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On graded prime and primary submodules

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Abstract

Let G be a multiplicative group. Let R be a G-graded commutative ring and M a G-graded R-module. Various properties of graded prime submodules and graded primary submodules of M are discussed. We have also discussed the graded radical of graded submodules of multiplication graded R-modules.

Key Words: Multiplication graded modules, graded prime submodules, graded primary submodules.

1. Introduction

Let G be a multiplicative group with identity e and R be a commutative ring with identity. Then R is called a G-graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \underset{g\in G}{\oplus} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. We denote this by $G(R)$. The elements of R_g are called homogeneous of degree g and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \underset{g\in G}{\cup} R_g$. If $a \in R$, then the element a can be written uniquely as $\sum_{g\in G} a_g$, where a_g is called the g-component of a in R_g. In this case, R_e is a subring of R and $1_R \in R_e$.

Let R be a G-graded ring and M an R-module. We say that M is a G-graded R-module (or graded R-module) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \underset{g\in G}{\oplus} M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \underset{g\in G}{\cup} M_g$ and the elements of $h(M)$ are called homogeneous. If $M = \underset{g\in G}{\oplus} M_g$ is a graded R-module, then for all $g \in G$ the subgroup M_g of M is an R_e-module. Let $M = \underset{g\in G}{\oplus} M_g$ be a graded R-module and N a submodule of M. Then N is called a graded submodule of M if $N = \underset{g\in G}{\oplus} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the g-component of N. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. Let M and M' be graded R-modules then the R-module homomorphism, $f : M \longrightarrow M'$ is called graded homomorphism of degree σ, $\sigma \in G$, if $f(M_\tau) \subseteq M'_{\tau\sigma}$ for all $\tau \in G$.

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Graded prime submodules and graded primary submodules of graded modules over graded commutative rings have been studied by various authors; see, for example, [3]–[5]. In [6] and [7], multiplication graded modules were characterized. In this paper, we characterize graded prime submodules and graded primary submodules of multiplication graded modules. Also, we characterize the graded radical of any graded submodule of multiplication graded module by Theorem 9.

Throughout this paper G is a multiplicative group, R is a commutative G-graded ring with a unit and M is a G-graded R-module.

Lemma 1 ***([3] and [4])*** *Let R be a G-graded ring, M a graded R-module and N a graded R-submodule of M. Then the following hold:*

(i) $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of $G(R)$,

(ii) If I is a graded ideal of $G(R)$, $r \in h(R)$ and $x \in h(M)$ then IN, rN and Rx are graded submodules of M.

Proof. (i) We have $(N :_R M)_g = (N :_R M) \cap R_g \subseteq (N :_R M)$ for all $g \in G$. Thus $\underset{g\in G}{\oplus}(N :_R M)_g \subseteq (N :_R M)$. For the reverse inclusion let $a = \sum\limits_{g\in G} a_g \in (N :_R M)$. It is enough to prove that $a_g M \subseteq N$ for all $g \in G$. So without loss of generality we may assume that $a = \sum\limits_{i=1}^{t} a_{g_i}$ where $a_{g_i} \neq 0$ for all $i = 1, 2, ..., t$ and $a_g = 0$ for all $g \notin \{g_1, ..., g_t\}$. As $a \in (N :_R M)$, we obtain $\sum\limits_{i=1}^{t} a_{g_i} M \subseteq N$. Since M is a graded module, we can assume that $m = \sum\limits_{j=1}^{n} m_{h_j}$ with $m_{h_j} \neq 0$ for all j and for any $m \in M$. Since $a \in (N : M)$ and $m_{h_j} \in M$, we have $am_{h_j} \in N$ for all j. So $\sum\limits_{i=1}^{t} a_{g_i} m_{h_j} = am_{h_j} \in N$. Now as N is a graded submodule of graded module M, we can conclude that $a_{g_i} m_{h_j} \in N$, and so $a_{g_i} M \subseteq N$.

