Assume that X has at least one K -rational point. If $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified, then $X_{K'}$ has good reduction for some finite unramified extension K' of K .*

2. Proof of the l -adic result. In this section we prove Theorem 0.1 and Corollary 0.5.

Since the statement of Theorem 0.1 allows finite unramified extensions, we often use the same symbol K for finite unramified extensions of the original K .

We first outline the proof of Theorem 0.1 briefly. Let Y be as in the statement of the Theorem 0.1. It is known that there exist rational maps $Y_{\overline{K}} \rightarrow X_{\overline{K}}$ and $X_{\overline{K}} \rightarrow Y_{\overline{K}}$ of degree 2 for some Kummer surface $X_{\overline{K}}$ defined over \overline{K} . We

- (1) analyze the first map and construct a model X of $X_{\overline{K}}$ over a finite unramified extension of K ,
- (2) using the fact that $H_{\text{ét}}^2(Y_{\overline{K}}, \mathbb{Q}_l)$ is unramified, show that $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified after taking a finite extension of K of ramification index 1, 2, 3, 4 or 6,
- (3) use Ito's result to obtain a good model (that is, a smooth proper model) \mathcal{X} of X after taking a finite unramified extension of K , and
- (4) analyze the second map to construct a smooth \mathcal{O}_K -scheme \mathcal{Y} , which will be a good model of Y .

The use of two different rational maps $Y_{\overline{K}} \rightarrow X_{\overline{K}}$ and $X_{\overline{K}} \rightarrow Y_{\overline{K}}$ seems to be essential. See Remark 2.7.

Step (1). Let Y be a K3 surface admitting a Shioda–Inose structure of product type. Then by definition $Y_{\overline{K}}$ admits an elliptic fibration $\overline{\Phi}: Y_{\overline{K}} \rightarrow \mathbb{P}_{\overline{K}}^1$ with a section and two singular fibers D, D' of type II^* . We will show that this fibration is defined over some finite unramified extension of K .

The singular fiber $D = \sum n_i C_i$ consists of 9 smooth rational curves $C_1, \dots, C_9 \subset X_{\overline{K}}$. The image Z of a section of Φ is also a smooth rational curve on $X_{\overline{K}}$. By Lemma 1.8 (1), these curves are defined over a finite unramified extension of K (which again we denote by K for simplicity). Then by Proposition 1.7, there exists a unique elliptic fibration $\Phi: Y \rightarrow \mathbb{P}^1$ defined over K with a section $\mathbb{P}^1 \xrightarrow{\sim} Z \hookrightarrow Y$ and singular fibers D and D' .

CLAIM 2.1. *Under this situation, Y is defined up to birational equivalence by an equation of the form*

$$(*) \quad y^2 = x^3 + ax + (b_{-1}t^{-1} + b_0 + b_1t)$$

for some $a, b_{-1}, b_0, b_1 \in K$ with $b_{-1}, b_1 \neq 0$.

PROOF. Since Y is an elliptic surface over \mathbb{P}^1 , it is defined up to birational equivalence by a minimal Weierstrass form

$$y^2 = x^3 + A(t)x + B(t)$$

in $\mathbb{P}^2 \times \mathbb{P}^1$ with coordinates $((x, y), t)$ with $A, B \in K[t]$. Comparing the (topological) Euler numbers of Y and the singular fibers (Kodaira [10, Theorem 12.2]), we have

$$\max\{3 \deg A, 2 \deg B\} \leq \chi(Y) = 24$$

where $\chi(Y)$ is the Euler number of Y . (To be precise, Kodaira proved this for complex varieties and $\chi = \chi_{\text{top}}$, but this is valid in our case (with $\chi = \chi_{l, \text{ét}}$) provided that the characteristic is different from 2, 3.) Hence we have $\deg A \leq 8$ and $\deg B \leq 12$. We may assume that the two singular fibers of type II^* are above $t = 0, \infty$. Since the fiber above $t = 0$ is of type II^* , we have $\text{ord}_t A \geq 4$ and $\text{ord}_t B = 5$. Similarly, since the fiber above $t = \infty$ is of type II^* , we

have $\deg A \leq 8 - 4 = 4$ and $\deg B = 12 - 5 = 7$. Consequently we obtain an equation

$$y^2 = x^3 + at^4x + (b_{-1}t^5 + b_0t^6 + b_1t^7)$$

with $b_{-1}, b_1 \neq 0$. The true defining equation of Y (which we omit) is obtained by performing successive blow-ups on the above formula. By a simple change of variables, we obtain the desired equation

$$y^2 = x^3 + ax + (b_{-1}t^{-1} + b_0 + b_1t).$$

(This argument is similar to the one given by Shioda [22, Section 4]. However, since we are working on a field not algebraically closed, our formula is slightly more complicated than his.) \square

Using the coordinates of (*) above, we define an involution $\iota: Y \rightarrow Y$ by $(x, y, t) \mapsto (x, -y, b'/t)$ where $b' = b_{-1}/b_1$. Then the fixed points of ι (over \overline{K}) are exactly the 2-torsion points of $\Phi^{-1}(\pm\beta)$ (there are four for each) where $\beta \in \overline{K}$ is a square root of b' . We can check by some computation that the number of 2-torsion points is always four even if $\Phi^{-1}(\pm\beta)$ are not smooth (elliptic). The quotient $Y/\langle\iota\rangle$ has 8 double points over \overline{K} and its minimal desingularization X is a K3 surface (Lemma 1.5); in fact, it is the Kummer surface which appears in the definition of Shioda–Inose structure, and the corresponding abelian surface is the product of two elliptic curves after taking base change to the algebraic closure (Shioda [22, Theorem 1.1]).

Now we proceed to Step (2). The étale cohomology of X is given by

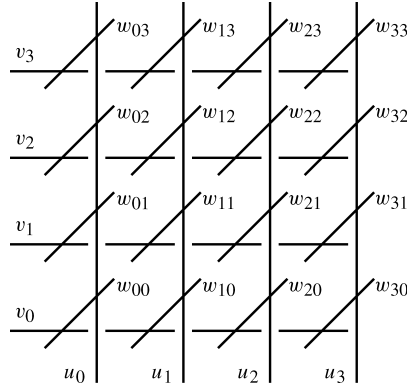
$$H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l) \cong H_{\text{ét}}^2(Y_{\overline{K}}, \mathbb{Q}_l)^{\langle\iota\rangle} \oplus \bigoplus_{i=1}^8 \mathbb{Q}_l(-1)[E_i].$$

Here the last term is the Tate twist of the permutation representation corresponding to the eight ι -fixed points (or the eight exceptional curves of $X \rightarrow Y/\langle\iota\rangle$). Let $H \subset G_K$ be the kernel of this permutation action and K_H/K the corresponding (finite) extension. Then the inertia subgroup of G_{K_H} acts on $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l)$ trivially. (This is the only place we need a (possibly) ramified extension.) In order to estimate the ramification index f of K_H/K , we use the next lemma.

