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On Good Triangulations in Three Dimensions*

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1 Introduction

Triangulation of a point set or a polyhedron is an important problem with applications for finite element simulations in CAD/CAM. Though a number of algorithms exist for triangulating a point set or a polyhedron in two and three dimensions [6, 1, 10, 12], few of them address the problem of guaranteeing the shape of the triangular elements, they generate. To reduce ill-conditioning as well as discretization error, finite element methods require triangular meshes of bounded aspect ratio [2, 11]. By aspect ratio of triangles or tetrahedra, one may consider the ratio of the radii of the circumscribing circle to that of inscribing circle (spheres in case of tetrahedra).

In 2-d, there are basically two approaches, known so far, to produce guaranteed quality triangulations. The first approach, based on *Constarined Delaunay Triangulations*, was first suggested by L.P.Chew [7]. He guarantees that all the triangles produced in the final triangulation have angles between 30° and 120°. In [8], we improved this algorithm with minor modifications to guarantee some of the triangles with better angle bounds. There is another approach based on *Grid Overlaying* which was first used by Baker et. al in [3] to produce a non-obtuse triangulation of a polygon. In [8], we proposed a simpler method based on this grid approach to triangulate a polygon with good angles. Recently, in [5], Bern et.al give algorithms for producing good triangulations which uses a special type of grid that simulates the planar division with the quadtree.

Though several good heuristics have been published, till date, there is no known algorithm which triangulates the convex hull of a three dimensional point set with guaranteed quality tetrahedra. This paper presents some results on good triangulations of the convex hull of a point set in three dimensions. The problem allows one to introduce new points to generate good tetrahedra with the restriction that all points added lie only inside or on the boundary of the convex hull. Good triangulations of convex polyhedra is a special case of this problem. We show that an algorithm based on constrained Delaunay triangulations as proposed by L.P. Chew for two dimensions can be extended to three dimensions to produce triangulations where all tetrahedra have certain guaranteed qualities.

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2 Preliminaries

2.1 Characterizing Bad Tetrahedra

In three dimensions, a tetrahedron which is not of bounded aspect ratio can be degenerate or bad in three possible ways as described in [4]. The following two parameters ω, κ characterize bad tetrahedra as follows. Let $\omega = \frac{R}{L}$ and $\kappa = \frac{L}{l}$, where R is the radius of the circumscribing sphere of a tetrahedron, L and l are the lengths of its longest and shortest edges respectively.

Type(i): $\omega = O(1), \kappa \gg 1$.
Type(ii): $\omega \gg 1$.
Type(iii): $\omega = O(1)$, $\kappa = O(1)$.

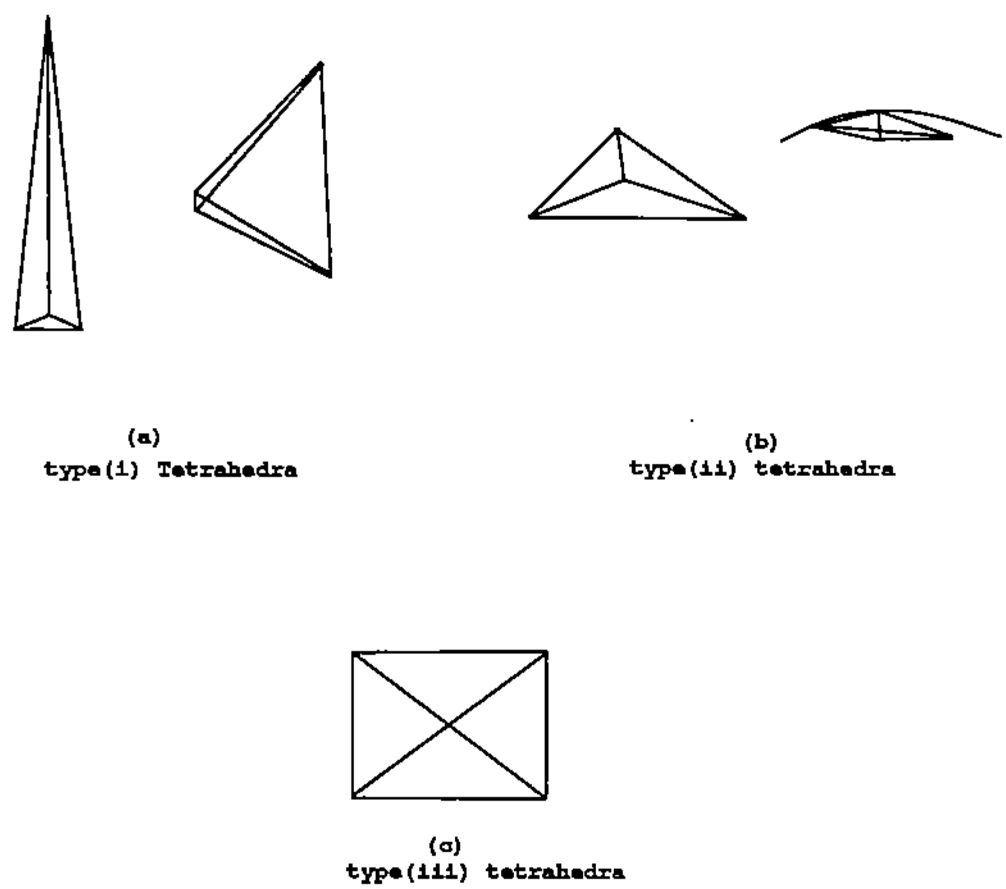


Figure 1: Bad tetrahedra

Definition: A sliver is a tetrahedron which is formed by four almost coplanar points and whose solid angles are very close to zero.

