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ON GRAPHS ASSOCIATED WITH MODULES OVER COMMUTATIVE RINGS

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Abstract. Let M be an R-module, where R is a commutative ring with identity 1 and let $G(V, E)$ be a graph. In this paper, we study the graphs associated with modules over commutative rings. We associate three simple graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ to M called full annihilating, semi-annihilating and star-annihilating graph. When M is finite over R, we investigate metric dimensions in $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$. We show that M over R is finite if and only if the metric dimension of the graph $ann_f(\Gamma(M_R))$ is finite. We further show that the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty if and only if M is a prime-multiplication-like R-module. We investigate the case when M is a free R-module, where R is an integral domain and show that the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty if and only if $M \cong R$. Finally, we characterize all the non-simple weakly virtually divisible modules M for which $Ann(M)$ is a prime ideal and $Soc(M) = 0$.

1. Introduction

The subject of associating a graph to an algebraic structure has become an exciting research topic and has attracted considerable attention over the last two decades, see for instance [1, 3, 4, 11, 21, 22, 27, 28]. Associating a graph to a commutative ring R was introduced by Beck in [10] and was further studied by D. D. Anderson and Naseer in [3]. A different approach of associating a graph $\Gamma(R)$ to R with vertices as $Z^*(R) = Z(R)\backslash\{0\}$, where $Z(R)$ is the set of all zero-divisors of R was given by D. F. Anderson and Livingston in [5]. Two vertices $x, y \in Z^*(R)$ of $\Gamma(R)$ are adjacent if and only if $xy = 0$. Redmond in [28] extended the zero-divisor graph of a commutative ring to an ideal-based zero-divisor graph of a commutative ring. For a given ideal I of R, he defined an undirected graph $\Gamma_I(R)$ with vertex set $\{x \in R - I \,|\, xy \in I$ for some $y \in$

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$R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. The concept of zero-divisor graphs has been also extended to modules over rings. Ghalandarzadeh and Malakooti Rad in [16] extended the notion of zero-divisor graph to the torsion graph associated with a module M over a ring R, whose vertices are the nonzero torsion elements of M such that two distinct vertices a and b are adjacent if and only if $(a : M)(b : M)M = 0$. Recent generalizations of zero-divisor graphs to module theory can be found in [9, 29].

On the other hand, the problem of metric dimension in graphs was first introduced in 1975 by Harary and Melter [18]. However, the metric dimension problem for hypercube was studied much earlier in 1963 by Erdos and Renyi [14]. The metric dimension in graphs has been extensively studied by various authors for many particular classes of graphs such as trees, cycles, complete graphs, grids, wheels, fans, unicyclic graphs, honeycombs and circulant graphs. Bailey and Cameron [7] established a relationship between the base size of automorphism group of a graph and its metric dimension. The relationship in [7] then motivated authors in [6, 8, 15] to study metric dimensions of distance regular graphs, such as Grassman graphs, Johnson and Kneser graphs and also bilinear form graphs. Recently in [25, 26], the concept of metric dimension in terms of locating number was introduced in zero-divisor graphs associated with commutative rings. The authors in [25, 26] have discussed various properties of locating numbers (metric dimensions) which includes the characterization of all finite rings, examination of two equivalence relations on the vertices of $\Gamma(R)$, relationship between the locating set (resolving set) and cut vertices of $\Gamma(R)$, investigation of metric dimension in $\Gamma(R)$ when R is a finite product of integral domains and so on. It is shown in [12, 19, 20] that determining the metric dimension of an arbitrary graph is an NP-complete problem. The problem is still NP-complete even if we consider some specific families of graphs, such as planar graphs [12] or Gabriel unit disk graphs [19].

Throughout, R is a commutative ring (with 1) and all modules are unitary unless otherwise stated. The symbols $\subseteq$ and $\subset$, has usual set theoretic meaning as containment and proper containment of sets. We will denote the ring of integers by $\mathbb{Z}$, the ring of integers modulo n by $\mathbb{Z}_n$ and a finite field on q elements by $\mathbb{F}_q$ respectively. For basic definitions from graph theory we refer to [13, 23, 30], and for module theory we refer to [2, 31].

2. Definitions and preliminaries

A simple graph $G(V, E)$ consists of a finite nonempty set $V(G)$ of objects called vertices together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of G called edges. A graph G is connected if there is a path between every two distinct vertices of G. The distance from a vertex v to u denoted by $d(v, u)$ is the length of the shortest path from v to u ($d(v, v) = 0$ and $d(v, u) = \infty$, if there is no such path). The diameter of G is $diam(G) = \sup\{d(v, u) \,|\, v, u \in V(G)\}$. A graph G is said to be complete if there is an edge between every pair of distinct vertices. A complete graph with n vertices is denoted by K_n. A graph G is said to be bipartite if its vertex set can be partitioned into two sets $V_1(G)$ and $V_2(G)$ such that every edge of G has one end in $V_1(G)$ and another in $V_2(G)$. A complete bipartite graph is one in which each vertex of one partite set is joined to every vertex of another partite set. We denote complete bipartite graph with partite sets of order m and n by $K_{m,n}$. A complete bipartite graph of the from $K_{1,n}$ is called a star graph. A graph G is Hamiltonian if it has a cycle which contains every vertex of the graph. Moreover, $N(v)$ denotes the set all vertices of G adjacent to the vertex v and $N[v] = N(v) \cup \{v\}$.

A set of vertices $S \subseteq V(G)$ resolves a graph G, and S is a resolving set of G, if every vertex is uniquely determined by its vector of distances to the vertices of S. More generally, for an ordered subset $S = \{v_1, v_2, \ldots, v_k\}$ of vertices in a connected graph G and a vertex $v \in V(G) \setminus S$ of G, the metric representation of v with respect to S is the k-vector $D(v|S) = (d(v, v_1), d(v, v_2), \ldots, d(v, v_k))$. The set S is resolving set for G if $D(v|S) = D(u|S)$ implies that $u = v$ for all pair of vertices in $V(G) \setminus S$. Equivalently, S is a resolving set for G if $D(v|S) \neq D(u|S)$ for all pair of distinct vertices $u, v \in V(G) \setminus S$. A resolving set S of minimum cardinality is the metric basis for G, and the number of elements in the resolving set of minimum cardinality is the metric dimension of G. The metric dimension of a graph G is denoted by $\dim(G)$. Note here that by Definition 2.1 of [25] the metric dimension of an empty graph is not defined.

The resolving set is also called the locating set, metric representation of a vertex is also called the locating code of a vertex and the metric dimension of a graph is also called the locating number of a graph.