LEMMA 2.2. *Let E be an elliptic curve defined over a local field K of residue characteristic $\neq 2, 3$. Then $K(E[2])/K$ is (at worst) tamely ramified and of ramification index at most 3.*

The same holds if E is a singular fiber of type I_2 or IV .

PROOF. We prove only the elliptic case. The remaining cases are similar. Since $E[2] \setminus \{0\}$ consists of 3 points, the extension $K(E[2])/K$ has a Galois group isomorphic to a subgroup of \mathfrak{S}_3 , and in particular the order of the Galois group divides 6. Hence the ramification is (at worst) tame and therefore the inertia group of $K(E[2])/K$ is cyclic. The order of a cyclic subgroup of \mathfrak{S}_3 is at most 3. \square

FIGURE 1. Curves u_i , v_j and w_{ij} .

An element of G_K belongs to H if and only if it fixes β and it fixes each 2-torsion point of both E_+ and E_- , where E_{\pm} are the fibers of Φ above $\pm\beta \in \mathbb{P}^1$. Let f_{\pm} be the ramification indices of $K'(E_{\pm}[2])/K'$ where $K' = K(\beta)$. By Lemma 2.2 we have $f_{\pm} \leq 3$.

If $K' = K$, then K_H is the compositum of $K(E_{\pm}[2])$ and hence of ramification index equal to $\text{lcm}(f_+, f_-)$ (by tameness).

If $K' \neq K$, then E_+ and E_- are conjugate under the nontrivial element of $\text{Gal}(K'/K)$ and hence $K'(E_+[2])$ and $K'(E_-[2])$ have the same ramification index $f_+ = f_-$ over K' . By tameness again, $K_H = K'(E_{\pm}[2])$ has ramification index f_{\pm} over K' . Hence K_H has ramification index f_{\pm} or $2f_{\pm}$ over K .

In each case we have $f \in \{1, 2, 3, 4, 6\}$.

Step (3). Since $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified as a representation of G_{K_H} , we can use Theorem 1.16 of Ito to obtain a good model \mathcal{X} over $\mathcal{O}_{K''}$ for some finite unramified extension K'' of K_H . (Note that X indeed has a K -rational point, for example the image of the intersection point of Z and D .)

Furthermore, by Lemma 1.10, the abelian surface \mathcal{A} over $\mathcal{O}_{K''}$ appearing in the proof of Theorem 1.16 (see the proof of Lemma 4.2) is the product of two elliptic curves over $\mathcal{O}_{K''}$ after replacing K'' by a finite unramified extension. This fact is used in the next step to obtain “rational curves” on \mathcal{X} .

Hereafter we write simply K instead of K'' . But note that this is a possibly ramified extension of the original K .

Now we turn to Step (4): the construction of a good model \mathcal{Y} from \mathcal{X} . This is the longest part of the proof. We first recall the construction of Shioda [22, Theorem 1.1], which describes Y (up to desingularization) as a quotient of X by an involution (instead of a double cover of X), and then extend this construction to the relative case (that is, over \mathcal{O}_K).

Fix a numbering $C_1[2] = \{p_i\}_{0 \leq i \leq 3}$ and $C_2[2] = \{q_j\}_{0 \leq j \leq 3}$: since C_1 and C_2 are defined and have good reduction over K , these points are defined over K after some finite unramified

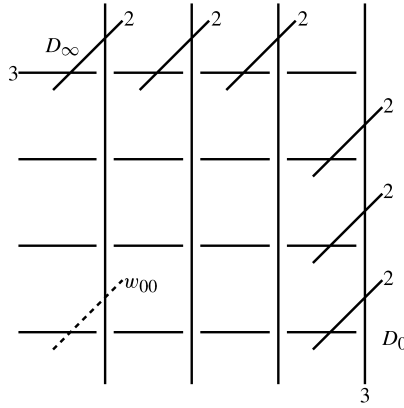


FIGURE 2. Divisors D_0 , D_∞ and w_{00} .

extension. The surface $X = \text{Km}(C_1 \times C_2)$ has 24 specific rational curves (defined over K): u_i , the strict transforms of the images of $p_i \times C_2$ under the quotient map; v_j , that of $C_1 \times q_j$; and the exceptional curves w_{ij} corresponding to the images of $p_i \times q_j$. The configuration of these curves are displayed in Figure 1: each w_{ij} meets u_i and v_j once, and there are no other intersections. We focus on three divisors

$$D_0 = v_0 + v_1 + v_2 + 2w_{30} + 2w_{31} + 2w_{32} + 3u_3,$$

$$D_\infty = u_0 + u_1 + u_2 + 2w_{03} + 2w_{13} + 2w_{23} + 3v_3$$

and w_{00} .

It is easily seen, from the configuration of these divisors displayed in Figure 2, that D_0 and D_∞ are disjoint divisors of type IV* with $w_{00} \cdot D_0 = w_{00} \cdot D_\infty = 1$. Then by Proposition 1.7 there exists an elliptic fibration $\Phi_X: X \rightarrow \mathbb{P}^1$ having D_0 and D_∞ as fibers and w_{00} as the image of a section.

Define two involutions ι_1 and ι_2 on X as follows. The multiplication-by- (-1) map on the generic fiber X_η (regarded as an elliptic curve over $\eta = \text{Spec } K(\mathbb{P}^1)$, the origin given by w_{00}) induces an involution ι_1 on X , which acts on each fiber also by (-1) . The multiplication-by- $(-1, 1)$ (or $(1, -1)$) map on $C_1 \times C_2$ induces an involution ι_2 on $X = \text{Km}(C_1 \times C_2)$. Put $\iota_X = \iota_1 \iota_2$.