Type(i) corresponds to tetrahedra which have a very short edge relative to other edges and have a circumscribing sphere which cannot have an arbitrarily large radius compared to the length of the longest edge. Specifically, type (i) includes needle like tetrahedra in which one of the solid angles is highly acute and the face opposite to it has a negligible area. It also includes slivers with a very short edge. See figure 1(a). Type(ii) corresponds to tetrahedra which have a circumscribing sphere with arbitrarily large radius compared to the longest edge. Specifically, type (ii) includes flat tetrahedra which have one of the solid angles highly obtuse. It also includes slivers which lies very close to the surface of its large circumscribing sphere. See figure 1(b). Type(iii) includes only slivers whose edges have lengths within a constant factor of each other and which cannot have a close incidence with the surface of the circumscribing sphere. See figure 1(c). We present an algorithm which triangulates a three dimensional point set inside a convex boundary with the guarantee that no tetrahedron can be of type (i) or type (ii).

2.2 2-d Algorithm

The core of the algorithm presented in this paper consists of Delaunay triangulation which is the straight line dual of the voronoi diagram. In two dimensions, the circumscribing circle of a triangle in a Delaunay triangulation of a point set does not contain any other points inside it. Similarly, in three dimensions, the circumscribing sphere of a tetrahedron in a Delaunay triangulation does not contain any other points inside it. This property of the Delaunay triangulation was utilized by L.P. Chew in two dimensions to produce good triangulations. He introduced the centers of those circumscribing circles which maintain a certain minimum distance from the three vertices of the corresponding triangle. Of course, the edges of the boundary have to satisfy certain length criteria. In his algorithm, Chew used edge lengths in between d and $\sqrt{3}d$ where any pair of input points is at least d units away from each other. In the modified algorithm of [8], we require edge lengths in between d and $1.5d$. This gives three distinct advantages.

1. It is easier to divide edges between d and $1.5d$ in practice.
2. It is not necessary to check whether the center point to be introduced is inside the boundary or not. It can be easily proved that with this edge length criteria, the centers which lie outside the boundary do not satisfy the distance criterion.
3. The triangles which have circumcenters outside the boundary have better bounds on their angles.

We present below this modified algorithm for good triangulation in two dimensions.

Algorithm 2d-TRI:
Input: Finite number of points in the plane within a polygonal boundary. The vertices of the polygonal boundary are included in the input point set.

Input Conditions: There exists a quantity d, such that no two given points are closer than d and no boundary edge is greater than $1.5d$ and less than d.

begin

Construct the Delaunay triangulation of the given points.

Repeat

Add the circumcenter v_l of a triangle $g = \Delta p_i p_j p_k$ satisfying the following property: v_l is at a distance of at least d from all the three points p_i, p_j, p_k.

Update the current triangulation by constructing the Delaunay triangulation of the augmented set of points.
Until there is no such triangle.
end

For a simple polygonal boundary with a certain lower bound (39°) on the minimum internal angles at the vertices, it is always possible to choose a d to satisfy the input conditions of the algorithm *2d-TRI*. Algorithm *2d-TRI* produces a planar triangulation T which has the following properties.

Property 1: All edges in T have lengths in between d and $2d$ and in particular all the boundary edges have lengths in between d and $1.5d$.

Property 2: The circumscribing circle of all triangles in T has radius less than d.

2.3 Geometric Lemmas

We use the following geometric lemmas in the next section.

Lemma 2.1: Let T be a Delaunay triangulation of a point set in two dimensions. Let R be the maximum radius of all the circumscribing circles of Delaunay triangles in T. The radius of any empty circle whose center lies inside T is less than or equal to R.

Proof: See Theorem 6.15[13]. ♣

Definition: Let c be a circle drawn on the surface of a sphere s. Let $\overline{p_1p_2}$ be the axis drawn through the center of c and which is perpendicular to c intersecting s at p_1 and p_2. The points p_1, p_2 are called the *poles* corresponding to the circle c.

Lemma 2.2: Let c be a circle with radius less than r drawn on the surface of a sphere s. Let the distance between c and its nearest pole be greater than d. The radius R of s must satisfy the condition $R < \frac{r^2+d^2}{2d}$.

Proof: Consider the circle c as shown in Figure 2 with the nearest pole p_1. Let a, b be the centers of s and c respectively. Obviously, $|\overline{ab}| < (R-d)$. Consider the right angled triangle $\triangle abt$ where t is a point on the circle c. Since the radius of c is less than r, we have $|\overline{bt}| < r$. Hence, $|\overline{at}^2| = R^2 = |\overline{ab}|^2 + |\overline{bt}|^2 < (R-d)^2 + r^2$ which gives $R < \frac{r^2+d^2}{2d}$. ♣

3 3-d Algorithm

We assume that a finite number of points are given in three dimensional space. We call the boundary of the convex hull of these points simply as the boundary. In what follows, by convex hull of a point set, we mean its interior along with its boundary. A point is called an *internal point* if it is not on the boundary and is called a *boundary point* otherwise. The facets of the boundary are referred to as *boundary facets* and the edges on the boundary facets are called *boundary edges*.