The concept of resolving set, metric representation and metric dimension in terms of locating set, locating code and locating number in zero-divisor graphs associated with commutative rings was introduced in [25] and has been further studied in [24, 26]. The authors in [24, 25, 26] have discussed various properties of metric dimensions which includes the characterization of all finite rings, examination of two equivalence relations on the vertices of $\Gamma(R)$, relationship between the resolving set and cut vertices of $\Gamma(R)$, investigation of metric dimension in $\Gamma(R)$ when R is a finite product of integral domains, when R is the finite product $R_1 \times R_2 \times \cdots \times R_n$, where $R_1, R_2, \ldots, R_n$ are n finite commutative rings with none of them being isomorphic to the Boolean ring $\prod_{i=1}^{n} \mathbb{Z}_2$, provided a combinatorial formula for computing the metric dimension of a zero-divisor graph $\Gamma(R \times \mathbb{F}_q)$ and so on.

Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two finite fields. Then the zero-divisor graph $\Gamma(\mathbb{F}_1 \times \mathbb{F}_2)$ associated with $\mathbb{F}_1 \times \mathbb{F}_2$ is either a star graph or a complete bipartite graph. Therefore, from Corollary 2.1 of [25], the metric dimension of $\Gamma(\mathbb{F}_1 \times \mathbb{F}_2)$ is $|\mathbb{F}_1| + |\mathbb{F}_2| - 4$. However, for $\mathbb{F}_1 = \mathbb{F}_2 = \mathbb{Z}_2$ the metric dimension is 1 because $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a path and the metric dimension of all finite paths by Lemma 2.1 of [25] is 1. For the rings $\mathbb{Z}_2 \times \mathbb{Z}_7$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_5$ and $\mathbb{Z}_3 \times \mathbb{F}_4$, it can be easily seen that the zero-divisor graphs associated with these rings are complete bipartite graphs $K_{1,6}$, $K_{3,4}$, $K_{1,4}$ and $K_{2,3}$. Therefore, from Corollary 2.1 of [25], the metric dimensions of these graphs are 5 and 3. Further, for the rings $\mathbb{Z}_2[x,y,z]/(x,y,z)^2$, $\mathbb{Z}_4[x,y]/(x^2,y^2,xy,2x,2y)$, $\mathbb{F}_8[x]/(x^2)$ and $\mathbb{Z}_4[x]/(x^3+x+1)$, the associated zero-divisor graph is a complete graph K_7. Therefore, from Lemma 2.2 of [25], the metric dimension is 6. For the rings $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_{p^2}[x]/(x^2)$, ($p \geq 2$ is a prime number), the associated zero-divisor graph is a complete graph on $p-1$ number of vertices. Therefore the metric dimension of the associated graph is $p-2$.

If $I = (0) \times \mathbb{Z}_3$ is an ideal of ring $R = \mathbb{Z}_9 \times \mathbb{Z}_3$, then the ideal based zero-divisor graph $\Gamma_I(R)$ defined in [28] with vertex set $V(\Gamma_I(R)) = \{(3,0),(3,1),(3,2),(6,0),(6,1),(6,2)\}$ is a complete graph K_6 on six vertices. Therefore metric dimension of $\Gamma_I(R)$ is 5. If I is a prime ideal of a ring R, then $\dim(\Gamma_I(R))$ is undefined. However, if $I = P_1 \cap P_2$, where P_1 and P_2 are prime ideals of a ring R, then $\dim(\Gamma_I(R))$ is finite which is in fact equal to $|V(\Gamma_I(R))| - 2$.

For more on the metric dimension of zero-divisor graphs, graphs determined by the equivalence classes of zero-divisors and ideal based zero-divisor graphs associated with commutative rings see [24, 25, 26]. In the remaining paper, we discuss the nature of graphs associated with modules and also determine the metric dimensions of these graphs, when M is finite over R. First we have the following definition.

Definition 2.1. Let M be an R-module. For an element $x \in M$, we define a set $[x : M_R] = \{r \in R : rM \subseteq Rx\}$, which clearly is an annihilator of the factor module M/Rx. The annihilator of a module M is defined as $Ann(M) = \{s \in R \mid sm = 0 \text{ for all } m \in M\}$. Clearly, for each $x \in M$, $Ann(M) \subseteq [x : M_R]$. Further, $Rx = M$ if and only if $[x : M_R] = R$. Since $Ann(M) \subseteq [x : M_R]$, based on the above definition we now classify the elements of M into three categories. An element $x \in M$ is a

(i) *full-annihilator*, if either $x = 0$ or $[x : M_R][y : M_R]M = 0$ for some nonzero $y \in M$ with $[y : M_R] \neq R$.

(ii) *semi-annihilator*, if either $x = 0$ or $[x : M_R] \neq 0$ and $[x : M_R][y : M_R]M = 0$ for some nonzero $y \in M$ with $0 \neq [y : M_R] \neq R$.

(iii) *star-annihilator*, if either $x = 0$ or $Ann(M) \subset [x : M_R]$ and $[x : M_R][y : M_R]M = 0$ for some nonzero $y \in M$ with $Ann(M) \subset [y : M_R] \neq R$.

We denote by the sets $A_f(M)$, $A_s(M)$ and $A_t(M)$ respectively the *full-annihilators*, *semi-annihilators* and *star-annihilators* for any module M over R. The name given to these sets is because of the containment $A_t(M) \subseteq A_s(M) \subseteq A_f(M)$. If $M = R$, then for each $x \in R$, $[x : R_R] = Ann(R/Rx) = Rx$. So $[x : R_R][y : R_R] = 0$ if and only if $xy = 0$. Therefore, x is a zero-divisor in R if and only if $[x : R_R][y : R_R]R = 0$ for some $y \neq 0 \in R$. Thus, we have the usual zero-divisors for R. So, for $M = R$ the full-annihilators, semi-annihilators and star-annihilators of M coincides with the zero-divisors of R.

Moreover, we let $\widehat{A_f(M)} = A_f(M) \backslash \{0\}$, $\widehat{A_s(M)} = A_s(M) \backslash \{0\}$ and $\widehat{A_t(M)} = A_t(M) \backslash \{0\}$ and associate three simple graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$

and $ann_t(\Gamma(M_R))$ to M over R called as full-annihilating, semi-annihilating and star-annihilating graphs of M over R and the vertices x and y are adjacent if and only if $[x : M_R][y : M_R]M = 0$. It is clear that $ann_t(\Gamma(M_R)) \subseteq ann_s(\Gamma(M_R)) \subseteq ann_f(\Gamma(M_R))$ as induced subgraphs. We will call all these graphs as annihilating graphs of M over R. It can be easily seen that for $M = R$, all the annihilating graphs are the zero-divisor graph of a commutative ring introduced by Anderson and Livingston in [5].