CLAIM 2.3. *This automorphism ι_X is a symplectic involution and the minimal desingularization of $X/\langle \iota_X \rangle$ is a K3 surface isomorphic to Y .*

PROOF. We look at the defining equation (*). (By the uniqueness of the minimal smooth model, we can ignore desingularizations and consider only generic equations.) Letting

$$u = (t + b'/t) \quad \text{and} \quad w = (t - b'/t)^{-1}y,$$

we see that X is defined by the equation

$$(u^2 - 4b')w^2 = x^3 + ax + (b_0 + b_1u),$$

which indicates two elliptic fibration structure: one over \mathbb{P}^1 with coordinate u , with singular fibers of type $\{\text{II}^*, \text{I}_c^*, \text{I}_{c'}^*\}$ or $\{\text{II}^*, \text{I}_0^*, \text{IV}^*\}$, and another over \mathbb{P}^1 with coordinate w . Letting $v = uw + b_1/2w$, we obtain a Weierstrass equation

$$v^2 = x^3 + ax + \left(b_0 + 4b'w^2 + \frac{b_1^2}{4}w^{-2}\right)$$

relative to the latter fibration, with two singular fibers of type IV^* over $w = 0, \infty$. Then, by the explicit calculation of Kuwata–Shioda [12, Sections 2.2 and 5.3]), we see that this fibration coincides with our (D_0, D_∞) -fibration and that the involution ι_X acts on this equation by $(w, x, v) \mapsto (-w, x, v)$. Then the quotient X/ι_X is birational to Y . \square

We now describe the fixed points of ι_X explicitly. Let P_{ij} and Q_{ij} ($i, j \in \{0, 1, 2, 3\}$) be the intersection of w_{ij} with u_i and v_j respectively. Let ϕ (resp. ψ) be the involution of u_3 (resp. v_3) which fixes P_{30} (resp. Q_{03}) and interchanges P_{31} and P_{32} (resp. Q_{13} and Q_{23}) (such an involution is unique). Denote by $P_{3\infty}$ (resp. $Q_{\infty 3}$) the fixed point of ϕ (resp. ψ) other than P_{30} (resp. Q_{03}).

CLAIM 2.4. *The fixed points of ι_X are $P_{00}, Q_{00}, P_{03}, Q_{03}, P_{30}, Q_{30}, P_{3\infty}$ and $Q_{\infty 3}$.*

PROOF. By Lemma 1.5, we only have to show that ι_X indeed fixes these eight points. We will show that both ι_1 and ι_2 fix these points.

It is clear that ι_2 fixes each u_i and v_j pointwise. Hence ι_2 in particular fixes each P_{ij} , Q_{ij} , $P_{3\infty}$ and $Q_{\infty 3}$.

By construction ι_1 fixes w_{00} pointwise and hence P_{00} and Q_{00} . Since ι_1 also fixes each fiber (not pointwise), ι_1 fixes the components u_0, w_{03}, v_3 , and similarly v_0, w_{30}, u_3 (all not pointwise). Hence ι_1 fixes P_{03}, Q_{03}, P_{30} and Q_{30} . As ι_1 acts by -1 on the group scheme $(D_\infty)^{\text{sm}}$ (which is the disjoint union of three components each isomorphic to \mathbb{G}_a), ι_1 interchanges u_1 and u_2 , hence w_{13} and w_{23} , and hence Q_{13} and Q_{23} . This means that ι_1 acts on v_3 by ψ . Hence it fixes $Q_{\infty 3}$. Similarly it fixes $P_{3\infty}$. \square

Let \tilde{X} be the blow-up of X at these eight points and $\tilde{\iota}_X$ the involution of \tilde{X} induced by ι_X . Then one can easily check that Y , which is isomorphic to the minimal desingularization of $X/\langle \iota_X \rangle$, is also isomorphic to $\tilde{X}/\langle \tilde{\iota}_X \rangle$.

We will now extend this construction to the relative case (over \mathcal{O}_K). By the construction of \mathcal{X} and the fact that the abelian surface \mathcal{A} is the fibered product of two elliptic curves over \mathcal{O}_K , the 24 rational curves on X extends naturally to closed subschemes on \mathcal{X} each of which is isomorphic to $\mathbb{P}^1_{\mathcal{O}_K}$. Using these subschemes, we define divisors $\mathcal{D}_0, \mathcal{D}_\infty$ and \mathcal{W}_{00} similarly as D_0, D_∞ and w_{00} . Also we define closed subschemes $\mathcal{P}_{ij}, \mathcal{Q}_{ij}, \mathcal{P}_{3\infty}$ and $\mathcal{Q}_{\infty 3}$ (each isomorphic to $\text{Spec } \mathcal{O}_K$) similarly as $P_{ij}, Q_{ij}, P_{3\infty}$ and $Q_{\infty 3}$.

Hereafter, we denote schemes over \mathcal{O}_K by calligraphic letters (e.g. \mathcal{C}) and their generic fibers and special fibers by italic letters equipped with suffixes $_K$ and $_k$ (e.g. C_K and C_k). For

sheaves on \mathcal{O}_K -schemes or morphisms of \mathcal{O}_K -schemes, we denote their restrictions to the generic and the special fibers by the same letter with suffixes K and k (e.g. Φ_K and Φ_k are the restrictions of Φ).

We use the next proposition, which is a relative version of Proposition 1.7.

PROPOSITION 2.5. *Let \mathcal{X} be a smooth proper scheme over \mathcal{O}_K such that X_K and X_k are K3 surfaces respectively over K and k . Let \mathcal{Z} and \mathcal{C}_i be subschemes of \mathcal{X} and let $\mathcal{D} = \sum n_i \mathcal{C}_i$ be a (finite) linear combination. Assume that*

- *each \mathcal{C}_i and \mathcal{Z} is isomorphic to $\mathbb{P}_{\mathcal{O}_K}^1$,*
- *the intersection of \mathcal{Z} and \mathcal{D} is a scheme isomorphic to $\text{Spec } \mathcal{O}_K$, and*
- *$\mathcal{D} = \sum n_i \mathcal{C}_i$ is a configuration of the type I_n ($n \geq 2$), I_n^* ($n \geq 0$), II^* , $\text{III}^{(*)}$ or $\text{IV}^{(*)}$ in Kodaira's notation. (Here, the intersection of two components should be the spectrum of a ring (\mathcal{O}_K) instead of the spectrum of a field. We exclude types I_0 , I_1 and II because their components are not \mathbb{P}^1 .)*

(It then follows that both $D_K \subset X_K$ and $D_k \subset X_k$ satisfy conditions of Proposition 1.7.)