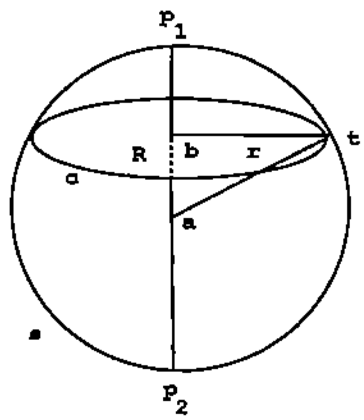


Figure 2: Lemma 2.2

Algorithm 3d-TRI:
Input: Finite number of points in three dimensional space.

begin

Let d_1 be the minimum of the distances between two points.
Let d_2 be the minimum distance from an internal point to a boundary facet.
Let d_3 be the minimum distance between two nonadjacent boundary facets.
Let $r = \frac{1}{6}min\{d_1, d_2, d_3\}$.
Triangulate each facet of the boundary using algorithm 2d-TRI in such a way that every edge has length in between r and $2r$ and every boundary edge has length in between r and $1.5r$.
Let P be the current point set. Construct a 3-d Delaunay triangulation $T(P)$ of the point set P.

repeat

Add the center v of the circumscribing sphere of a tetrahedron t_i in $T(P)$ satisfying the following properties:
(i) all four vertices of t_i is at a distance of at least $2r$ from v,
(ii) the center v lies inside the boundary.
Set $P = P \cup v$ and update the Delaunay triangulation $T(P)$.

until there is no such tetrahedron.
end

With the above choice of r and with the assumption that all the face-angles of the facets on the boundary satisfy the minimum angle criterion, it is possible to triangulate them by *2d-TRI* maintaining the edge lengths as stated. In the following Lemma, we prove that the above procedure terminates.

Lemma 3.1: Algorithm *3d-TRI* terminates.

Proof: Algorithm *2d-TRI* terminates since the points added by it are always at a certain distance from all other points. There can be only finitely many such points inside the given polygonal boundary. Extending this argument to Algorithm 3d-TRI, we can observe that all the circumcenters of the tetrahedra which are added as new points are at a distance of at least $2r$ from all other points. There can be only finitely many such points inside the convex hull of the input points, which assures the termination of the Algorithm 3d-TRI. ♣

Lemma 3.2: Any point on a boundary facet which does not lie on a boundary edge must be at a distance of at least $\frac{\sqrt{7}}{4}r$ from all edges of that facet.

Consider a point p on a facet f. Let e be any edge of f. Note that the edge e is divided into smaller edges $e_1, e_2, ..., e_n$ through the triangulation of the boundary facets adjacent to e. Drop a perpendicular from p on the line supporting e. If the perpendicular intersects the edge e, let e_l be the edge of the triangulation on e which is intersected by it. According to the property 1, all boundary edges of the triangulation of f must have lengths in between r and $1.5r$. Further, the point p is at least r units away from the end points of e_l. Thus, the minimum distance between p and e_l is at least $\frac{\sqrt{7}}{4}r$. In case, the perpendicular dropped from p does not intersect e, it must intersect some other edge e' of f'. In that case, the distance between p and e must be greater than the distance between p and e'. We can estimate the minimum distance between p and e by estimating the same between p and e'. While estimating the distance between p and e', if it occurs that the perpendicular dropped from p does not intersect e', we will have another edge to estimate the minimum distance between p and e'. Since there are finite number of edges and since each time we go to a next edge, its distance from p gets smaller than the previous one, there must be an edge of f which is intersected by the perpendicular dropped from p. Let e'' be the first such edge encountered in the above process. As argued above, the distance between p and e'' is at least $\frac{\sqrt{7}}{4}r$. Hence, the distance between p and e is at least $\frac{\sqrt{7}}{4}r$. Thus, any point on a boundary facet which does not lie on a boundary edge must be at a distance of at least $\frac{\sqrt{7}}{4}r$ from all edges of that facet.♣

Lemma 3.3: All edges in the triangulation produced by the algorithm *3d-TRI* have lengths greater than $l_{\min}$ where $l_{\min} = \min(r, \frac{\sqrt{7}}{2}r \sin\frac{\theta_m}{2})$. Here θ_m is the minimum dihedral angle between two adjacent boundary facets.