If M is finite over R, then $ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R))$, whereas the graph $ann_t(\Gamma(M_R))$ can be different from $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$. In the following example, we show that for a finite module $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ over $\mathbb{Z}$, $ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = K_7$ and $ann_t(\Gamma(M_R)) = K_5$.

Example 2.2. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Then, M over $\mathbb{Z}$ consists of eight elements as $\{(0,0),(1,0),(0,1),(0,2),(0,3),(1,1),(1,2),(1,3)\}$. Let $m_1 = (1,0)$, $m_2 = (0,1)$, $m_3 = (0,2)$, $m_4 = (0,3)$, $m_5 = (1,1)$, $m_6 = (1,2)$, and $m_7 = (1,3)$ be nonzero elements of M. It can be easily verified that $[m_2 : M_R] = [m_3 : M_R] = [m_4 : M_R] = [m_5 : M_R] = [m_7 : M_R] = 2\mathbb{Z}$ and $[m_1 : M_R] = [m_6 : M_R] = 4\mathbb{Z} = Ann(M)$. Thus, $\widehat{A_f(M)} = \widehat{A_s(M)} = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7\}$ and $\widehat{A_t(M)} = \{m_2, m_3, m_4, m_5, m_7\}$. Since $[m_i : M_R][m_j : M_R]M = 0$, for all $1 \leq i, j \leq 7$, it follows that $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ are complete graphs with seven vertices but $ann_t(\Gamma(M_R))$ is a complete graph with five vertices.

The above examples lead to a natural question: what is the nature of graphs $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ when M is infinite over R.

The following example illustrates that the graphs $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ are different when M is infinite over R.

Example 2.3. Let $M = \oplus_{i=1}^{n}\mathbb{Z}$ and $R = \mathbb{Z}$. Then, for all non-zero $x, y \in M$, $[x : M_R][y : M_R]M = 0$ with $[y : M_R] \neq R$. So, the graph $ann_f(\Gamma(M_R))$ is complete with vertices as $\widehat{M}$ and by definition it follows that the graph $ann_s(\Gamma(M_R))$ is empty. Thus in general for infinite modules over commutative rings the graphs $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ are different.

3. Graphs associated with multiplication-like modules over R

In this section, we characterize all the finite modules over commutative rings. Moreover, we characterize all the graphs associated with multiplication-like modules, prime multiplication modules and indecomposable modules.

The following observation shows that the graph $ann_f(\Gamma(M_R))$ is connected and has exceedingly small diameter which is analogous to the case for graphs $\Gamma(R)$ and $\Gamma_I(R)$ found in [[5], [28], Theorem 2.3 and Theorem 2.4].

Lemma 3.1. *Let M be an R-module. Then $ann_f(\Gamma(M_R))$ is a connected graph and $diam(ann_f(\Gamma(M_R))) \leq 3$.*

Proof. Let $x, y \in \widehat{A_f(M)}$ with $x \neq y$. We have the following cases.

Case 1. $[x : M_R][y : M_R]M = 0$. Then, $x - y$ is a path.

Case 2. $[x : M_R][y : M_R]M \neq 0$. If $[x : M_R]^2 M = 0$ and $[y : M_R]^2 M = 0$ then, $x - z - y$ is a path of length 2, for each $0 \neq z \in Rx \cap Ry$, $[x : M_R] \subseteq [x : M_R] \cap [y : M_R]$.

Case 3. $[x : M_R][y : M_R]M \neq 0$, $[y : M_R]^2 M \neq 0$ and $[x : M_R]^2 M = 0$. Then, there exists $b \in \widehat{A_f(M)} \backslash \{x, y\}$ such that $[b : M_R][y : M_R]M = 0$. If $[b : M_R][x : M_R]M = 0$, then $x - b - y$ is a path of length 2. If $[b : M_R][x : M_R]M \neq 0$, then for each $0 \neq c \in Rb \cap Rx$, $[c : M_R] \subseteq [b : M_R] \cap [x : M_R]$, $x - c - y$ is a path of length 2.

Case 4. $[x : M_R][y : M_R]M \neq 0$, $[x : M_R]^2 M \neq 0$ and $[y : M_R]^2 M = 0$. The proof follows from Case 3.

Case 5. $[x : M_R][y : M_R]M \neq 0$, $[x : M_R] \neq 0$ and $[y : M_R] \neq 0$. Then, there exists $a \in \widehat{A_f(M)} \backslash \{x, y\}$ with $[a : M_R][x : M_R]M = 0$ and $b \in \widehat{A_f(M)} \backslash \{x, y\}$ such that $[b : M_R][y : M_R]M = 0$.

Subcase 1. $[a : M_R] = [b : M_R]$. Then, $x - a - y$ is a path of length 2.

Subcase 2. $[a : M_R] \neq [b : M_R]$ and $[a : M_R][b : M_R]M = 0$, then $x - a - b - y$ is a path of length 3 and hence $d(x, y) \leq 3$. If $[a : M_R][b : M_R]M \neq 0$, then there exists $0 \neq d \in Ra \cap Rb$ such that $x - d - y$ is a path of length 2.

Thus $ann_f(\Gamma(M_R))$ is connected and $diam(ann_f(\Gamma(M_R))) \leq 3$. □

Let M be a nonzero R-module. Then M is a *prime module* if whenever N is a nonzero submodule of M and A is an ideal of R such that $NA = 0$, then $MA = 0$. That is, $Ann(M) = Ann(N)$ for all nonzero submodules N of M. Also, an R-module M is called a *multiplication module* if each submodule of M is of the form IM, where I is an ideal of R. A multiplication module M is *multiplication-like module* if for each nonzero submodule N of M, $Ann(M) \subset Ann(M/N)$. It is clear that for each nonzero submodule N of a multiplication module M, $N = Ann(M/N)M$ and $Ann(M) \subset Ann(M/N)$. Thus, it follows that every multiplication module is a multiplication-like module.

We have the following observation.

Lemma 3.2. *Let M be an R-module. Then M is multiplication-like if and only if $Ann(M) \subset [m : M_R]$ for each $0 \neq m \in M$.*

Proof. Suppose M is a multiplication-like module. Then

$$Ann(M) \subset Ann(M/N)$$

for each submodule N of M. Now using Definition 2.1, it follows $Ann(M) \subset [m : M_R]$ because for $m \in M$, we have $[m : M_R] = Ann(M/Rm)$ and clearly Rm is a submodule of M by taking the action of R on Rm.

Conversely, suppose that for each $0 \neq m \in M$, we have $Ann(M) \subset [m : M_R]$. Let N be a submodule of M. We show that M is a multiplication module. For each $0 \neq x \in N$, there exists an ideal $[x : N_R]$ of R such that $[x : N_R]M \subseteq Rx$.