(ii) Trivial. □

Definition 1 *Let R be a G-graded ring, M a graded R-module and N a graded R-submodule of M. We say that N is a graded prime submodule of M if $N \neq M$ and whenever $a \in h(R)$ and $m \in h(M)$ with $am \in N$, then either $m \in N$ or $a \in (N :_R M)$.*

Here we will note that there exists graded R-submodule N of some graded R-module M, where $(N :_R M)$ is a graded prime ideal but N is not prime. For this we give an example.

Example 1 *Let $R = \mathbb{Z} = R_0$ be as $\mathbb{Z}$-graded ring and $M = \mathbb{Z} \times \mathbb{Z}$ be a $\mathbb{Z}$-graded R-module with $M_0 = (\mathbb{Z} \times \{0\})$ and $M_1 = (\{0\} \times \mathbb{Z})$. Then the submodule $N = 4\mathbb{Z} \times \{0\}$ is a graded submodule. In this case the graded ideal $(N :_R M) = 0$ is graded prime in R but the graded submodule N is not graded prime in M, since $2(2, 0) \in N$ but $2 \notin (N :_R M)$ and $(2, 0) \notin N$.*

Theorem 1 *Let R be a G-graded ring, M a graded R-module and N a graded R-submodule of M. Then N*

is a graded prime submodule of M if and only if whenever $IV \subseteq N$ with I is a graded ideal of $G(R)$ and V is a graded submodule of M implies $I \subseteq (N :_R M)$ or $V \subseteq N$.

Proof. Let N be a graded prime submodule and $IV \subseteq N$ for the graded ideal I of $G(R)$ and the graded submodule V of M. Assume that $I \nsubseteq (N :_R M)$. Then there exist an element $a_g \in I_g \backslash (N :_R M)$ for $g \in G$, such that $a_g v_h \in N$ for all $v_h \in V_h$ $(h \in G)$. Since N is a graded prime submodule, we get $v_h \in N$ for all $v_h \in V_h$ $(h \in G)$. Thus $V \subseteq N$.
For the converse let $r \in h(R)$, $m \in h(M)$ such that $rm \in N$. So we have $(r)(m) \subseteq N$ where (r) is a graded ideal of $G(R)$ and (m) a graded submodule of M. By our asumption we obtain $(r) \subseteq (N :_R M)$ or $(m) \subseteq N$. Hence $r \in (N :_R M)$ or $m \in N$. □

Proposition 1 *Let N be a graded submodule of the graded R-module M. Then N is graded prime if and only if for every graded submodule K of M such that $N \subset K \subseteq M$, $(N :_R K) = (N :_R M)$.*

Proof. Suppose that N is a graded prime submodule of M and let K be a graded submodule K of M such that $N \subset K \subseteq M$. It is clear that $(N :_R M) \subseteq (N :_R K)$. Now we will prove the reverse inclusion. Let $r_g \in (N :_R K)_g$ for $g \in G$. Then there exist an element $n \in h(M)$ where $n \in K \backslash N$ such that $r_g n \in N$. Since N is a graded prime, we obtain $r_g \in (N :_R M)$. Thus $(N :_R K) \subseteq (N :_R M)$.

Now, suppose that for every submodule K of M such that $N \subset K \subseteq M$ we have $(N :_R K) = (N :_R M)$. Assume that $rm \in N$ but $m \notin N$ for $r \in h(R)$, $m \in h(M)$. If we set $K = N + (m)$ we obtain $N \subset K \subseteq M$. And now by our assumption $(N :_R K) = (N :_R M)$. Hence $r \in (N :_R M)$. □

Definition 2 *A graded R-module M is said to be a multiplication graded module if for every graded submodule N, there exists a graded ideal I of $G(R)$ such that $N = IM$.*

Theorem 2 *Every homomorphic image of a multiplication graded module is a multiplication graded module.*

