On Graphs with Linear Ramsey Numbers

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Abstract: For a fixed graph H, the *Ramsey number* $r(H)$ is defined to be the least integer N such that in any 2-coloring of the edges of the complete graph K_N, some monochromatic copy of H is always formed. Let $\mathcal{H}(n, \Delta)$ denote the class of graphs H having n vertices and maximum degree at most Δ. It was shown by Chvatál, Rödl, Szemerédi, and Trotter that for each Δ there exists $c(\Delta)$ such that $r(H) < c(\Delta)n$ for all $H \in \mathcal{H}(n, \Delta)$. That is, the Ramsey numbers grow *linearly* with the size of H. However, their proof relied on the well-known regularity lemma of Szemerédi and only gave an upper bound for $c(\Delta)$ which grew like an exponential tower of 2′*s* of height Δ. This was remedied substantially in a recent paper of Eaton, who showed that one could take $c(\Delta) < 2^{2^{c\Delta}}$ for some fixed c. Eaton, however, also used a variant of the regularity lemma in her proof. In this paper, we avoid the use of the regularity lemma altogether, and show that one can in fact take, for some fixed c, $c(\Delta) < 2^{c\Delta\,(\log \Delta)^2}$ in the general case, and even

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$c(\Delta) < 2^{c\Delta \log \Delta}$ if H is bipartite. In particular, we improve an old upper bound on the Ramsey number of the n-cube due to Beck. We also show that for a fixed $c' > 0$, and for all n and Δ, there are graphs $H' \in \mathcal{H}(n, \Delta)$ with $r(H') > 2^{c'\Delta} n$, which is not so far from our upper bound. In addition, we indicate how the upper bound result can be extended to the larger class of so-called p-arrangeable graphs, introduced by Chen and Schelp. J Graph Theory 35: 176–192, 2000

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1. INTRODUCTION

For any graph H, we will denote by $r(H)$ the least integer N such that in any 2-coloring of the edges of K_N, the complete graph on N vertices, some monochromatic copy of H must always be formed. The existence of $r(H)$ is guaranteed by the classic theorem of Ramsey, and indeed, we will refer to $r(H)$ as the Ramsey number of H. For dense graphs $H, r(H)$ tends to grow exponentially in the size of H. For example, the extreme case of $H = K_n$ has $r(K_n)$ lying roughly between $2^{n/2}$ and 4^n (see [9] for more precise bounds).

However, for relatively sparse graphs, $r(H)$ grows much more modestly. A particular class which has been investigated from this perspective is $\mathcal{H}(n, \Delta)$, the class of graphs H having n vertices and maximum degree at most Δ. It was shown by Chvatál, Rödl, Szemerédi, and Trotter [6] that for each Δ there exists a constant $c(\Delta)$ so that for all $H \in \mathcal{H}(n, \Delta)$, we have

$$r(H) \leq c(\Delta)n.$$

That is, the Ramsey numbers for these H grow *linearly* in the size of H. Unfortunately, their estimate for $c(\Delta)$ was very weak, since the proof used the powerful regularity lemma of Szemerédi [17] (it grew like an exponential tower of 2's of height Δ). In fact, a beautiful recent result of Gowers [8] shows that the best bounds in general that are possible using the original Szemerédi lemma have this form, i.e., there are graphs whose decompositions require such gigantic numbers. Very recently, Eaton [7] improved the upper bound for $c(\Delta)$ to a function of the form $2^{2^{c\Delta}}$ by using a more effective variant of the regularity lemma. For bipartite graphs H a similar, doubly exponential bound follows from a weakening of the regularity lemma due to Komlós (cf. Corollary 7.6 in [13]).

In this note we dispense with the regularity lemma altogether, and obtain a bound of the form $c(\Delta) < 2^{c\Delta(\log\Delta)^2}$ for a suitable constant $c > 0$. We also show (cf. Section 5) that for all n and Δ there are graphs $H' \in \mathcal{H}(n, \Delta)$ such that

$$r(H') > 2^{c'\Delta}n$$

for a fixed constant $c' > 0$. Moreover, we can make H' bipartite. Thus, the upper and lower bounds are becoming reasonably close.