Let $I = \sum_{0 \neq x \in N} [x : N_R]$. Then, $0 \neq IM = N$. Thus, it follows that M is a multiplication module and hence a multiplication-like module. $\square$

Now, we characterize all the finite modules over commutative rings. We show that M is finite over R if and only if metric dimension of the graph $ann_f(\Gamma(M_R))$ is finite. In fact the following result is the generalization of Theorem 3.1 of [25].

Theorem 3.3. $\dim(ann_f(\Gamma(M_R)))$ *is finite if and only if M is finite over R.*

Proof. Suppose M is finite over R. Then, clearly the set $A_f(M)$ is finite. It follows that the graph $ann_f(\Gamma(M_R))$ is finite, which implies that the number $\dim(ann_f(\Gamma(M_R)))$ is finite.

Conversely, suppose $ann_f(\Gamma(M_R))$ is finite. Then, $\dim(ann_f(\Gamma(M_R)))$ is finite. If there exists $0 \neq x \in M$ such that $[x : M_R] = Ann(M)$, then $[x : M_R][y : M_R]M = 0$ for all $0 \neq y \in M$, that is, $\widehat{A_f(M)} = \widehat{M}$, where $\widehat{M}$ is a set of nonzero elements of M. Therefore, M is finite. If $[x : M_R] \neq Ann(M)$ for any $0 \neq x \in M$, then by Lemma 3.2, M is a multiplication-like module.

Suppose M is infinite. Since M is a multiplication-like module, for each nonzero submodule N of M, $Ann(M/N) \neq 0$, that is, $[x : M_R] \neq 0$ for all $x \in M$. Since $ann_f(\Gamma(M_R))$ is finite and nonempty, there exists some $x, y \in \widehat{M}$ such that $[x : M_R][y : M_R]M = 0$. It follows that there is a path $x - y$ in $ann_f(\Gamma(M_R))$. Let $r \in R$ and assume $ry \neq 0$. It is clear that $[ry : M_R] \subseteq [y : M_R]$. So, $[x : M_R][ry : M_R]M \subseteq [x : M_R][y : M_R]M = 0$. Thus $x - ry$ is a path in $ann_f(\Gamma(M_R))$ and therefore $Rx \subseteq A_f(M)$ is finite, since $0 \neq [x : M_R] \subseteq Rx$, $[x : M_R]$ is also finite. Let $z \in [x : M_R]$ such that $0 \neq zM$. Then, zM is finite and there exists an ideal A of R such that $0 \neq AM \subseteq zM$. If M is not finite, then there is an element $m_1 \in M$ such that $T = \{m \in M : zm_1 = zm\}$ is infinite. Clearly, $N = \{m \in M : zm = 0\}$ is a nonzero submodule and is infinite. Since M is multiplication, there exists an ideal B of R such that $0 \neq BM \subseteq N$. Let $jm^* \in JM = \{\sum_{\text{finite}} j_i m_i : j_i \in J, m_i \in M\}$. Then, $[jm^* : M_R] \subseteq Rjm^* \subseteq JM$. So, for each $0 \neq m \in N, [m : M_R][jm^* : M_R]M \subseteq [m : M_R]JM \subseteq [m : M_R]zM \subseteq z[m : M_R]M \subseteq zN$ (because N is a submodule of a multiplication module). Therefore, $N \subseteq \widehat{A_f(M)}$, a contradiction. Thus M must be finite. $\square$

Remark 3.4. For a finite module M over R, we have

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)).$$

So, if $ann_f(\Gamma(M_R))$ is finite, then clearly $\dim(ann_s(\Gamma(M_R)))$ is finite. Now, if $\dim(ann_s(\Gamma(M_R)))$ is finite, then by Lemma 3.1, the diameter of $ann_s(\Gamma(M_R))$ is not more than 3. Therefore, by [25, Theorem 2.2] the number of vertices of $ann_s(\Gamma(M_R))$ is finite, which implies that the graph $ann_s(\Gamma(M_R))$ is also finite.

Remark 3.5. (i) By [25, Lemma 2.1],

$$\dim(ann_f(\Gamma(M_R))) = \dim(ann_s(\Gamma(M_R))) = \dim(ann_t(\Gamma(M_R))) = 1$$

if and only if the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are paths on $|\widehat{A_f(M)}|$, $|\widehat{A_s(M)}|$ and $|\widehat{A_t(M)}|$ number of vertices.

(ii) If $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are cycles on $|\widehat{A_f(M)}|$, $|\widehat{A_s(M)}|$ and $|\widehat{A_t(M)}|$ number of vertices, then, by [25, Lemma 2.3],

$$\dim(ann_f(\Gamma(M_R))) = \dim(ann_s(\Gamma(M_R))) = \dim(ann_t(\Gamma(M_R))) = 2.$$

(iii) By [25, Lemma 2.2],
$\dim(ann_f(\Gamma(M_R))) = |\widehat{A_f(M)}| - 1$, $\dim(ann_s(\Gamma(M_R))) = |\widehat{A_s(M)}| - 1$ and $\dim(ann_t(\Gamma(M_R))) = |\widehat{A_t(M)}| - 1$ if and only if the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are complete on $|\widehat{A_f(M)}|$, $|\widehat{A_s(M)}|$ and $|\widehat{A_t(M)}|$ number of vertices.

(iv) If $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are complete bipartite graphs or star graphs (other than $K_{1,1}$), then by [25, Corollary 2.1]
$\dim(ann_f(\Gamma(M_R))) = |\widehat{A_f(M)}| - 2$, $\dim(ann_s(\Gamma(M_R))) = |\widehat{A_s(M)}| - 2$ and $\dim(ann_t(\Gamma(M_R))) = |\widehat{A_t(M)}| - 2$.

Example 3.6. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_6$. Then, $[2 : M_R] = 2\mathbb{Z}$, $[3 : M_R] = 3\mathbb{Z}$ and $[4 : M_R] = 4\mathbb{Z}$ with $Ann(M) = 6\mathbb{Z}$. Thus, $\widehat{A_f(M)} = \widehat{A_s(M)} = \widehat{A(M)} = \{2, 3, 4\}$. Clearly $[2 : M_R][3 : M_R]M = 0$ and $[3 : M_R][4 : M_R]M = 0$. Therefore, we have $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ as paths on three vertices. Thus we conclude that $\dim(ann_f(\Gamma(M_R))) = \dim(ann_s(\Gamma(M_R))) = \dim(ann_t(\Gamma(M_R))) = 1$.