Then there exists an “elliptic fibration” $\Phi: \mathcal{X} \rightarrow \mathbb{P}_{\mathcal{O}_K}^1$ having \mathcal{D} as a “singular fiber” and \mathcal{Z} as the image of a section, that is, Φ satisfies the following:

- Φ is a proper surjection.
- $\Phi_K: X_K \rightarrow \mathbb{P}_K^1$ and $\Phi_k: X_k \rightarrow \mathbb{P}_k^1$ are elliptic fibrations in the usual meaning.
- The composite $\mathcal{Z} \hookrightarrow \mathcal{X} \rightarrow \mathbb{P}_{\mathcal{O}_K}^1$ is an isomorphism.
- There exists an \mathcal{O}_K -valued point $s \in \mathbb{P}_{\mathcal{O}_K}^1(\mathcal{O}_K)$ such that $\Phi^{-1}(s) = \mathcal{D}$.

Moreover if \mathcal{D}' is as in Proposition 1.7 (2) then \mathcal{D}' is another “singular fiber”.

To prove this, we need a well-known lemma on cohomology of fibers. For a proof see [13, Theorem 5.3.20].

LEMMA 2.6. *Let \mathcal{X} be a proper \mathcal{O}_K -scheme and \mathcal{F} a coherent sheaf on \mathcal{X} , flat over \mathcal{O}_K . Then*

- (1) $\dim_k H^p(X_k, \mathcal{F}_k) \geq \dim_K H^p(X_K, \mathcal{F}_K)$,
- (2) the equality in (1) holds if and only if $H^p(\mathcal{X}, \mathcal{F})$ is a free \mathcal{O}_K -module such that the canonical morphism $H^p(\mathcal{X}, \mathcal{F}) \otimes_{\mathcal{O}_K} k \rightarrow H^p(X_k, \mathcal{F}_k)$ is an isomorphism, and
- (3) the morphism in (2) is an isomorphism if and only if $H^{p+1}(\mathcal{X}, \mathcal{F})$ is a free \mathcal{O}_K -module.

PROOF OF PROPOSITION 2.5. First we show that $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D})) \cong \mathcal{O}_K^{\oplus 2}$.

We make use of the cohomology long exact sequence of the sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{X}}(\mathcal{D}) \rightarrow \mathcal{O}_{\mathcal{X}}(\mathcal{D}) \otimes \mathcal{O}_{\mathcal{D}} \rightarrow 0.$$

First note that $\mathcal{O}_{\mathcal{X}}(\mathcal{D})|_{\mathcal{D}} \cong \mathcal{O}_{\mathcal{D}}$: This isomorphism is true restricted on the generic fiber D_K since $(\mathcal{C}_i)_K \cdot D_K = 0$ for each component \mathcal{C}_i of \mathcal{D} , and then it is true on \mathcal{D} since $\text{Pic } \mathcal{C}_i$ is isomorphic to $\text{Pic}(\mathcal{C}_i)_K$ (since $\mathcal{C} \cong \mathbb{P}_{\mathcal{O}_K}^1$). Connectedness of X_K and X_k yields $H^0(X_K, \mathcal{O}_{X_K}) = K$ and $H^0(X_k, \mathcal{O}_{X_k}) = k$. Hence by Lemma 2.6 we have $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) =$

\mathcal{O}_K , and again by the lemma $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is free over \mathcal{O}_K . Since cohomology commutes with taking the generic fiber (which is a flat base change), $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \otimes_{\mathcal{O}_K} K = H^1(X_K, \mathcal{O}_{X_K}) = 0$ and hence $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$. Similarly, $H^0(\mathcal{D}, \mathcal{O}_{\mathcal{D}}) = \mathcal{O}_K$, and $H^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$ is free over \mathcal{O}_K . Combining these information, we obtain an exact sequence

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_K \rightarrow H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D})) \rightarrow \mathcal{O}_K \\ &\rightarrow 0 \rightarrow H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D})) \rightarrow H^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}). \end{aligned}$$

It follows that $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D})) \cong \mathcal{O}_K^{\oplus 2}$. So this “linear system” defines a morphism $\Phi: \mathcal{X} \rightarrow \mathbb{P}_{\mathcal{O}_K}^1$.

Next we will show that this construction is compatible with that of Proposition 1.7 (1), that is, $\Phi_K: X_K \rightarrow \mathbb{P}_K^1$ and $\Phi_k: X_k \rightarrow \mathbb{P}_k^1$ is the same as those constructed in Proposition 1.7 (1). This will show that Φ is the morphism wanted in the proposition.

Again by the compatibility with flat base change, we have $H^0(X_K, \mathcal{O}_{X_K}(D_K)) \cong H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D})) \otimes_{\mathcal{O}_K} K$. For the special fiber, $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D}))$ is a free \mathcal{O}_K -module (since it is a submodule of a free module $H^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$), so by the lemma we have $H^0(X_k, \mathcal{O}_{X_k}(D_k)) \cong H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D})) \otimes_{\mathcal{O}_K} k$. These equalities show that Φ_K and Φ_k are those constructed in Proposition 1.7 (1).

The last assertion is proved by following the proof of Proposition 1.7 (2). \square

We return to the proof of Theorem 0.1. Proposition 2.5 shows that there exists an “elliptic fibration” $\Phi_{\mathcal{X}}: \mathcal{X} \rightarrow \mathbb{P}_{\mathcal{O}_K}^1$. One defines $\iota_{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{X}$ similarly and observes (following the proof of Claim 2.4) that the fixed points are the union of $\mathcal{P}_{00}, \mathcal{Q}_{00}, \mathcal{P}_{03}, \mathcal{Q}_{03}, \mathcal{P}_{30}, \mathcal{Q}_{30}, \mathcal{P}_{3\infty}$ and $\mathcal{Q}_{\infty 3}$. Let $\mathcal{Y} = \tilde{\mathcal{X}}/\langle \tilde{\iota}_{\mathcal{X}} \rangle$ where $\tilde{\mathcal{X}}$ is the blow-up of \mathcal{X} at the (union of) fixed points and $\tilde{\iota}_{\mathcal{X}}$ is the involution on $\tilde{\mathcal{X}}$ induced by $\iota_{\mathcal{X}}$. We shall show that this \mathcal{Y} is a smooth proper model of Y .