Proof: Initially, all internal points are at a distance of at least $6r$ units from every other points. Two boundary points, lying on non adjacent facets, are at least $6r$ units away from each other. These conditions are ensured by the particular choice of r. A boundary point is at a distance of at least r from every other point on the same facet which is ensured by the algorithm *2d-TRI*. The points added by the algorithm *3d-TRI* are always at a distance of at least $2r$ from every other point. Thus, all points except the points on the adjacent facets are at a distance of at least r from each other. To estimate the minimum distance between any two points on the adjacent boundary facets, consider two such points p_i, p_j lying on the adjacent facets f_i, f_j respectively. Let e be the edge shared by f_i and f_j. Drop a perpendicular from p_i on e. Let it meet e at p_m. Consider the triangle $\triangle p_i p_j p_m$. Let the minimum dihedral angle between any two adjacent facets be θ_m. It is easy to prove that the angle between $\overline{p_i p_m}$ and $\overline{p_j p_m}$ in the triangle $\triangle p_i p_j p_m$ must be at least θ_m. From the above discussion, it is obvious that $|\overline{p_i p_m}| > \frac{\sqrt{7}}{4}r$ and $|\overline{p_j p_m}| > \frac{\sqrt{7}}{4}r$. Thus, the distance between p_i and p_j is at least $\frac{\sqrt{7}}{2}r\sin\frac{\theta_m}{2}$. Hence, all edges in the final triangulation produced by the algorithm *3d-TRI* have lengths greater than $l_{\min} = \min(r, \frac{\sqrt{7}}{2}r\sin\frac{\theta_m}{2})$. ♣

Lemma 3.4: Any point p present as a vertex in the triangulation produced by the algorithm *3d-TRI* is at a distance of at least $\frac{\sqrt{7}}{4}r\sin\theta_m$ from any boundary facet on which p does not lie. Here, θ_m is a measure of angle such that all dihedral angles of the input boundary are within θ_m and $180° - \theta_m$.

Proof: If p is an inner point, we already know p is at least r units away from every boundary facet. By the choice of r, any point on a boundary facet is at least r units away from any other nonadjacent facet. We prove that if p lies on a boundary facet but not on a boundary edge, it is at a distance of at least $\frac{\sqrt{7}}{4}r\sin\theta_m$ from all adjacent facets. Let p lie on f_i and let f_j be any facet adjacent to f_i. In Lemma 3.2, we proved that the distance of p from any line supporting an edge of the facet f_i , is at least $\frac{\sqrt{7}}{4}r$. Let l be the distance of p from the line where f_i and f_j meet. The distance d of p from f_j is given by $d = l\sin\theta$ where θ is the dihedral angle between f_i and f_j. Putting the minimum value of l and θ gives the lower bound on d. Thus, the distance of a point from any facet which does not contain it, is at least $d_{\min} = \min(r, \frac{\sqrt{7}}{4}r\sin\theta_m) = \frac{\sqrt{7}}{4}r\sin\theta_m$. ♣

4 Qualities of Tetrahedra

Definition: A tetrahedron in the final triangulation is called to have a *good circumcenter* if the center of its circumscribing sphere lies inside or on the boundary (convex hull boundary). Conversely, a tetrahedron is called to have a *bad circumcenter* if the center of its circumscribing sphere lies outside the boundary.
We classify the tetrahedra with bad circumcenters into two classes, namely class A and class B.

Definition: A tetrahedron t with a bad circumcenter is called a class A tetrahedron if there exists a facet intersected by the circumscribing sphere of t in such a way that the foot of the perpendicular dropped from the circumcenter to the plane containing the facet lies inside the facet. Any other tetrahedron with a bad circumcenter is called a class B tetrahedron. See figure 3 and figure 4.

Assuming a lower and upper bounds on the dihedral angles between adjacent boundary facets, we can prove that all tetrahedra produced by *3d-TRI* cannot be of type (i) or type (ii). Though, we cannot avoid type (iii) tetrahedra, occurrences of them in practice are rare as stated in [4]. Finally, in most of the cases these type (iii) tetrahedra or slivers can often be avoided by introducing a suitable point inside the circumscribing sphere. See [4]. In what follows, we assume that all dihedral angles between adjacent boundary facets are greater than θ_m and less than $180° - \theta_m$.

Lemma 4.1: No tetrahedron with good circumcenter can be of type (i) or type (ii).