Remark 3.7. Let M be an R-module and $M = M_1 \oplus M_2$, where M_1 and M_2 are non-isomorphic simple submodules of M. Then, for $M_1 = M_2 = \mathbb{Z}_2$, $ann_f(\Gamma(M_R))$ is a path. In all other cases, $ann_f(\Gamma(M_R))$ is a complete bipartite graph. Therefore, by [25, Corollary 2.1], $\dim(ann_f(\Gamma(M_R))) = |\widehat{A_f(M)}| - 2$.

Remark 3.8. Let $p \geq 3$ be a prime number. Clearly, $ann_f(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p))$ is a complete graph on $p^2 - 1$ vertices. It follows that $ann_f(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p))$ is isomorphic to K_{p^2-1}. Therefore, from [25, Lemma 2.2], it follows that the graph $ann_f(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p))$ is Hamiltonian if and only if $\dim(ann_f(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p))) = p^2 - 2$.

From Example 2.2, we have $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ as complete graphs on seven vertices, where as $ann_t(\Gamma(M_R))$ is a complete graph on five vertices. Therefore, by Lemma 2.2 of [25],

$$\dim(ann_f(\Gamma(M_R))) = \dim(ann_s(\Gamma(M_R))) = 6 \neq 4 = \dim(ann_t(\Gamma(M_R))).$$

If M is a multiplication-like module over R, then all the annihilating graphs associated with M have same metric dimension, as can be seen below.

Theorem 3.9. *Let M be a multiplication-like R-module. Then*

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = ann_t(\Gamma(M_R)).$$

Proof. Suppose M is a multiplication-like module. If $ann_f(\Gamma(M_R)) = \phi$, then clearly $ann_s(\Gamma(M_R)) = ann_t(\Gamma(M_R)) = \phi$. Assume that $ann_f(\Gamma(M_R)) \neq \phi$ and fix a vertex x in $ann_f(\Gamma(M_R))$. Then, there exists $0 \neq y \in M$ such that $[x : M_R][y : M_R]M = 0$, (that is the vertices x and y are connected by a path $x - y$ in $ann_f(\Gamma(M_R))$). Since for each $0 \neq m \in M$, $Ann(M) \subset [m : M_R]$, so $x \in ann_t(\Gamma(M_R))$. It follows that $x - y$ is a path in $ann_t(\Gamma(M_R))$. Thus,

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = ann_t(\Gamma(M_R)). \qquad \square$$

Corollary 3.10. *If M is a multiplication R-module, then*

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = ann_t(\Gamma(M_R)).$$

Proof. Since every multiplication module is a multiplication-like module, the result follows from Theorem 3.9. $\square$

Remark 3.11. From Theorem 3.9, for a multiplication-like R-module M, it follows that all the annihilating graphs coincide. Thus the metric dimensions of all these graphs are the same, that is,

$$\dim(ann_f(\Gamma(M_R))) = \dim(ann_s(\Gamma(M_R))) = \dim(ann_t(\Gamma(M_R))).$$

So is the case for the multiplication R-module.

By [25, Theorem 3.1], it is clear that for a commutative ring R, the metric dimension of graph $\Gamma(R)$ is undefined if and only if R is an integral domain. That is, the graph $\Gamma(R)$ is empty if and only if R is an integral domain.

In the following result, we see that the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty if and only if M is a prime multiplication-like module.

Theorem 3.12. *Let M be an R-module. Then M is a prime multiplication-like module if and only if the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty.*

Proof. Suppose M is a prime multiplication-like module. Then for every $0 \neq x \in M$, we have $Ann(M) \subset [x : M_R]$. It follows that $[x : M_R][y : M_R]M \neq 0$ for each $0 \neq x, y \in M$. So,

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = ann_t(\Gamma(M_R)) = \phi.$$

Conversely, suppose that graphs

$$ann_f(\Gamma(M_R)),\ ann_s(\Gamma(M_R)) \text{ and } ann_t(\Gamma(M_R))$$

are empty. Then, by Theorem 3.9, M is a multiplication-like module. Assume that M is not prime. Then, by [17, Corollary 1.6], $Ann(M)$ is not a prime ideal. Therefore, $ABM = 0$, for some ideals A and B with $Ann(M) \subset A, B$. Since $AM \neq 0$ and $BM \neq 0$, there exist $0 \neq x \in AM$ and $0 \neq y \in BM$ such that $[x : M_R] \subseteq Rx \subseteq AM$ and $[y : M_R] \subseteq Ry \subseteq BM$. Then, $[x : M_R][y : M_R]M \subseteq ABM = 0$. Therefore, $ann_f(\Gamma(M_R)) \neq \phi$, a contradiction. Thus M is a prime multiplication-like module. $\square$

Remark 3.13. From Theorem 3.12, it follows that metric dimension of all the annihilating graphs associated with M over R are undefined if and only if M over R is a prime multiplication-like module.

The following is an immediate consequence of Theorem 3.12.

Corollary 3.14. *Let M be an R-module. Then M is a prime multiplication module if and only if M is a multiplication-like module for which $Ann(M)$ is a prime ideal.*

A nonzero R-module M is called an *indecomposable* if M cannot be written as a direct sum of nonzero submodules.

We have the following observation regarding decomposable modules.

Lemma 3.15. *Let $M = M_1 \oplus M_2$ be decomposable R-module, where M_1 and M_2 are nonzero R-modules. If $ann_f(\Gamma(M_{1R}))$ is a complete graph, then $ann_f(\Gamma(M_R))$ is also a complete graph.*

Proof. Let $0 \neq x \in \widehat{A_f(M_1)}$. Then there exists $0 \neq y \in M_1$ such that $[x : M_{1R}][y : M_{1R}]M_1 = 0$, where $[x : M_{1R}] = Ann(M_1/Rx)$ and $[y : M_{1R}] = Ann(M_1/Ry)$. Clearly,

$$[(x,0) : M_R] = Ann(\frac{M_1 \oplus M_2}{R(x,0)}) = Ann(\frac{M_1}{Rx} \oplus M_2), \tag{1}$$

$$[(y,0) : M_R] = Ann(\frac{M_1 \oplus M_2}{R(y,0)}) = Ann(\frac{M_1}{Ry} \oplus M_2). \tag{2}$$

Moreover,

$$[x : M_R] = Ann(\frac{M_1 \oplus M_2}{Rx \oplus (0)}) = Ann(\frac{M_1}{Rx} \oplus M_2), \tag{3}$$

$$[y : M_R] = Ann(\frac{M_1 \oplus M_2}{(0) \oplus Ry}) = Ann(M_1 \oplus \frac{M_2}{Ry}). \tag{4}$$

It follows that $[x : M_R] \subseteq Ann(M_2)$ and $[y : M_R] \subseteq Ann(M_1)$. Thus, $[x : M_R][y : M_R]M = 0$.