Proof. Let M be a multiplication graded R-module and $f : M \longrightarrow M'$ a graded homomorphism. Set $f(M) = K$ and let N be a graded submodule of K. Then $f^{-1}(N) = IM$ for some graded ideal I of $G(R)$. Therefore we have $N = f(IM) = If(M) = IK$. □

Corollary 1 *Let M be a multiplication graded R-module and N a graded submodule of M. Then M/N is a multiplication graded R-module.*

Theorem 3 *Suppose M is a multiplication graded R-module. Then N is a graded prime submodule of M if and only if $(N :_R M)$ is a graded prime ideal of $G(R)$.*

Proof. First suppose that N is a graded prime submodule and $ab \in (N :_R M)$ for $a, b \in h(R)$. This part is true in general for any graded R-module M. The reader can look to [3, Proposition 2.7].

Conversely suppose that $(N :_R M)$ is a graded prime ideal and let I be a graded ideal of $G(R)$ and V be a graded submodule of M such that $IV \subseteq N$. Since M is a multiplication module, there exists a graded ideal J of $G(R)$ such that $V = JM$. Therefore we get $N \supseteq IV = IJM$. So $IJ \subseteq (N :_R M)$ and since $(N :_R M)$ is a graded prime ideal, $I \subseteq (N :_R M)$ or $J \subseteq (N :_R M)$. Thus $I \subseteq (N :_R M)$ or $JM = V \subseteq N$. So N is a graded prime ideal by Theorem 1. □

Definition 3 *Suppose M is a multiplication graded R-module, $N = IM$ and $K = JM$ are graded submodules of M where I and J are graded ideals of $G(R)$. The product of N and K is denoted by NK and is defined by $(IJ)M$. From this we can define the product of two elements $m, m' \in h(M)$ as, if $Rm = IM$ and $Rm' = JM$ then $mm' = (IM)(JM) = (IJ)M$.*

Theorem 4 *Let $N = IM$ and $K = JM$ be graded submodules of the multiplication graded R-module M, where I and J be graded ideals of $G(R)$. Then the product of N and K is independent of the expresion of N and K as products.*

Proof. Let $N = I_1M = I_2M$ and $K = J_1M = J_2M$ for graded ideals I_i, J_i of $G(R)$, $i = 1, 2$. Then $NK = (I_1J_1)M = I_1(J_1M) = I_1(J_2M) = J_2(I_1M) = J_2(I_2M) = (I_2J_2)M$. □

Definition 4 *Let R be a G-graded ring and M be a multiplication graded R-module. We say that an element $0 \neq a \in h(M)$ is a graded zero divisor if there exist $0 \neq b \in h(M)$ such that $ab = 0$, i.e. if we set $(a) = IM$ and $(b) = JM$ for some graded ideals I, J of $G(R)$ then $0 = ab = (IJ)M$.*

It is known that if we let I, J as graded ideals of the G-graded ring R and P a graded prime ideal then $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. Now we will extend this property to the graded prime submodules of multiplication graded modules.

Theorem 5 *Let M be a multiplication graded R-module and N be a graded submodule of M. Then N is graded prime if and only if $AB \subseteq N$ implies $A \subseteq N$ or $B \subseteq N$ for each graded submodule A and B of M.*

Proof. The proof is similar to the proof of Theorem 3. □

Corollary 2 *Let M be a multiplication graded R-module and N be a graded submodule of M. Then N is graded prime if and only if $mm' \subseteq N$ implies $m \in N$ or $m' \in N$ for any $m, m' \in h(M)$.*

Theorem 6 *Let M be a multiplication graded R-module and N a graded submodule of M. Then N is graded prime if and only if M/N has no graded zero divisor.*

Proof. Let N be a graded prime submodule and $\bar{a}, \bar{b} \in h(M/N)$ such that $\bar{a}\bar{b} = \bar{0}$. Then there exist graded ideals I, J of $G(R)$ such that $(\bar{a}) = I(M/N)$ and $(\bar{b}) = J(M/N)$. And so we have $(IJ)(M/N) = \bar{0}$. Thus $(IJ)M \subseteq N$ and since N is graded prime, we have $I \subseteq (N :_R M)$ or $J \subseteq (N :_R M)$. Hence $IM \subseteq N$ or $JM \subseteq N$. So it follows $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$.