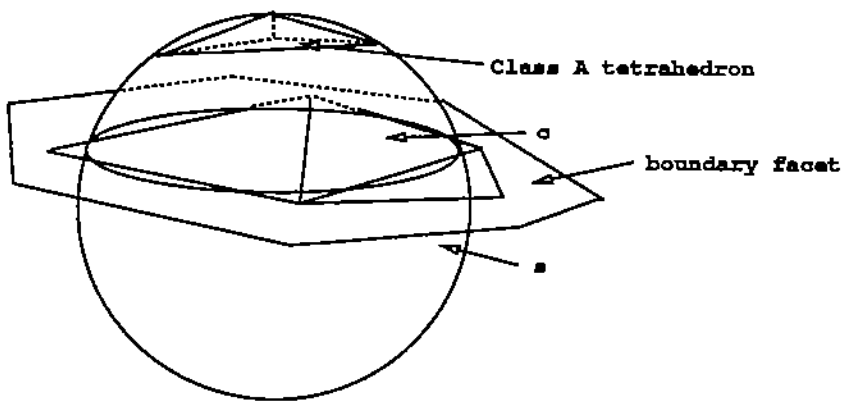


Figure 3: class A tetrahedron

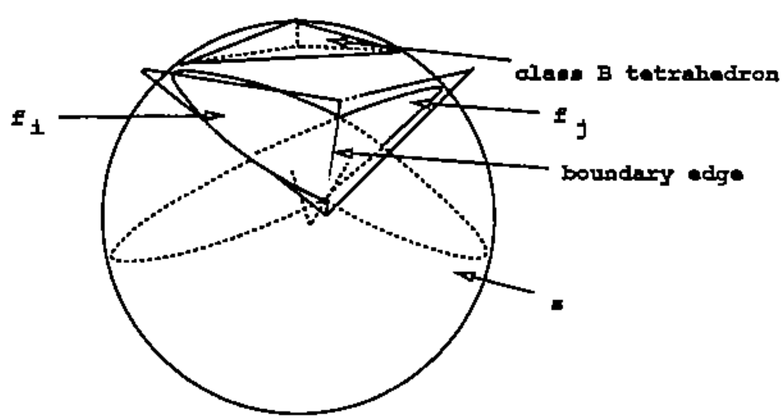


Figure 4: class B tetrahedron

Proof: All tetrahedra in the final triangulation having good circumcenters must have circumscribing spheres with radii less than $2r$, because otherwise these circumcenters would have been introduced as new points. Hence, all these tetrahedra have edges of length less than $4r$. By Lemma 3.3, all edges have lengths greater than $\min(r, \frac{\sqrt{7}}{2} r \sin \frac{\theta_m}{2})$. Thus, κ for these tetrahedra can be at most $\max(4, \frac{8}{\sqrt{7}\sin\frac{\theta_m}{2}})$. Assuming a lower bound on the dihedral angles of the input boundary, we get κ for these tetrahedra to be of $O(1)$ which violates the condition for type (i) tetrahedra. Further, ω for these tetrahedra can be at most $\max(4, \frac{4}{\sqrt{7}\sin\frac{\theta_m}{2}}) = O(1)$ which prohibits them to be of type (ii). ♣

Lemma 4.2: No class A tetrahedron can be of type (i) or type (ii).

Proof: Let t be a class A tetrahedron with the circumscribing sphere s. By the definition of class A tetrahedron, there exists a boundary facet such that the foot of the perpendicular dropped from the center of s on the plane of the facet lies inside that facet. Let f be such a facet intersected by s in a circle $c = s \cap f$. By Lemma 3.4, a vertex p of t which does not lie on f must be at a distance of at least $\frac{\sqrt{7}}{4} r \sin\theta_m$ from f where θ_m is defined as before. The center of the circle c lies inside f. Thus, the center must lie inside the triangulation T of f produced by the algorithm *2d-TRI.* Further, c must be an empty circle since s does not include any point of f inside it. See figure 3. By property 2, all triangles of T have circumscribing circles of radii less than r. Hence, according to Lemma 2.1, c must have a radius less than or equal to r. The vertex p lying on s must be at a distance of at least $\frac{\sqrt{7}}{4} r \sin\theta_m$ from c. Further, the vertex p and the center of s lie on the opposite sides of c. This implies c is at a distance of at least $\frac{\sqrt{7}}{4} r \sin\theta_m$ from its nearest pole. Thus, according to Lemma 2.2, s must have a radius less than or equal to $k_1 r$ where $k_1 = (\frac{\sqrt{7}\sin\theta_m}{8} + \frac{2}{\sqrt{7}\sin\theta_m})$. This puts an upper bound of $2k_1 r$ on the lengths of the edges of t_i. By Lemma 3.2, all edges of t_i are greater than $k_2 r$ where $k_2 = \min(1, \frac{\sqrt{7}}{2}\sin\frac{\theta_m}{2})$. Hence, ω, κ for t_i is $O(1)$ assuming a lower bound on θ_m (A lower bound on θ_m puts an upper and a lower bound on the dihedral angles between adjacent boundary facets). This prohibits it to be of type (i) or type (ii). ♣

Lemma 4.3: Let t be a class B tetrahedron with the circumscribing sphere s. There must exist two boundary facets f_i, f_j intersected by s with the following criterion:
Let c be any circle drawn on s which is normal to the line where f_i, f_j meet. The feet of the perpendiculars dropped from the center of c on the supporting planes P_i and P_j of f_i and f_j do not lie on the line segments $c \cap f_i$, $c \cap f_j$.

Proof: Consider a boundary facet f_i which have the convex hull and the center of s on opposite sides. Since t has a bad circumcenter, such a facet always exists. Drop a perpendicular from the center of s on the supporting plane of f_i. The foot of this perpendicular lies outside f_i since t is a class B tetrahedron. This is possible if there exists another boundary facet f_j which has the convex hull and the foot of this perpendicular on opposite sides. Consider the great circle c' of s whose supporting plane is normal to the edge shared by f_i and f_j. Due to the above facts, the feet of the perpendiculars dropped from the center of s on the supporting planes P_i and P_j of f_i and f_j cannot lie on the line segments $c' \cap f_i$