Properness and flatness of \mathcal{Y} over \mathcal{O}_K is clear from the construction. We also know that Y and Y' are nonsingular, where we denote by Y' the surface obtained by performing similar operations on the special fiber X_k of \mathcal{X} . Hence it suffices to check that the generic fiber and the special fiber of \mathcal{Y} are isomorphic to Y and Y' respectively. Since we have assumed that the residue characteristic is not equal to the order of $\iota_{\mathcal{X}}$ ($=2$), $(\tilde{\mathcal{X}}/\langle \tilde{\iota}_{\mathcal{X}} \rangle) \times_{\mathcal{O}_K} k$ is isomorphic to $(\tilde{\mathcal{X}} \times_{\mathcal{O}_K} k)/\langle \tilde{\iota}_k \rangle$. Since this blow-up commutes with base change by Lemma 1.2, we have $\tilde{\mathcal{X}} \times_{\mathcal{O}_K} k \cong (X_k)^\sim$, and hence $(\tilde{\mathcal{X}} \times_{\mathcal{O}_K} k)/\langle \tilde{\iota}_k \rangle \cong (X_k)^\sim/\langle \tilde{\iota}_k \rangle \cong Y'$. The generic case is easier, as we do not need Lemma 1.2 since blow-up always commutes with flat base change.

This concludes the proof of Theorem 0.1.

REMARK 2.7. In the proof, we used two different rational maps $Y_{\bar{K}} \rightarrow X_{\bar{K}}$ and $X_{\bar{K}} \rightarrow Y_{\bar{K}}$. Here we explain why we needed both.

Let us try to construct X from Y via the map $X_{\bar{K}} \rightarrow Y_{\bar{K}}$. We can determine the branch locus of $X_{\bar{K}} \rightarrow Y_{\bar{K}}$ explicitly (it is the union of the components of odd multiplicity in two fibers of type Π^*) and so we can define X and $X \rightarrow Y$ over a finite unramified extension of K . However, for the relationship of their cohomology groups, we merely obtain

$$H_{\text{ét}}^2(Y_{\bar{K}}, \mathbb{Q}_l) \cong H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_l)^{(l)} \oplus \bigoplus \mathbb{Q}_l(-1)[E_i]$$

and we cannot deduce that $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified, even after taking some (ramified) extension on K .

Next let us try to construct \mathcal{Y} from \mathcal{X} via the map $Y_{\overline{K}} \rightarrow X_{\overline{K}}$. According to the construction of Shioda–Inose [23, Section 2], X admits an elliptic fibration with (at least) three singular fibers of types $\{\text{II}^*, \text{I}_c^*, \text{I}_{c'}^*\}$ or $\{\text{II}^*, \text{I}_0^*, \text{IV}^*\}$, and the branch locus of the morphism $Y \rightarrow X$ is the union of the components of multiplicity 1 in the fibers of type I_c^* or IV^* . Unfortunately, the types of singular fibers might differ between the generic and the special fibers of \mathcal{X} , and we have trouble constructing \mathcal{Y} as a double cover of \mathcal{X} .

REMARK 2.8. We can give an explicit but far from best possible bound for the degree of the extension needed.

Checking the proofs of Theorems 0.1 and 1.16, we see that the only places we need field extensions are (i) where we use Lemmas 1.8 (twice) and 1.10, and (ii) where we take the kernel of the permutation action on 8 fixed points. The degree of extension in (ii) has trivial bound $8!$. For each time in (i), it suffices to take an extension K' so that $G_{K'}$ acts trivially on $\text{NS}(W_{\overline{K}})$ where W is one of Y, X, A . The same arguments as in Remark 4.3 give explicit bounds.

Combining these, we have a bound $3^{484+484+36} \cdot 8!$, which is $< 10^{484}$.

(In the positive characteristic case, this gives a bound for the separable degree, but we could not control the inseparable degree.)

REMARK 2.9. We shall note the difficulties in the case residue characteristic p is equal to 2 or 3. The case $p = 2$ had to be excluded as involutions in characteristic 2 behaves so badly (for example, a “Kummer surface” (the desingularization of $A/\langle -1 \rangle$ for an abelian surface A) even may not be a K3 surface). If $p = 3$, things are better than $p = 2$, but elliptic surfaces in characteristic (2 or) 3 might have wild fibers which do not appear in the other characteristics, and this causes many troubles, so we had to exclude this case also.

We conclude this section with the proof of Corollary 0.5.

PROOF OF COROLLARY 0.5. Any singular K3 surface has Shioda–Inose structure of product type such that corresponding elliptic curves C_1, C_2 have complex multiplication (Shioda–Inose [23, Theorem 4]). Any elliptic curve with complex multiplication is defined over some number field and has good reduction over some extension. Using the construction of \mathcal{Y} from \mathcal{X} above, the corollary follows. \square

3. p -adic criterion. We now focus on p -adic cohomology. We denote by K_0 the unramified closure of \mathbb{Q}_p in K .

We first overview the proofs of Theorems 0.2 and 0.3. As in the l -adic case, the main idea for the Kummer case (resp. Shioda–Inose case) is to reduce to the abelian case (resp. Kummer case). However, there are some additional difficulties, and we need our l -adic results to overcome these difficulties. Given a Kummer surface X as in Theorem 0.2, we

(Km1) construct an abelian surface A corresponding to it over a finite extension of K ,

- (Km2) using the crystallineness hypothesis, show that $H_{\text{ét}}^2(A_{\overline{K}}, \mathbb{Q}_p)$ is crystalline after taking a finite unramified extension of K and a quadratic twist of A ,
 (Km3) hence obtain a good model \mathcal{A} (by Coleman–Iovita [4, Theorem 4.7]),
 (Km4) construct a good model \mathcal{X} using the rational map $A_{\overline{K}} \rightarrow X_{\overline{K}}$, and
 (Km5) show that we can take the extension in step (Km1) to be unramified.

Similarly, given a K3 surface Y admitting a Shioda–Inose structure of product type as in Theorem 0.3, we

- (SI1) construct a Kummer surface X corresponding to it over a finite extension of K ,
 (SI2) using the crystallineness hypothesis, show that $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline after taking a finite extension of K of ramification index 1, 2, 3, 4 or 6,
 (SI3) hence by Theorem 0.2, after some finite unramified extension, obtain a good model \mathcal{X} ,
 (SI3') which we may assume, after further finite unramified extension, to be obtained from the product of two elliptic curves and hence have 24 specific smooth “rational curves”,
 (SI4) construct a good model \mathcal{Y} using the rational map $X_{\overline{K}} \rightarrow Y_{\overline{K}}$, and
 (SI5) show that we can take the extension in step (SI1) to be unramified.

Steps (Km1) and (SI1) are easy. As in the l -adic case, it suffices to take an extension over which certain (finitely many) curves are rational. Note that we do not, at this moment, require the extension to be unramified.