In fact, in the case of a bipartite graph $H \in \mathcal{H}(n, \Delta)$, we can further narrow the gap by dropping one logarithmic factor in the exponent (cf. Section 3). In particular, we improve an old upper bound on the Ramsey number of the n-cube due to Beck [2].

Part of the motivation for this work arose from an attempt to attack the following conjecture of Burr and Erdős [3]: For all δ there exists a constant $c(\delta)$ such that for all graphs H on n vertices in which every subgraph has minimum degree at most δ, we have

$$r(H) \leq c(\delta)n.$$

Burr and Erdős offered \$25 for settling this conjecture, but they also wrote "However, it seems to be quite difficult, and probably further work must continue to be in the direction of partial results." While this conjecture still remains unresolved, Chen and Schelp [5] have introduced the class of so-called "p-arrangeable" graphs, and showed that they also have linearly growing Ramsey numbers. In Section 4, we indicate how our methods apply to this larger class as well.

Throughout this paper we will be using the notation $N_S(v)$ for the set of neighbors of a vertex v which belong to a set S, and $N_S(T) = \bigcup_{v \in T} N_S(v)$, where T is another set of vertices.

2. UPPER BOUND

We first settle on some notation. If G is a graph with vertex set V, and $U \subset V$, then $G[U]$ will denote the induced subgraph of G in U, and $e(U)$ will denote its number of edges. The *edge density* $d(U)$ of U is defined by

$$d(U) = \frac{e(U)}{\binom{|U|}{2}}$$

The maximum degree of G is denoted by $\Delta(G)$. If X and Y are two disjoint subsets of V then $G[X, Y]$ denotes the induced (bipartite) subgraph of G on $X \cup Y, e(X, Y)$ stands for its number of edges and the density of the pair (X, Y) is defined by

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}$$

We will say that G is (ρ, d)-dense if for all $U \subset V$ with $|U| \geq \rho|V|$, we have $d(U) \geq d$. Similarly, we will say that G is bi-(ρ, d)-dense if for all $X \subset V, Y \subset V$ with $X \cap Y = \emptyset, |X| \geq \rho|V|, |Y| \geq \rho|V|$, we have $d(X, Y) \geq d$. It follows by a simple averaging argument that if G is not (ρ, d)-dense, then there is a set U of order $|U| = \lfloor \rho|V| \rfloor$ with $d(U) < d$. Similarly, if G is not bi-(ρ, d)-dense, then there are disjoint sets $X \subset V, Y \subset V$ of order $|X| = |Y| = \lfloor \rho|V| \rfloor$, with $d(X, Y) < d$.

We assume that H has no isolated vertices. Thus, for $\Delta = 1, H$ is matching, and it is an easy exercise to show that $r(H) = 3n/2 - 1$ (it also follows as a special case of Theorem 9 in [4]).

Theorem 1. *For some positive constant c, and for all integers $\Delta \geq 2$, and all $n \geq \Delta + 1$, if $H \in \mathcal{H}(n, \Delta)$ then*

$$r(H) < 2^{c\Delta(\log\Delta)^2} n.$$

Before going into details, a rough sketch of the proof is as follows. For a large N, let $E(K_N) = G_R \cup G_B$ be any 2-coloring of the edges of K_N. If the graph G_R on the set of *Red* edges is *not* (ρ, d)-dense for appropriate ρ and d, then G_R must have a large subset U with maximum degree at most $|U|/(2\Delta)$ which, in turn, will imply (by an easy graph packing result—see Lemma [3] below) that H and G_R can be packed edge disjointly in K_N, i.e., K_N has a *Blue* copy of H.

On the other hand, if G_R *is* (ρ, d)-dense then we will show that it contains a large subgraph B_R which is bi-(ρ', d')-dense (again, for suitable ρ' and d'). From this, it will follow that B_R must contain a copy of H, which of course, gives us a *Red* copy of H.