Using (1), (2), (3) and (4), we have $[(x,0) : M_R] \subseteq [x : M_R]$, $[(y,0) : M_R] \subseteq [y : M_R]$ and $[(x,0) : M_R]M_2 = [(y,0) : M_R]M_2 = 0$. Therefore, $[(x,0) : M_R][(y,0) : M_R]M = 0$, which implies $(x,0) \in \widehat{A_f(M)}$ and the vertices $(x,0)$, $(y,0)$ are adjacent in $ann_f(\Gamma(M_R))$, which further implies that if the graph $ann_f(\Gamma(M_{1R}))$ is complete, then the graph $ann_f(\Gamma(M_R))$ is also complete. $\square$

The following result shows that if the graph $ann_f(\Gamma(M_R))$ is empty, then M is always indecomposable.

Theorem 3.16. *Let M be an R-module. If* $\dim(ann_f(\Gamma(M_R)))$ *is undefined, then M is indecomposable.*

Proof. Suppose $\dim(ann_f(\Gamma(M_R)))$ is undefined. Then $ann_f(\Gamma(M_R)) = \phi$. Therefore, by Theorem 3.12, M is a prime multiplication-like module. If $M = M_1 \oplus M_2$, where M_1 and M_2 are nonzero R-modules, then by Lemma 3.15, $ann_f(\Gamma(M_R)) \neq \phi$, a contradiction. □

The following is a consequence of Theorem 3.16.

Corollary 3.17. *Every prime multiplication-like R-module is an indecomposable R-module.*

Let M be an R-module. If for some ideal I of R, $am = 0$ for all $a \in I$, $m \in M$, then we say M is annihilated by I. In this situation we can make M into an R/I-module by defining an action of the quotient ring R/I on M.

In the following result, we show that the graph $ann_t(\Gamma(M_R)))$ coincides with the graph $ann_t(\Gamma(M_{R/I}))$ while the graph $ann_f(\Gamma(M_R))$ coincides with $ann_f(\Gamma(M_{R/I}))$.

Proposition 3.18. *Let M be an R-module with $I = Ann(M)$. Then*

$$ann_t(\Gamma(M_R))) = ann_t(\Gamma(M_{R/I})),$$

and

$$ann_f(\Gamma(M_R))) = ann_f(\Gamma(M_{R/I})).$$

Proof. To prove the result, it is enough to show that the graphs $ann_t(\Gamma(M_R))$ and $ann_t(\Gamma(M_{R/I}))$ coincide. That is, we show that the vertices x and y are adjacent in $ann_t(\Gamma(M_R))$ if and only if they are adjacent in $ann_t(\Gamma(M_{R/I}))$. Let $x \in \widehat{A_t(M)}$. Then, there exists $0 \neq y \in M$ such that $Ann(M) \subset [x : M_R]$ and $[x : M_R][y : M_R]M = 0$ with $Ann(M) \subset [y : M_R] \subset R$. It is clear here that $I = Ann(M) \subseteq [x : M_R] \cap [y : M_R]$. Thus, $([x : M_R]/I)([y : M_R]/I)M = 0$ (because $Ann(M/Rx)$ over R/I is $[x : M_R]/I$ and $Ann(M/Ry)$ over R/I is $[y : M_R]/I$). It follows that $x \in \widehat{A_t(M)}$ if and only if $x \in A(\widehat{M_t(R/I)})$ and the vertices x and y are adjacent in $ann_t(\Gamma(M_R))$ if and only if they are adjacent in $ann_t(\Gamma(M_{R/I}))$. Thus, $ann_t(\Gamma(M_R))$ and $ann_t(\Gamma(M_{R/I}))$ are equal. Similarly it can be proved that $ann_f(\Gamma(M_R))) = ann_f(\Gamma(M_{R/I}))$. □

4. Graphs associated with divisible and free modules over *R*

We start this section with the following observation on the action of R on M.

Lemma 4.1. *Let M be an R-module. Then the following hold.*

(i) *If the action of R on M is faithful, then*

$$ann_t(\Gamma(M_R)) = ann_s(\Gamma(M_R)).$$

(ii) *If the action of R on M is not faithful, then*

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)).$$

Proof. (i) Since the action of R on M is faithful, so the annihilator ideal is a nonzero ideal. That is, $Ann(M) \neq (0)$. Let $x \in \widehat{A_t(M)}$. Then, $[x : M_R] \neq 0$ and there exists $0 \neq y \in M$ such that $Ann(M) \subset [x : M_R]$, $[x : M_R][y : M_R]M = 0$ with $Ann(M) \subset [y : M_R] \subset R$. It follows that $x \in \widehat{A_s(M)}$ and the vertices x and y are adjacent in $ann_t(\Gamma(M_R))$ if and only if they are adjacent in $ann_s(\Gamma(M_R))$. Therefore, $ann_t(\Gamma(M_R)) = ann_s(\Gamma(M_R))$.

(ii) Similar to part (i). □

In the following result, we consider the graphs associated with free modules over an integral domain R. We show that the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty if and only if $R \cong M$. Moreover, we show graphs $ann_t(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ are empty and the graph $ann_t(\Gamma(M_R))$ is complete if and only if $M \ncong R$.

Proposition 4.2. *Let M be a free R-module, where R is an integral domain. Then the following hold.*

(i) $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ *and* $ann_t(\Gamma(M_R))$ *are empty graphs if and only if* $R \cong M$.

(ii) $ann_t(\Gamma(M_R))$ *and* $ann_s(\Gamma(M_R))$ *are empty graphs and the graph* $ann_f(\Gamma(M_R))$ *is complete if and only if* $M \ncong R$.

Proof. (i) Suppose the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty. Then

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = ann_t(\Gamma(M_R)).$$

Therefore, by Theorem 3.12, M is a prime multiplication-like module. Further, by Theorem 3.16, M is an indecomposable module and so $M \cong R$.

Conversely, if M and R are isomorphic, it is clear that all the annihilating graphs are empty.

(ii) Suppose that $ann_t(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = \phi$ and $ann_f(\Gamma(M_R))$ is a complete graph. Then $\widehat{A_f(M)} \neq \phi$. Therefore, $M \ncong R$.