Conversely assume that M/N has no graded zero divisors and $ab \subseteq N$ for some $a, b \in h(M)$. Then $\bar{a}\bar{b} = \bar{0}$. Therefore, we get $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$. Hence $a \in N$ or $b \in N$. □

Definition 5 *Let M be a graded R-module:*

(i) For a given graded prime ideal P of $G(R)$, we consider the set

$T_P(M) = \{m \in M : \exists c \in h(R) \setminus P$ such that $cm = 0\}$. If $M = T_P(M)$ we will say that M is graded P-torsion.

(ii) For a given prime ideal P of $G(R)$, if there exists $x \in h(M)$ and $c \in h(R) \setminus P$ such that $cM \subseteq Rx$ then we will say that M is graded P-cyclic.

(iii) M is called faithful if $aM = 0$ implies $a = 0$ for $a \in h(R)$.

(iv) The annihilator of an element m of M is defined as, $ann(m) = (0 :_R m) = \{r \in h(R) : rm = 0\}$.

Note that the set $T_P(M)$ is a graded R-submodule of M.

Theorem 7 *If M is a graded R-module, then M is multiplication graded if and only if for every graded prime ideal P of $G(R)$, M is graded P-torsion or graded P-cyclic.*

Proof. One can look to [7, Theorem 5.9]. □

Theorem 8 *Let M be a faithful graded R-module. Then M is a multiplication graded module if and only if*

(i) $\underset{\lambda\in\Lambda}{\cap}(I_\lambda M) = \left(\underset{\lambda\in\Lambda}{\cap} I_\lambda\right) M$ where I_λ is a graded ideal of $G(R)$ and

(ii)For any graded submodule N of M and graded ideal A of $G(R)$ such that $N \subset AM$ there exist a graded ideal B with $B \subset A$ and $N \subseteq BM$.

Proof. Suppose that the two conditions satisfied. Let N be a graded submodule of M and $S = \{I : I$ is a graded ideal of $G(R)$ such that $N \subseteq IM\}$. This set is nonempty since $R \in S$. Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a nonempty subset of S, by (i) we have $\underset{\lambda\in\Lambda}{\cap} I_\lambda \in S$. Since S is totally ordered by inverse inclusion, we can use the Zorn's Lemma. So S has a minimal member, say A. Then $N \subseteq AM$ Suppose that $N \neq AM$, by (ii) there exists a graded ideal B of $G(R)$ such that $B \subset A$ and $N \subseteq BM$. In this case $B \in S$ and this contradicts with the choice of A. Thus $N = AM$.