To carry out the proof of the second part, we will need two lemmas. Lemma 1, in a sense, replaces the Szemerédi regularity lemma. A similar technique was used in [12].

Lemma 1. *For all numbers s, β, ρ, d such that $0 < \beta, \rho, d < 1, s \geq \log_2(4/d)$ and $(1 - \beta)^{2s} \geq 2/3$, the following holds. If G is a $((2\rho)^s \beta^{s-1}, d)$-dense graph on N vertices, then there exists $U \subset V$ with $|U| \geq \rho^{s-1}\beta^{s-2}N$ such that $G[U]$ is bi-$(\rho, d/2)$-dense.*

Proof. What we will actually show is that if for all U with $|U| \geq \rho^{s-1}\beta^{s-2}N, G[U]$ is *not* bi-$(\rho, d/2)$-dense, then G is *not* $((2\rho)^s \beta^{s-1}, d)$-dense. Specifically, assume that

(*) For all $U \subset V$ with $|U| \geq \rho^{s-1}\beta^{s-2}N$, there are disjoint $X, Y \subset U$ with $|X| \geq \rho|U|, |Y| \geq \rho|U|$ such that $d(X, Y) < d' = d/2$.

Our ultimate goal is to prove that condition $(*)$ implies the existence of a set $Z \subset V$ with $|Z| \geq (2\rho)^s \beta^{s-1}N$ and $d(Z) < d$.

We will prove by induction that for all $1 \leq t \geq s$ there are disjoint sets $W_i, 1 \leq i \leq 2^t$, with $|W_i| = x_t$ satisfying

$$\sum_{i<j} e(W_i, W_j) < \frac{d' x_t^2 \binom{2^t}{2}}{(1 - \beta)^{2t-2}}, \tag{1}$$

where $x_t = \rho^t \beta^{t-1} N$. Then we will take $Z = \bigcup_{i=1}^{2^t} W_i$.

First, let us take $U = V$ itself. Thus, $|U| = N \geq \rho^{s-1}\beta^{s-2}N$, so by $(*)$ there are $W_1, W_2 \subset U, |W_1| \geq \rho N, |W_2| \geq \rho N$ with $d(W_1, W_2) < d'$. By averaging, we can assume $|W_1| = |W_2| = x_1 = \rho N$. Thus, we have $e(W_1, W_2) < d' x_1^2$, which is inequality (1) for $t = 1$.

Next, consider W_1. Since $|W_1| = x_1 = \rho N \geq \rho^{s-1}\beta^{s-2}N$, we can apply $(*)$. This means we can find $A_1, B_1 \subset W_1$ with $|A_1| \geq \rho x_1, |B_1| \geq \rho x_1$ so that $d(A_1, B_1) < d'$. Again, by averaging we can get much smaller sets $A_1' \subset A_1, B_1' \subset B_1$, with $|A_1'| = |B_1'| = \rho\beta x_1 = x_2$ such that $d(A_1', B_1') < d'$, i.e., $e(A_1', B_1') < d'(\rho\beta x_1)^2$. Now, remove A_1' and B_1' from W_1 to form $W_1^{(1)}$. Then $|W_1^{(1)}| = x_1 - 2\rho\beta x_1 = x_1(1 - 2\rho\beta) = \rho(1 - 2\rho\beta)N \geq \rho\beta N \geq (\rho\beta)^{s-1}N$, where the first inequality follows from our assumption, which, in particular implies that $\beta \leq 1/3$. Thus, we can apply $(*)$ again to get $A_2', B_2' \subset W_1^{(1)}$ with $|A_2'| = |B_2'| = \rho\beta x_1 = x_2$ and $d(A_2', B_2') < d'$, i.e., $e(A_2', B_2') < d' x_2^2$. Now form $W_1^{(2)}$ by removing A_2' and B_2' from $W_1^{(1)}$, and continue. We will continue as long as what is left, say $W_1^{(l-1)}$, has size $|W_1^{(l-1)}| \geq \beta x_1$. Each time we remove a set of size $2x_2$ from $W_1^{(j-1)}$ to form $W_1^{(j)}$. Thus, when we finally get stuck, we will have $l \geq \frac{x_1(1-\beta)}{2x_2}$ and the final remainder $W_1^{(l)}$ will have size at most $\beta x_1 = \rho\beta N$.