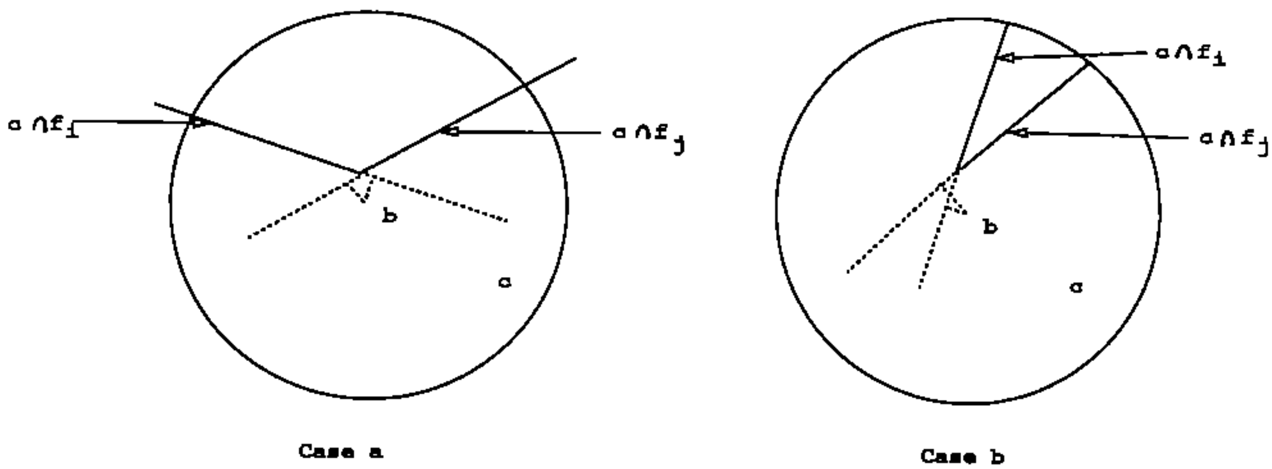


Figure 5: Lemma 4.3.

and $c' \cap f_j$. See figure 5. This immediately implies that the above condition is true for any circle c on s which has a supporting plane parallel to c'.♣

Lemma 4.4: No class B tetrahedron can be of type (i) or type (ii).

Proof: Let t be a class B tetrahedron. Let the circumscribing sphere s of t intersect the boundary edge e shared by the facets f_i and f_j which satisfy the criterion as stated in Lemma 4.3. The endpoints of the edge segment e_n on e which is intersected by s cannot be inside s. Let w, y be the points where s intersects e_n. Further, let a and R denote the center and radius of s respectively.

Case(i): The tetrahedron t has a vertex p which lies neither on facet f_i nor on facet f_j. Consider the circle c on s whose supporting plane is perpendicular to e_n and which passes through p. Let R' be the radius of c. Join the center b of c with the point u where c meets e_n. Extend the line $\overline{bu}$ beyond u until it intersects the boundary of c at v as shown in figure 6. Let $|\overline{bu}| = x$. Certainly, $|\overline{uv}| = R' - x$. Let d denote the minimum distance of p from the two facets f_i and f_j. There are two subcases as shown in figure 6. In subcase $i(a)$, the center of c lies in the sides of the planes containing f_i, f_j which are opposite to those containing the convex hull. It is not difficult to see that in this subcase $d \le |\overline{uv}| = R' - x$. Since, $R \ge R'$, we have $d \le R - x$. To estimate a lower bound on x, drop a perpendicular $\overline{az}$ from the center a of s on e_n. This perpendicular has the same length as $\overline{bu}$. Consider the triangle Δawy. It is easy to see that $|\overline{az}| = \sqrt{R^2 - \frac{|\overline{wy}|^2}{4}}$. Since e_n can have a length of at most $1.5r$, we have $x = |\overline{az}| \ge \sqrt{R^2 - \frac{9r^2}{16}}$. Thus, $d \le R - \sqrt{R^2 - \frac{9r^2}{16}}$. We already know $d \ge \frac{\sqrt{7}}{4} r \sin\theta_m$ (Lemma 3.4). Hence,