Step (Km2): Let X be a Kummer surface such that $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline and A an abelian surface over K such that $\text{Km } A \cong X$. We show that (after taking some finite unramified extension of K) there exists an abelian surface A' such that $\text{Km } A' \cong \text{Km } A$ and $H_{\text{ét}}^1(A'_{\overline{K}}, \mathbb{Q}_p)$ is semi-stable, and show that it is then automatically crystalline.

There exists a finite Galois extension L/K such that $V = H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_p)$ is a semi-stable representation of G_L . By replacing K by its unramified closure in L , we may assume that L/K is totally ramified. Put $D = D_{\text{st},L}(V) = (B_{\text{st}} \otimes V)^{G_L}$. Since $D_{\text{st},L}$ commutes with exterior product for semi-stable representations of G_L , we have

$$D_{\text{st},K} \left(\bigwedge^2 V \right) = \left(D_{\text{st},L} \left(\bigwedge^2 V \right) \right)^G = \left(\bigwedge^2 D \right)^G,$$

where $G = \text{Gal}(L/K)$. Since $\bigwedge^2 V = H_{\text{ét}}^2(A_{\overline{K}}, \mathbb{Q}_p)$ is a crystalline (hence semi-stable) representation of G_K by assumption, and since V is semi-stable representation of G_L , we have

$$\dim_{K_0} \left(\bigwedge^2 D \right)^G = \dim_{\mathbb{Q}_p} \bigwedge^2 V = \dim_{L_0} \bigwedge^2 D.$$

Since $L_0 = K_0$, it follows that G acts on $\bigwedge^2 D$ trivially. Then by the next lemma, there exists a subgroup $G' \subset G$ of index at most 2 such that G' acts on D trivially.

LEMMA 3.1. *Let W be a vector space over a field with $\dim W \geq 3$ and f a linear automorphism of W . If $\bigwedge^2 f$ acts as the identity on $\bigwedge^2 W$ then f is either the identity or (-1) times the identity.*

PROOF. This is an easy exercise of linear algebra. \square

If $G' = G$ then $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_p)$ is already semi-stable, so we can take $A' = A$. Assume $G' \subsetneq G$. We follow the construction in the proof of Theorem 4.1. Let M be the quadratic extension of K corresponding to G' . Then we have $\dim_{K_0} D^{G'} = \dim_{K_0} D = \dim_{\mathbb{Q}_p} V$, which means that V is semi-stable as a representation of $G_M = G'$. Put $G'' = G/G' = \text{Gal}(M/K)$. Let $G'' \cong \{\pm 1\}$ act on A by ± 1 and consider another abelian surface $A' = (A \times_K M)/G''$ over K , where G'' acts diagonally on $A \times_K M$. It is clear from the construction that $\text{Km } A \cong \text{Km } A'$. The surfaces A'_M and A_M are naturally isomorphic and hence $H_{\text{ét}}^1(A'_{\overline{K}}, \mathbb{Q}_p) \cong H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_p)$ as a representation of G_M . The action of $g \in G_K$ on $H_{\text{ét}}^1(A'_{\overline{K}}, \mathbb{Q}_p)$ is equal to ± 1 times the action of g on $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_p)$, where the sign is positive if $g \in G_M$ and negative otherwise.

Put $V' = H_{\text{ét}}^1(A'_{\overline{K}}, \mathbb{Q}_p)$ and $D' = D_{\text{st},L}(V') = (V' \otimes B_{\text{st}})^{G_L}$. Then as in the proof of Theorem 4.1, $D_{\text{st},K}(V') = (D')^{G_K}$ has an appropriate dimension over K_0 . Thus V' is a semi-stable representation of G_K .

It remains to show that the monodromy N of $D_{\text{st},K}(V')$ is zero. From the crystalline hypothesis the monodromy of $D_{\text{st},K}(\bigwedge^2 V')$ is zero, that is, $N \wedge 1 + 1 \wedge N = 0$ on $\bigwedge^2 D_{\text{st},K}(V')$. By applying Lemma 3.1 to $f = \exp N = \sum N^k/k!$ (which is a finite sum since N is nilpotent) we obtain $\exp N = \pm 1$ and hence $N = 0$. This means V' is crystalline, thus concludes step (Km2).

Step (SI2): Assume surfaces Y and X are given, where X is the minimal desingularization of Y/ι . As in the l -adic case we have

$$H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p) \cong H_{\text{ét}}^2(Y_{\overline{K}}, \mathbb{Q}_p)^{(\iota)} \oplus \bigoplus_{i=1}^8 \mathbb{Q}_p(-1)[E_i],$$

where the last term is the Tate twist of the permutation representation corresponding to the eight ι -fixed points. As before, let $H \subset G_K$ be the kernel of this permutation action and K_H/K the corresponding (finite) extension, which is of ramification index 1, 2, 3, 4 or 6. Then $\bigoplus \mathbb{Q}_p(-1)[E_i]$, being the Tate twist of a trivial representation, is a crystalline representation of G_{K_H} . Also $H_{\text{ét}}^2(Y_{\overline{K}}, \mathbb{Q}_p)^{(\iota)}$, being a direct summand of a crystalline representation $H_{\text{ét}}^2(Y_{\overline{K}}, \mathbb{Q}_p)$, is crystalline. So step (SI2) is done.

Steps (Km3) and (SI3) are just applying the indicated results.

For step (SI3'), we need a p -adic analogue of Lemma 1.10. But we can reduce this to l -adic Lemma 1.10: if $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_p)$ is crystalline, then A has good reduction and hence $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_l)$ is unramified.

Steps (Km4) and (SI4) are the same as in the l -adic case.

For steps (Km5) and (SI5), we need a p -adic analogue of Lemma 1.8. The next proposition reduces this to l -adic Lemma 1.8. Note that the potential good reduction assumption is satisfied since we have already proved steps (Km1–Km4) and (SI1–SI4).

PROPOSITION 3.2. *Let X be a K3 surface over K with potential good reduction. Assume that $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline. Then $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified for any prime $l \neq p$.*

PROOF. Take a finite Galois extension K'/K such that $X_{K'}$ has good reduction. Then the action of I_K on $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l)$ factors through $I(K'/K) = I_K/I_{K'}$. Let \mathcal{X} be a good model of $X_{K'}$ over $\mathcal{O}_{K'}$, and let $\overline{X} = \mathcal{X} \times_{\mathcal{O}_{K'}} k'$.