For the converse let $\{I_\lambda\}_{\lambda\in\Lambda}$ be any nonempty collection of graded ideals of $G(R)$ and let $I = \underset{\lambda\in\Lambda}{\cap} I_\lambda$. Clearly $IM \subseteq \underset{\lambda\in\Lambda}{\cap}(I_\lambda M)$. Now let $x \in \underset{\lambda\in\Lambda}{\cap}(I_\lambda M)$ and $K = \{r \in h(R) : rx \in IM\}$. Suppose $K \neq R$. Then there exist a graded maximal ideal P of $G(R)$ such that $K \subseteq P$. Clearly $x \notin T_P(M)$ since otherwise there exist $c \in h(R) \setminus P$ such that $cx = 0 \in IM$, therefore $c \in K \subseteq P$ and this is a contradiction. Hence M is a graded P-cyclic, that is there exists $m \in h(M)$ and $c \in h(R) \setminus P$ such that $cM \subseteq Rm$. Then $cx \in c\left[\underset{\lambda\in\Lambda}{\cap}(I_\lambda M)\right] = \underset{\lambda\in\Lambda}{\cap}(cI_\lambda M)$. From this $cx \in \underset{\lambda\in\Lambda}{\cap}(I_\lambda cM) \subseteq \underset{\lambda\in\Lambda}{\cap}(I_\lambda Rm) = \underset{\lambda\in\Lambda}{\cap}(I_\lambda m)$. For each $\lambda \in \Lambda$ there exists $a_\lambda \in I_\lambda$ such that $cx = a_\lambda m$. Choose $\alpha \in \Lambda$, for each $\lambda \in \Lambda$ we get $a_\alpha m = a_\lambda m$ so that $(a_\alpha - a_\lambda)m = 0$. Now, $c(a_\alpha - a_\lambda)M = (a_\alpha - a_\lambda)cM \subseteq (a_\alpha - a_\lambda)Rm = 0$. Since M is a faithful graded module we have $c(a_\alpha - a_\lambda) = 0$. Therefore $ca_\alpha = ca_\lambda \in I_\lambda$ and hence $ca_\alpha \in I_\lambda$. Thus $c^2x = ca_\lambda m = ca_\alpha m \in IM$. It follows that $c^2 \in K \subseteq P$, therefore $c \in P$. Thus $K = R$, so $x \in IM$. Now let N be a graded submodule of M and A be a graded ideal of $G(R)$ such that $N \subset AM$. Then there exist a graded ideal C of $G(R)$ such that $N = CM$. Let $B = A \cap C$ then $B \subset A$ and $BM = (A \cap C)M = AM \cap CM = N$. □

Lemma 2 *Let M be a faithful multiplication graded R-module and P be a graded prime ideal of $G(R)$. If $am \in PM$ for $a \in h(R)$, $m \in h(M)$ then $a \in P$ or $m \in PM$.*

Proof. Let $am \in PM$ for some $a \in h(R)$, $m \in h(M)$ and suppose that $a \notin P$. Consider the set $K = \{r \in h(R) : rx \in PM\}$. Now assume that $K \neq R$, then there exist a graded maximal ideal Q such that $K \subseteq Q$. $m \notin T_Q(M)$, if not there exist $c \in h(R) \backslash Q$ such that $cm = 0 \in PM$. This is $c \in K \subseteq Q$ and this is contradiction. Since M is a multiplication graded module, M is a graded Q-cyclic module. Thus we have an element $c \in h(R) \backslash Q$ such that $cM \subseteq Rm$. In this case there exist elements $m' \in M$, $s \in R$ such that $cm' = sm$ and $cam' = pm$. Then $asm = cam' = pm$, thus $(as - p)m = 0$. Hence $c(\operatorname{ann}(m))M = 0$ and since M is faithful we get $c(\operatorname{ann}(m)) = 0$. From this $cas = cp \in P$, so $s \in P$. Then $cm' = sm \in PM$, this means $c \in K$, a contradiction. □

Corollary 3 *The following statements are equivalent for a proper graded submodule N of a multiplication graded R-module M:*

(i) N is a graded prime submodule;

(ii) $(N :_R M)$ is a graded prime ideal of $G(R)$;

(iii) $N = PM$ for some graded prime ideal P of $G(R)$ with $ann(M) \subseteq P$.

Proof. $(i) \Rightarrow (ii)$ Clear from [3, Theorem 2.11 (ii)],

$(ii) \Rightarrow (iii)$ If we let $(N :_R M) = P$, it is clear.

$(iii) \Rightarrow (i)$ Let P be a graded prime ideal of $G(R)$ such that $N = PM$ and $\operatorname{ann}(M) \subseteq P$. Then M is a faithful graded $R/\operatorname{ann}(M)$-module. The result follows from Lemma 2. □

Recall that a graded ideal I of the graded ring R, the graded radical of I, $\sqrt{I}$, is the set of all $x \in R$ such that for each $g \in G$ there exists a positive integer $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element of R, then $r \in \sqrt{I}$ if and only if $r^n \in I$ for some positive integer n (see, [8, definition 1.1]). The graded radical of a graded submodule N of a graded module M, denoted by $M\text{-}rad(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M then $M\text{-}rad(N) = M$. One can see easily that in any multiplication graded module, every graded submodule is contained in a graded maximal submodule. By the following theorem, we characterize the graded radical of any graded submodules of multiplication graded module with elements.