$$\frac{\sqrt{7}r}{4}\sin\theta_m \le R - \sqrt{R^2 - \frac{9r^2}{16}},$$

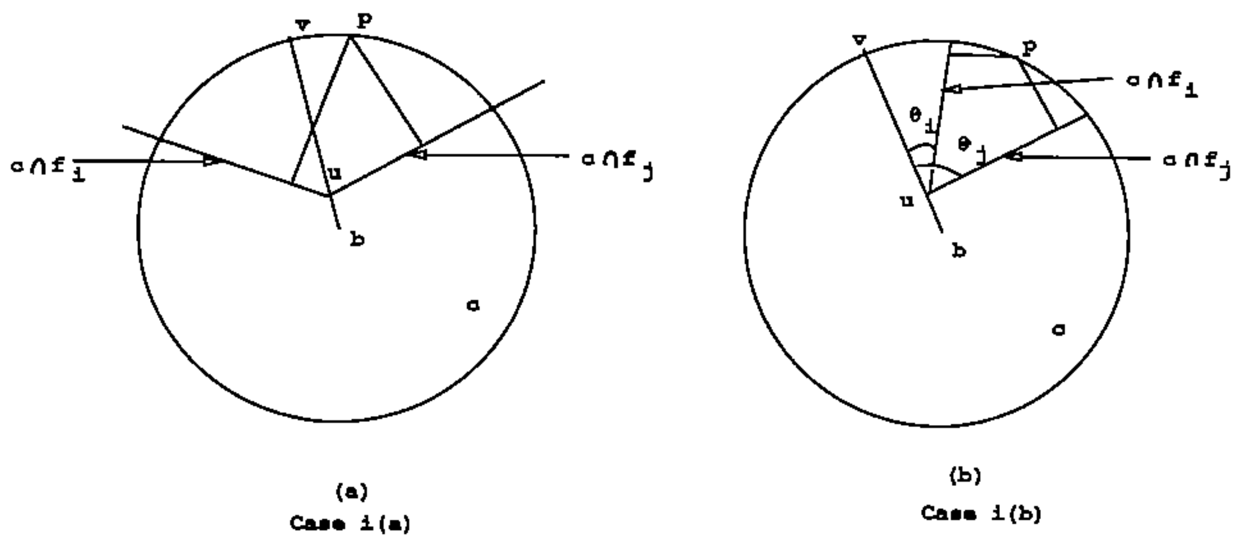


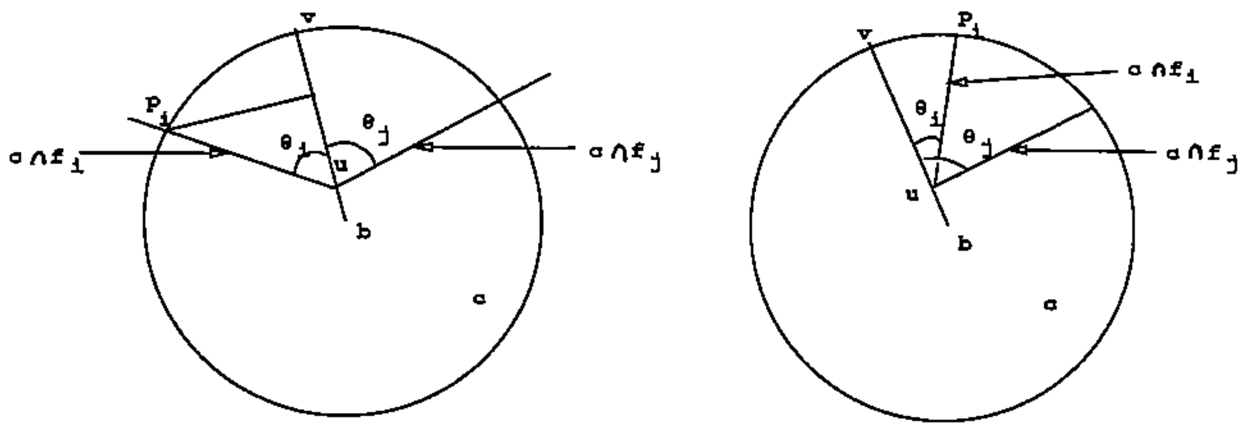
Figure 6: Lemma 4.4, case (i).

$$R \;\leq\; \frac{7\sin^2\theta_m + 9}{8\sqrt{7}\sin\theta_m} r.$$

Now, consider the subcase $i(b)$. In this subcase, one of the supporting planes of f_i and f_j has the center of c and the convex hull on its opposite sides and the other one has them on same side. Without loss of generality, assume that the supporting plane of f_i has them on same side as shown in figure 6(b). Certainly, the line segments $c \cap f_i$ and $c \cap f_j$ make angles less than equal to $90°$ with $\overline{uv}$. Otherwise, f_i, f_j do not satisfy the criterion as stated in Lemma 4.3. In this subcase, we have $d \leq R - x$, since the distance of v from the supporting plane of f_j is greater than that of p from the same plane. Thus, in both of the subcases $i(a)$ and $i(b)$, we have,

$$R \leq \frac{7\sin^2\theta_m + 9}{8\sqrt{7}\sin\theta_m} r.$$

Case(ii): All vertices of the tetrahedron t lie either on f_i or on f_j. This immediately implies that one of the vertices of t lies on f_i but not on f_j and another on f_j but not on f_i. Consider the vertex p_i lying on f_i but not on f_j. Let c be the circle passing through p_i with the supporting plane being perpendicular to e_n. As in the previous case, let b be the center of c, u be the foot of the perpendicular dropped from b to e_n, and v be the point of intersection of the line $\overline{bu}$ and the circle c such that u is in between b and v. Again, we have two subcases as shown in figure 7. Consider the subcase $ii(a)$. It is easy to see that $|\overline{p_i u}| \leq \frac{|\overline{uv}|}{\cos\theta_i}$, where θ_i is the angle between $\overline{p_i u}$ and $\overline{uv}$. We proved in lemma 3.2 that the distance of any point on a boundary facet which does not lie on any of its edges is at least $\frac{\sqrt{7}}{4} r$ away from any of its edges. Thus, $|\overline{p_i u}| \geq \frac{\sqrt{7}}{4} r$. Hence, $\frac{\sqrt{7}}{4} r \leq \frac{R'-x}{\cos\theta_i} \leq \frac{R-x}{\cos\theta_i}$, where



(a)
Case ii(a)

(b)
Case ii(b)

Figure 7: Lemma 4.4, case (ii).