Take an arbitrary element $\sigma \in I(K'/K)$ and denote by \mathcal{X}^σ the scheme obtained from \mathcal{X} by the base change $\sigma^*: \text{Spec } \mathcal{O}_{K'} \rightarrow \text{Spec } \mathcal{O}_{K'}$. Denote by Γ the scheme-theoretic closure of $\Delta(X_{K'})$ in $\mathcal{X} \times_{\mathcal{O}_{K'}} \mathcal{X}^\sigma$, where $\Delta: X_{K'} \rightarrow X_{K'} \times_{K'} X_{K'}$ is the diagonal map. Since $\mathcal{X} \times_{\mathcal{O}_{K'}} \mathcal{X}^\sigma$ is regular there exists a resolution $\mathcal{E}_\bullet \rightarrow \mathcal{O}_\Gamma$ of finite length by locally free modules of finite rank. Put

$$\begin{aligned} \tilde{\Gamma} &= \sum_i (-1)^i \text{ch}_2(\mathcal{E}_i) \in \text{CH}^2(\mathcal{X} \times_{\mathcal{O}_{K'}} \mathcal{X}^\sigma)_{\mathbb{Q}} \quad \text{and} \\ \overline{\Gamma} &= \tilde{\Gamma}|_{\overline{X} \times_{k'} \overline{X}} = \sum_i (-1)^i \text{ch}_2(\mathcal{E}_i|_{\overline{X} \times_{k'} \overline{X}}) \in \text{CH}^2(\overline{X} \times_{k'} \overline{X})_{\mathbb{Q}} \end{aligned}$$

(these do not depend on the choice of the resolution).

Then by the Riemann–Roch theorem (see [19, Lemma 2.17]) the restriction of $\tilde{\Gamma}$ on the generic fiber coincides with $\Delta(X_{K'})$. Hence we have a commutative diagram

$$\begin{array}{ccc} H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l) & \xleftarrow{=} & H_{\text{ét}}^i(\overline{X}_{k'}, \mathbb{Q}_l) \\ \sigma_* \downarrow & & \overline{\Gamma}^* \downarrow \\ H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l) & \xleftarrow{=} & H_{\text{ét}}^i(\overline{X}_{k'}, \mathbb{Q}_l) \end{array}$$

(Saito [19, Corollary 2.20]) and hence an equality

$$\text{Tr}(\sigma_* | H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l)) = \text{Tr}(\overline{\Gamma}^* | H_{\text{ét}}^i(\overline{X}_{k'}, \mathbb{Q}_l)).$$

By the Lefschetz trace formula we have

$$\sum_i (-1)^i \text{Tr}(\overline{\Gamma}^* | H_{\text{ét}}^i(\overline{X}_{k'}, \mathbb{Q}_l)) = (\overline{\Gamma}, \Delta(\overline{X})) = \sum_i (-1)^i \text{Tr}(\overline{\Gamma}^* | H_{\text{crys}}^i(\overline{X})),$$

where the intersection number is taken in $\overline{X} \times_{k'} \overline{X}$.

Since the isomorphism in the crystalline conjecture is compatible with pull-backs, cup products with cycle classes and direct images (Tsuji [25], [26, Theorem A2] and Berthelot–Ogus [2, Proposition 3.4]), it is compatible with the action of a correspondence. So we have

a commutative diagram

$$\begin{array}{ccc} D_{\text{crys}, K'}(H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_p)) & \xrightarrow{=} & H_{\text{crys}}^i(\overline{X}) \\ \sigma_* \downarrow & & \overline{\Gamma}^* \downarrow \\ D_{\text{crys}, K'}(H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_p)) & \xrightarrow{=} & H_{\text{crys}}^i(\overline{X}). \end{array}$$

Since $H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_p)$ is a crystalline representation for all i (for $i = 2$ this is the assumption, for $i = 0, 4$ it is clear and for $i = 1, 3$ the cohomologies vanish), we have

$$D_{\text{crys}, K'}(H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_p)) = D_{\text{crys}, K}(H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_p)) \otimes_{K_0} K'_0$$

and hence $I(K'/K)$ acts on $D_{\text{crys}, K'}(H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_p))$ trivially. By the above diagram, $\overline{\Gamma}^*$ acts on $H_{\text{crys}}^i(\overline{X})$ trivially. So

$$\text{Tr}(\overline{\Gamma}^* | H_{\text{crys}}^i(\overline{X})) = \dim_{K'_0} H_{\text{crys}}^i(\overline{X}) = \dim_{\mathbb{Q}_p} H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_p).$$

Finally (by comparing both sides with the Betti numbers) we have

$$\dim_{\mathbb{Q}_p} H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_p) = \dim_{\mathbb{Q}_l} H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l).$$

Combining these equalities we obtain

$$\sum_i \text{Tr}(\sigma_* | H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l)) = \sum_i \dim_{\mathbb{Q}_l} H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_l)$$

(note again that $H^1 = H^3 = 0$). Thus each element in $I(K'/K)$ acts on $H_{\text{ét}}^*(X_{\overline{K}}, \mathbb{Q}_l)$ by trace equal to the dimension of this \mathbb{Q}_l -vector space. It then follows that the action of this group is trivial. \square

4. Appendix: Good reduction of Kummer surfaces. We record the proof of Theorem 1.16, a result of Ito [9].

First we review the relation between Kummer surfaces and abelian surfaces.

Let F be a field of characteristic $\neq 2$. Let A be an abelian surface over \overline{F} , and $X = \text{Km}(A)$. Let $G = \{\text{id}, \iota\}$ where ι is the multiplication-by- (-1) map of A . The surface X is, by definition, obtained by the blow-ups at 16 singular points of A/G . However we can also obtain X from A in the following way.

Let \tilde{A} be the blow-up of A at $A[2]$. Since $A[2]$ is the fixed points of the action of G , we can extend the action of G on \tilde{A} . Then the quotient variety \tilde{A}/G is naturally isomorphic to X . We have a cartesian diagram

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & \tilde{A}/G \cong X \\ \downarrow & & \downarrow \\ A & \longrightarrow & A/G, \end{array}$$

where the horizontal maps are the quotient maps and the vertical maps are the blow-ups at 16 points. Let Z be the exceptional divisor of the blow-up $X \rightarrow A/G$. This is the union of 16

curves of self-intersection -2 . By construction, $\tilde{A} \rightarrow X$ is a double covering whose branch locus is Z .

We first prove the following special case of Theorem 1.16.