Theorem 9 *Let N be a proper graded submodule of a multiplication graded R-module M. If $A = (N :_R M)$ then $M\text{-}rad(N) = \left(\sqrt{A}\right) M$.*

Proof. Without loss of generality we can assume that M is a faithful multiplication graded R-module. Let $\wp$ denote the collection of all graded prime ideals P of $G(R)$ such that $A \subseteq P$. If $B = \sqrt{A}$ then $B = \bigcap_{P \in \wp} P$ and hence, by Theorem 8, $BM = \bigcap_{P \in \wp} (PM)$. Let $P \in \wp$, if $M = PM$ then $M\text{-}rad(N) \subseteq PM$. If $M \neq PM$ then

$N = AM \subseteq PM$ implies $M\text{-}rad(N) \subseteq PM$, by Corollary 3. It follows that $M\text{-}rad(N) \subseteq BM$. Conversely, suppose that K is a graded prime submodule of M containing N. By Corollary 3 there exist a graded prime ideal Q of $G(R)$ such that $A \subseteq Q$ and $K = QM$. Since $AM = N \subseteq QM = K \neq M$ it follows that $A \subseteq Q$, by Lemma 2, and hence $B \subseteq Q$. Thus $BM \subseteq K$ and so $BM \subseteq M\text{-}rad(N)$. Hence $M\text{-}rad(N) = BM$. □

Definition 6 *Let R be a G-graded ring, M a graded R-module and N a graded R-submodule of M. We say that N is a graded primary submodule of M if $N \neq M$ and whenever $a \in h(R)$ and $m \in h(M)$ with $am \in N$, then either $m \in N$ or $a^k \in (N :_R M)$ for some positive integer k.*

Theorem 10 *Let R be a G-graded ring, M a graded R-module and N a graded R-submodule of M. Then N is a graded primary submodule of M if and only if whenever $IV \subseteq N$ with I be a graded ideal of $G(R)$ and V a graded submodule of M implies $V \subseteq N$ or $I^n \subseteq (N :_R M)$ for some positive integer n.*

Proof. Let N be a graded primary submodule and $IV \subseteq N$ for some graded ideal I of $G(R)$ and V a graded submodule of M. Assume that $V \nsubseteq N$ and $I^n \nsubseteq (N :_R M)$ for all positive integer n. Then there exist elements $a \in I^n \backslash\ (N :_R M)$ and $v \in V \backslash N$ such that $av \in N$. Since N is a graded primary submodule, we get a contradiction.

For the converse let $r \in h(R)$, $m \in h(M)$ such that $rm \in N$. So we have $(r)(m) \subseteq N$ where (r) is a graded ideal of $G(R)$ and (m) is a graded submodule of M. By our asumption we obtain $(m) \subseteq N$ or $(r)^n \subseteq (N :_R M)$ for some positive integer n. Hence $r^n \in (N :_R M)$ or $m \in N$. □

Theorem 11 *Suppose M is a multiplication graded R-module, then N is a graded primary submodule of M if and only if $(N :_R M)$ is a graded primary ideal of $G(R)$.*

Proof. First suppose that N is a graded primary submodule. This part is true in general for any graded R-module M. The reader can look to [4, Proposition 2.5].