For the converse, let $M = \oplus_{\lambda\in\Omega} R$, where Ω is an index set with $|\Omega| \geq 2$. Let $0 \neq x = (x_\lambda)_{\lambda\in\Omega} \in M$, where $x_\lambda \in R$, for each $\lambda \in \Omega$. Then, $x_\mu \neq 0$, for some $\lambda \neq \mu \in \Omega$ and also $[x : M_R]M = \oplus_{\lambda\in\Omega}[x : M_R] \subseteq Rx = R(x_\lambda)_{\lambda\in\Omega}$. If $[x : M_R] \neq 0$ and $0 \neq z \in [x : M_R]$, then we put $y_\mu = 0$ and $y_\lambda = z$, for each $\mu \neq \lambda$. Therefore, $(y_\lambda)_{\lambda\in\Omega} \oplus_{\lambda\in\Omega} [x : M_R]$, and so there exist $l \in R$ such that $(y_\lambda)_{\lambda\in\Omega} = l(x_\lambda)_{\lambda\in\Omega}$. It follows that $0 = y_\mu = tx_\mu$ and $z = tx_\lambda$, for each $\mu \neq \lambda$. Since R is an integral domain and $x_\mu \neq 0$, $t = 0$ which implies that $z = 0$, a contradiction. Thus, $[x : M_R] = 0$ for each $x \in M$. Hence the graph $ann_f(\Gamma(M_R))$ is complete and since M is faithful R-module, by Lemma 4.1, $ann_t(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = \phi$. □

Let M be an R-module. Then we say M is *divisible* if $rM = M$ for all $0 \neq r \in R$. If R is a principal integral domain, then M is injective if and only if it is divisible. Over R, the divisible modules are exactly the injective modules. However, over other domains divisible modules need not to be injective. Further, we say that M is a *virtually divisible* module if $Ann(M/N) = Ann(M)$ for each proper submodule N of M. Also, M is a *weakly virtually divisible module* if $Ann(M/Rn) = Ann(M)$ for each proper cyclic submodule Rn of M (that is, $[x : M_R] = Ann(M)$ for each $0 \neq x \in M$ with $Rx \neq M$).

In the following result, we give the nature of all the annihilating graphs associated with weakly virtually divisible R-modules.

Theorem 4.3. *Let M be weakly virtually divisible R-module such that M is not cyclic. Then the following hold.*

(i) *$ann_t(\Gamma(M_R))$ is an empty graph and $ann_f(\Gamma(M_R))$ is a complete graph.*

(ii) *If the action of R on M is faithful, then $ann_s(\Gamma(M_R))$ is an empty graph.*

(iii) *If the action of R on M is not faithful, then $ann_s(\Gamma(M_R))$ is a complete graph.*

Proof. (i) Since M is not cyclic and M is weakly virtually divisible module, $[x : M_R] = Ann(M)$, which implies that the graph $ann_t(\Gamma(M_R))$ is empty and the graph $ann_f(\Gamma(M_R)))$ is complete.

(ii) If R acts on M such that the action on M is faithful, then by Lemma 4.1, $ann_t(\Gamma(M_R)) = ann_s(\Gamma(M_R))$. So $ann_s(\Gamma(M_R))$ is an empty graph.

(iii) If the action of R on M is not faithful, then by Lemma 4.1, $ann_f(\Gamma(M_R))$ $= ann_s(\Gamma(M_R))$. By (i), $ann_f(\Gamma(M_R))$ is a complete graph. Thus, it follows that the graph $ann_s(\Gamma(M_R))$ is also complete. $\square$

An R-module M is called *simple* if $M \neq (0)$ and it has no submodules except (0) and M. An R-module M is a *semi-simple* module if it is a direct sum of simple modules. Also, an R-module M is called a homogenous *semi-simple* R-module if it is a direct sum of isomorphic simple R-modules, that is, $Ann(M)$ is a maximal ideal of R.

Remark 4.4. Let R be a field and M a homogeneous semi-simple R-module. If M is simple, then all the annihilating graphs of M over R are empty. If M is not simple, then $ann_t(\Gamma(M_R))$ is an empty graph and the graphs $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ are complete.

Now, we have the following observation regarding divisible and virtually divisible modules over an integral domain.

Lemma 4.5. *A module M over R is virtually divisible if and only if $P = Ann(M)$ is a prime ideal and M is a divisible R/P-module.*

Proof. Suppose $P = Ann(M)$ is a prime ideal of R and M is a divisible R/P-module. Then, clearly M is virtually divisible.

Conversely, suppose M is virtually divisible. Let $ab \in P$, where $a, b \in R$. Let $aM \neq 0$. Then, clearly aM is a nonzero submodule of M. If $aM \neq M$, then $Ann(M/aM) = Ann(M) = P$ (because M is virtually divisible module). So $a \in Ann(M/aM) = Ann(M)$, a contradiction. Thus, $aM = M$ and so $bM = baM = 0$. It follows that $b \in Ann(M) = P$. Therefore P is a prime ideal.

Further, let $0 \neq r \in R/P$. Then $rM \neq 0$. If $rM \neq M$, then by the same reasoning as above we have a contradiction. Thus $rM = M$ (that is, $(r + P)M = M$) and so M is a divisible R/P-module. □

In the next result, we show that if M is virtually divisible R-module and simple, then all the annihilating graphs associated with M are empty. Further, we show that if M is a non simple virtually divisible R-module, then the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ are complete, where as $ann_t(\Gamma(M_R))$ is an empty graph.

Theorem 4.6. *Let R be an integral domain and let M be an R-module. If M is virtually divisible R-module, then the following hold.*

(i) *If M is simple, then all the annihilating graphs associated with M over R are empty.*
(ii) *If M is not simple, then $ann_t(\Gamma(M_R))$ is empty and the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ are complete.*

Proof. Let M be a virtually divisible R-module. By Lemma 4.5, $P = Ann(M)$ is a prime ideal and M is a divisible R/P-module. If $P = 0$, then M is a divisible R-module. If $P \neq 0$, then P is a maximal ideal and so M is a homogeneous semi-simple module. Now the result follows from Remark 4.4. □

Remark 4.7. Let R be an integral domain and let M be an R-module. If M is a divisible R-module and simple, then all the annihilating graphs of M over R are empty. However, if M is a divisible R-module but not simple, then the graph $ann_t(\Gamma(M_R))$ is empty, where as the graphs $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ are complete.

The *socle* of a module M over ring R is denoted by $Soc(M)$ and is defined by $Soc(M) = \sum\{N : N \textit{ is a simple submodule of } M\}$. The following result characterizes all non-simple weakly virtually divisible modules.

Theorem 4.8. *The graph $ann_f(\Gamma(M_R))$ is complete if and only if M is a non-simple weakly virtually divisible module for which $Ann(M)$ is a prime ideal and $Soc(M) = 0$.*

Proof. Suppose M is a non-simple weakly virtually divisible R-module. Then, $Ann(M) = [x : M_R]$ for each $x \in M$. It follows that the graph $ann_f(\Gamma(M_R))$ is complete.