THEOREM 4.1. *Let A be an abelian surface over K and $X = \text{Km}(A)$. Then X has good reduction if and only if $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified.*

LEMMA 4.2. *Let A be an abelian surface over K and $X = \text{Km}(A)$. If A has good reduction, then X has good reduction.*

PROOF. The Néron model \mathcal{A} of A over \mathcal{O}_K is an abelian scheme over \mathcal{O}_K ([3, Proposition 1.4/2]). By the Néron mapping property, the multiplication-by- (-1) map on A has a natural extension to an involution $\iota: \mathcal{A} \rightarrow \mathcal{A}$. In the special fiber, ι is the multiplication-by- (-1) map and has exactly 16 fixed points because the residue characteristic p is not equal to 2. Let $\tilde{\mathcal{A}}$ be the blow-up of \mathcal{A} at $\mathcal{A}[2]$: this is smooth over \mathcal{O}_K by Lemma 1.2. Then ι induces an involution $\tilde{\iota}$ on $\tilde{\mathcal{A}}$. Taking the quotient of $\tilde{\mathcal{A}}$ by $\tilde{\iota}$, we obtain a proper smooth model of X over \mathcal{O}_K . \square

PROOF OF THEOREM 4.1. Since $H_{\text{ét}}^2(A_{\overline{K}}, \mathbb{Q}_l)$ is a direct summand of $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l)$, the inertia group I_K acts trivially on $H_{\text{ét}}^2(A_{\overline{K}}, \mathbb{Q}_l)$, which is isomorphic to $\bigwedge^2 H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_l)$. By Lemma 3.1, we have a homomorphism $f: I_K \rightarrow \{\pm 1\}$ which the action of I_K on $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_l)$ factors through. If f is trivial we are done by applying Theorem 1.15 and Lemma 4.2. Assume that f is not trivial. The idea is to take a quadratic twist of A to get an abelian surface A' over K such that A' has good reduction over K and that $\text{Km}(A) \cong \text{Km}(A')$.

Let L be a ramified quadratic extension of K . Since the kernel of f corresponds to the (unique) ramified quadratic extension LK^{un} of K^{un} , the homomorphism $\tilde{f}: G_K \rightarrow \text{Gal}(L/K) \cong \{\pm 1\}$ extends f . Let $G = \{\pm \text{id}\}$ be a group of automorphisms of A , and fix the unique isomorphism $\text{Gal}(L/K) \cong G$ (thus we have an action of $\text{Gal}(L/K)$ on A). We take a quotient $A' = (A \times_K L) / \text{Gal}(L/K)$, where $\text{Gal}(L/K)$ acts diagonally on $A \times_K L$.

We shall see that this A' satisfies the desired conditions. By the above construction we have $A_L \cong A'_L$, and hence I_L acts on $H_{\text{ét}}^1(A'_L, \mathbb{Q}_l)$ trivially. Since I_K acts on $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_l)$ by $I_K \rightarrow I_K/I_L \cong \text{Gal}(L/K) \cong \{\pm 1\}$, and since the involution $-1 \in G$ acts by -1 , it follows that I_K acts on $H_{\text{ét}}^1(A'_{\overline{K}}, \mathbb{Q}_l)$ trivially. Hence A' has good reduction over \mathcal{O}_K by Theorem 1.15.

It is easy to see $\text{Km}(A) \cong \text{Km}(A')$: the effect of the quadratic twist vanishes after we take the quotients by G . It now suffices to apply Lemma 4.2. \square

Now we prove Theorem 1.16 by reducing to Theorem 4.1.

PROOF OF THEOREM 1.16. By Theorem 4.1, it suffices to show that, for some finite unramified extension K' of K , $X_{K'}$ can be written as $X_{K'} = \text{Km } A$ for an abelian surface A over K' .

Let $A_{\overline{K}}$ be an abelian surface over \overline{K} such that $X_{\overline{K}} = \text{Km}(A_{\overline{K}})$, and let $Z_{\overline{K}}$ be the exceptional divisor of the blow-up $X_{\overline{K}} \rightarrow A_{\overline{K}}/\{\pm 1\}$. By the description given at the beginning of this section, there is a double covering of $X_{\overline{K}}$ whose branch locus is $Z_{\overline{K}}$. By Lemma 1.8,

all the components of $Z_{\overline{K}}$ and the covering are defined over some finite unramified extension K' of K . Write $X_{\overline{K}} = X_{K'} \times_{K'} \overline{K}$ and $Z_{\overline{K}} = Z_{K'} \times_{K'} \overline{K}$. Since we know that the inverse image of $Z_{K'}$ is, over \overline{K} , the disjoint union of 16 rational curves of self-intersection -1 , we can blow down this inverse image to obtain a variety A over K' . We see that $A \times_{K'} \overline{K}$ is the abelian surface $A_{\overline{K}}$ over \overline{K} by the uniqueness of the double covering (Lemma 1.3). Moreover, since the origin of $A_{\overline{K}}$ is the image of one of the contracted curves that is defined over K' , it is a K' -rational point. Therefore, A is an abelian surface over K' such that $X_{K'} = \text{Km}(A)$ over K' . \square

REMARK 4.3. We can give an explicit bound for the degree of the field extension in the case of characteristic 0.

By the proof of Theorem 1.16, it suffices to estimate the degree of K' such that $G_{K'}$ acts trivially on $\text{NS}(X_{\overline{K}})$. The rank of $\text{NS}(X_{\overline{K}})$ is less than or equal to 22 ($= \dim_{\mathbb{Q}_l} H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_l)$). Since every divisor on $X_{\overline{K}}$ can be defined over a finite extension of K , the image of G_K in $GL(\text{NS}(X_{\overline{K}}))$ is torsion. So it remains to give a bound for the order of a torsion subgroup of $GL(22, \mathbb{Z})$.

Take any prime number $l \geq 3$ (which we do not assume to be different from the characteristic). By the exact sequence

$$1 \rightarrow 1 + IM(22, \mathbb{Z}_l) \rightarrow GL(22, \mathbb{Z}_l) \rightarrow GL(22, \mathbb{F}_l) \rightarrow 1$$

and the fact that $1 + IM(22, \mathbb{Z}_l)$ is torsion-free, a torsion subgroup of $GL(22, \mathbb{Z})$ has order $\leq |GL(22, \mathbb{F}_l)| \leq l^{22^2}$. By choosing $l = 3$ we can take 3^{484} as a bound. Of course this bound is too rough.

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