$x = [\overline{bu}]$. Similarly, considering the vertex p_j of t lying on f_j but not on f_i, we can prove that $\frac{\sqrt{7}}{4}r \leq \frac{R-x}{\cos\theta_j}$ where θ_j is the angle between $f_j \cap c$ and $\overline{uv}$. Obviously, $\theta = \theta_i + \theta_j$ is the dihedral angle between f_i and f_j. Since one of θ_i, θ_j is less than or equal to 90° and the *cos* function decreases monotonically from 0° to 90°, we have $\frac{\sqrt{7}}{4}r \leq \frac{R-x}{\cos\frac{\theta}{2}}$. By the same argument as in case(i), we get $x \geq \sqrt{R^2 - \frac{9}{16}r^2}$. Hence,

$$\begin{aligned} \frac{\sqrt{7}}{4}r &\leq \frac{R - \sqrt{R^2 - \frac{9}{16}r^2}}{\cos\frac{\theta}{2}} \\ R &\leq \frac{\frac{9}{16}r^2 + \frac{7}{16}r^2\cos^2\frac{\theta}{2}}{\frac{\sqrt{7}}{2}r\cos\frac{\theta}{2}}. \end{aligned}$$

Assuming an upper bound on $\theta \leq (180° - \theta_m)$ we have,

$$R \leq \frac{7\sin^2\frac{\theta_m}{2} + 9}{8\sqrt{7}\sin\frac{\theta_m}{2}}r.$$

Now, consider the subcase $ii(b)$. The angles between $\overline{uv}$ and the line segments $c \cap f_i$ and $c \cap f_j$ are less than 90° since otherwise f_i, f_j violate the condition of Lemma 4.3. Without loss of generality assume that $\theta_i < \theta_j$. Certainly, the distance between v and $c \cap f_j$ is greater than that between p_i and $c \cap f_j$. This implies $d \leq R - x$ which gives the same upper bound on R as we derived in case(i).

Thus, all class B tetrahedra have a circumscribing sphere of radius $k_1 r$ where $k_1 = O(1)$ assuming lower and upper bounds on the dihedral angles between adjacent boundary facets. This with the fact that edges of all tetrahedra have lengths greater than $k_2 r$ where $k_2 = O(1)$

(recall Lemma 3.3), makes ω and κ of these tetrahedra to be of $O(1)$ and thus prohibits them to be of type (i) or type (ii). ♣

Theorem 4.1: Algorithm *3d-TRI* produces a triangulation T of the convex hull of a three dimensional point set, where no tetrahedron in T can be of type (i) or type (ii), if there are upper and lower bounds on the dihedral angles between adjacent boundary facets of the convex hull.

4.1 Conclusion

Though, in our algorithm we avoided type (i) or type (ii) tetrahedra, we could not avoid some special type of slivers or type (iii) tetrahedra. Our immediate goal is to find a new method or modify this algorithm so that we can avoid these slivers too. The problem with the avoidance of these slivers is that an upper bound on the radius of circumscribing sphere and a lower bound on lengths of the edges of a tetrahedron do not prohibit it to be a type (iii) tetrahedron. A lower bound on the radius of the inscribing sphere together with an upper bound on the radius of the circumscribing sphere of a tetrahedron avoids such tetrahedra. But, currently we do not know how to achieve these both bounds simultaneously.

References

[1] D. Avis and H. ELGindy, (1986), "Triangulating Simplicial Point Sets in Space", *Proc. 2nd. Ann. ACM Symposium on Computational Geometry*, pp. 133-141.

[2] Babuska and A.K.Aziz, (1976), "On the Angle Condition in the Finite Element Method", *SIAM Numerical Analysis*, 13, 214-226.

[3] B. S. Baker, E. Grosse, and C.S. Rafferty, (1988), "Nonobtuse Triangulation of Polygons", *Discrete and Computational Geometry*, 3, 147-168.

[4] T.J. Baker, (1989), "Automatic Mesh Generation for Complex Three-Dimensional Regions Using a Constrained Delaunay Triangulation", *Engineering with Computers*, 5, 161-175.

[5] M. Bern, D. Eppstein, and J. Gilbert, (1990), "Provably Good Mesh Generation", Manuscript.

[6] B. Chazelle and L. Palios, (1990), "Triangulating a Non-convex Polytope", *Discrete and Computational Geometry*, 5, pp. 505-526.

[7] L. Paul Chew, "Guaranteed-Quality Triangulation Meshes", Technical Report, Cornell University.

[8] T. Dey, (1990), "Good Triangulations in Plane", *Proc. of Second Canadian Conference in Computational Geometry*, 102-106.

[9] H. Edelsbrunner,(1989), "Spatial Triangulations with Dihedral Angle Conditions", *Proc. of Intl. Workshop on Discrete Algorithms and Complexity, Fukuoka, Japan*, 83-89.

[10] H. Edelsbrunner, F.P. Preparata and D.B. West, (1986), "Tetrahedrizing Point Sets in Three Dimensions", Tech. Report UIUCDCS-R-86-1310.

[11] I. Fried, (1972), "Condition of Finite Element Matrices Generated from Nonuniform Meshes", *AIAA J.*, 10, pp. 219-221.

[12] B. Joe., (1989), "Three-dimensional Triangulations from Local Transformations", *SIAM J. Sci. Stat. Comput.*, 10, pp. 718-741.

[13] F.P. Preparata, and M.I. Shamos, "Computational Geometry, An Introduction", (1985), *Springer-Verlag.*

[14] D.T. Lee, and A.K. Lin,(1986), "Generalized Delaunay triangulation for planar graphs", *Discrete and Computational Geometry,* 1, 201-217.