Conversely, suppose the graph $ann_f(\Gamma(M_R))$ is complete. This implies $ann_s(\Gamma(M_R))$ is a complete graph with vertices $\widehat{M}$. Therefore, for $x \neq y \in \widehat{M}$,

$[x : M_R]$ and $[y : M_R]$ are the two ideals of R such that $[x : M_R][y : M_R]M = 0$. Since, $Ann(M)$ is a prime ideal, either $[x : M_R]M = 0$ or $[y : M_R]M = 0$, that is, for each $0 \neq x, y \in M$, either $[x : M_R] = Ann(M)$ or $[y : M_R] = Ann(M)$. If $Ann(M) \subset [x_0 : M_R]$ for some $0 \neq x_0 \in M$, we show that $Rx_0 = \{0, x_0\}$. Let $rx_0 \neq x_0$, where $r \in R$. Since $[x_0 : M_R]M \subseteq Rx_0$ and $r[x_0 : M_R]M \subseteq Rrx_0$, we have $r[x_0 : M_R] \subseteq [rx_0 : M_R] = Ann(M)$. Therefore, $rM = 0$ and so $rx_0 = 0$. Thus, $Rx_0 = \{0, x_0\}$ which is a simple submodule of M and $Soc(M) \neq 0$, a contradiction. Therefore, $[x : M_R] = Ann(M)$ for each $x \in M$, that is, M is a weakly virtually divisible module. □

We conclude this section with the following open problems.

1. Let M be an R-module. Then $ann_t(\Gamma(M_R))$ is a connected graph and $diam(ann_t(\Gamma(M_R))) \leq 3$.

2. Let M be an R-module. Then $\dim(ann_t(\Gamma(M_R)))$ is a finite number if and only if the graph $ann_t(\Gamma(M_R))$ is finite.

3. The number $\dim(ann_t(\Gamma(M_R)))$ is finite if and only if M is finite over R.

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References

[1] S. Akbari and A. Mohammadian, *On zero-divisor graph of finite rings*, J. Algebra **314** (2007), no. 1, 168–184.

[2] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1992.

[3] D. D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra **159** (1993), no. 2, 500–517.

[4] D. F. Anderson, R. Levy, and J. Shapiro, *Zero-divisor graphs, von Neumann regular rings, and Boolean algebras*, J. Pure Appl. Algebra **180** (2003), no. 3, 221–241.

[5] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217**, (1999), no. 2, 434–447.

[6] R. F. Bailey, J. Caceres, D. Garijo, A. Gonzalez, A. Marquezc, K. Meagherd, and M. L. Puertas, *Resolving sets for Johnson and Kneser graphs*, European J. Combin. **34** (2013), no. 4, 736–751.

[7] R. F. Bailey and P. J. Cameron, *Base size, metric dimension and other invariants of groups and graphs*, Bull. Lond. Math. Soc. **43** (2011), no. 2, 209–242.

[8] R. F. Bailey and K. Meagher, *On the metric dimension of Grassmann graphs*, Discrete Math. Theor. Comput. Sci. **13** (2011), no. 4, 97–104.

[9] M. Baziar, E. Momtahan, and S. Safaeeyan, *A zero-divisor graph for modules with respect to their (first) dual*, J. Algebra Appl. **12** (2013), no. 2, 1250151, 11 pp.

[10] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), no. 1, 208–226.

[11] F. DeMeyer and K. Schneider, *Automorphisms and zero-divisor graphs of commutative rings*, Internat. J. Commutative Rings **1** (2002), no. 3, 93–106.

[12] J. Díaz, O. Pottonen, M. Serna, and E. J. van Leeuwen, *On the complexity of metric dimension*, On the complexity of metric dimension. , 419–430, Lecture Notes in Comput. Sci., 7501, Springer, Heidelberg, 2012.

[13] R. Diestel, *Graph Theory*, Springer-Verlag, New York, 1997.

[14] P. Erdos and A. Renyi, *On two problems of information theory*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 229–243.
[15] M. Feng and K. Wang, *On the metric dimension of bilinear forms graphs*, Discrete Math. **312** (2012), no. 6, 1266–1268.
[16] Sh. Ghalandarzadeh and P. Malakooti Rad, *Torsion graph over multiplication modules*, Extracta Math. **24** (2009), no. 3, 281–299.
[17] A. Haghany and M. R. Vedadi, *Endoprime modules*, Acta Math. Hungar. **106** (2005), no. 1-2, 89–99.
[18] F. Harary and R. A. Melter, *On the metric dimension of a graph*, Ars Combin. **2** (1976), 191–195.
[19] S. Hoffmann and E Wanke, *Metric dimension for Gabriel unit disk graphs is NP-complete*, Lecture Notes Comp. Sci. **7718** (2013), 90–92.
[20] S. Khuller, B. Raghavachari, and A. Rosenfeld, *Landmarks in graphs.* Discrete Appl. Math. **70** (1996), no. 3, 217–229.
[21] R. Levy and J. Shapiro, *The zero-divisor graph of von Neumann regular rings*, Comm. Algebra **30** (2002), no. 2, 745–750.
[22] S. B. Mulay, *Cycles and symmetries of zero-divisors*, Comm. Algebra **30** (2002), no. 7, 3533–3558.
[23] S. Pirzada, *An Introduction to Graph Theory*, Universities Press, OrientBlackSwan, Hyderabad, India, 2012.
[24] S. Pirzada and R. Raja, *On the metric dimension of a zero-divisor graph*, Comm. Algebra, to appear.
[25] S. Pirzada, R. Raja, and S. P. Redmond, *Locating sets and numbers of graphs associated to commutative rings*, J. Algebra Appl. **13** (2014), no. 7, 1450047, 18 pages.
[26] R. Raja, S. Pirzada, and S. P. Redmond, *On locating numbers and codes of zero-divisor graphs associated with commutative rings*, J. Algebra Appl. **15** (2016), no. 1, 1650014, 22 pages.
[27] S. P. Redmond, *The zero-divisor graph of a noncommutative ring*, Internat. J. Commutative Rings **1** (2002), no. 4, 203–211.
[28] ———, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra **31** (2003), no. 9, 4425–4443.
[29] S. Safaeeyan, M. Bazair and E. Momtahan, *A generalization of the zero-divisor graph for modules*, J. Korean Math. Soc. **51** (2014), no. 1, 87–98.
[30] D. B. West, *Introduction to Graph Theory*, 2nd ed. USA: Prentice Hall, 2001.
[31] R. Wisbauer, *Foundations of Modules and Ring Theory*, Gordon and Breach Reading, 1991.

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