For the converse let $(N :_R M)$ is a graded primary ideal of $G(R)$ and $IV \subseteq N$ for some graded ideal I of $G(R)$ and V a graded submodule of M. Since M is a multiplication module, there exists a graded ideal J of $G(R)$ such that $V = JM$. So we get $(IJ)M \subseteq N$ and it follows $IJ \subseteq (N :_R M)$. By our assumption $J \subseteq (N :_R M)$ or $I^n \subseteq (N :_R M)$ for some positive integer n. Hence $V \subseteq N$ or $I^n \subseteq (N :_R M)$ for some positive integer n. Consequently, N is graded primary submodule by Theorem 10. □

Theorem 12 *Let M be a multiplication graded R-module and N, A and B be graded submodules of M. Then N is graded primary if and only if $AB \subseteq N$ implies $B \subseteq N$ or $A^n \subseteq N$ for some positive integer n.*

Proof. Let N be a graded primary submodule of M and $AB \subseteq N$ for some graded submodules A and B of M. Suppose $A = IM$ and $B = JM$, for some graded ideals I, J of $G(R)$. Then we obtain $(IJ)M \subseteq N$. So $IJ \subseteq (N :_R M)$. Since $(N :_R M)$ is graded primary, we get $J \subseteq (N :_R M)$ or $I^n \subseteq (N :_R M)$ for some positive integer n. Thus $B \subseteq N$ or $A^n \subseteq N$ for some positive integer n.

Conversely suppose for each graded submodule A and B of M such that $AB \subseteq N$ implies $B \subseteq N$ or $A^n \subseteq N$ for some positive integer n. Let $IV \subseteq N$ for some graded ideal I of $G(R)$ and a graded submodule

V of M. Since M is a multiplication graded R-module, there exist a graded ideal J of $G(R)$ such that $V = JM$. Then we obtain $N \supseteq IV = I(JM) = (IM)(JM)$ and therefore $JM \subseteq N$ or $(IM)^n = I^nM \subseteq N$ for some positive integer n. Hence $V \subseteq N$ or $I^n \subseteq (N :_R M)$ for some positive integer n. □

Corollary 4 *Let M be a multiplication graded R-module and N be a graded submodule of M. Then N is graded primary if and only if for any $m, m' \in h(M)$ such that $mm' \subseteq N$ implies $m' \in N$ or $m^n \subseteq N$ for some positive integer n.*

Definition 7 *A homogeneous element m of the multiplication graded module M is called nilpotent if there exists a positive integer n such that $m^n = 0$.*

Theorem 13 *Let M be a multiplication graded R-module and N a graded submodule of M. Then N is graded primary if and only if $M \neq N$ and every graded zero divisor in M/N is nilpotent.*

Proof. Let N be graded primary submodule of M. Then $M/N \neq 0$. Assume that $\bar{a} = a + N \in M/N$ is a graded zero divisor. Then there exists a homogeneous element $\bar{0} \neq \bar{b} = b + N \in M/N$ such that $\bar{a}\bar{b} = \bar{0}$. Then $ab \subseteq N$ and $b \notin N$. So we get $a^k \subseteq N$ for some positive integer k by Corollary 4. Hence $\bar{a}^k = \bar{0}$.

Conversely suppose that $0 \neq M/N$ and every graded zero divisor element in M/N is nilpotent. Now let $a, b \in h(M)$ such that $ab \subseteq N$ and $b \notin N$. Then $(a + N)(b + N) = (ab + N) = \bar{0}$ and $b + N \neq \bar{0}$. By our assumption we get $a^k + N = (a + N)^k = \bar{0}$ for some positive integer k. Thus $a^k \subseteq N$. □

Theorem 14 *Let M be a multiplication graded R-module and N a graded R-submodule of M. If N is graded primary submodule of M then M-$rad(N)$ is a graded prime submodule of M.*

Proof. Assume that N is a graded primary submodule of M. Then the graded ideal $(N :_R M)$ of $G(R)$ is graded primary, and by [8, Lemma 1.8] $\sqrt{(N :_R M)}$ is a graded prime ideal. Now if we use Theorem 9 and Corrollary 3 we obtain M-$rad(N)$ as a graded prime submodule. □

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