We also carry out the same process for W_2, ending with $C_1', D_1', \ldots, C_l', D_l'$. Here, for all $i = 1, \ldots, l$,

$$|A_i'| = |B_i'| = |C_i'| = |D_i'| = x_2, \qquad e(A_i', B_i') < d' x_2^2, \quad \text{and} \quad e(C_i', D_i') < d' x_2^2$$

We would like to find a pair $X_i = A_i' \cup B_i', Y_j = C_j' \cup D_j'$ so that $e(X_i, Y_j)$ is small. First, throw out $W_1^{(l)}$ and $W_2^{(l)}$, leaving $\overline{W_1} = W_1 \backslash W_1^{(l)}, \overline{W_2} = W_2 \backslash W_2^{(l)}$, so that $|\overline{W_1}| = 2lx_2 = |\overline{W_2}|$.

Consider the sum $\sum_{i,j} e(X_i, Y_j) \leq e(W_1, W_2) < d' x_1^2$. Since the sum has l^2 terms then for some choice of i and j we have

$$e(X_i, Y_j) < \frac{d' x_1^2}{l^2} \leq \frac{4d' x_2^2}{(1-\beta)^2}.$$

Thus, setting $T_1 = A_i', T_2 = B_i', T_3 = C_j', T_4 = D_j'$ we obtain

$$\sum_{1 \leq i < j \leq 4} e(T_i, T_j) < 2d' x_2^2 + \frac{4d' x_2^2}{(1-\beta)^2} < \binom{4}{2} \frac{d' x_2^2}{(1-\beta)^2}.$$

Now we consider the general case. Suppose that for $k < s$ we have defined $W_1, W_2, \ldots, W_{2^k}$ with $|W_i| = x_k = \rho^k \beta^{k-1} N$ so that

$$\sum_{i<j} e(W_i, W_j) < \frac{d' x_k^2 \binom{2^k}{2}}{(1-\beta)^{2k-2}}. \tag{2}$$

We will focus on W_i, and apply $(*)$ repeatedly. As before, this will give us pairs $A_j'(i)$, $B_j'(i)$, $1 \leq j \leq l$, with $|A_j'(i)| = |B_j'(i)| = x_{k+1} = \rho\beta x_k$, and with the remainder set $W_i^{(l)}$ having $|W_i^{(l)}| < \beta x_k$. We also have $e(A'_j(i), B'_j(i)) < d'x_{k+1}^2$. We do this for all the W_i, $1 \le i \le 2^k$. The previous argument now shows that $l \ge \frac{x_k(1-\beta)}{2x_{k+1}}$.

Consider the sum

$$S = \sum_{i_1,\ldots,i_{2^k}} e(Z_{i_1}(1), Z_{i_2}(2), \ldots, Z_{i_{2^k}}(2^k))$$

where $Z_{i_j}(j) = A'_{i_j}(j) \cup B'_{i_j}(j), 1 \le i_j \le l, 1 \le j \le 2^k$, and $e(Z(1), Z(2), \ldots, Z(2^k))$ denotes the total number of edges spanned by the 2^k-partite graph with vertex sets $Z(1), \ldots, Z(2^k)$. Now, each edge in $E(Z_{i_r}(r), Z_{i_s}(s))$ is counted $l^{2^k} - 2$ times in the sum. Thus, by (2), we infer that

$$S < \frac{d'x_k^2 \binom{2^k}{2} l^{2^k-2}}{(1-\beta)^{2k-2}}.$$

Since the sum S has l^{2^k} terms then for some choice of $i_1, i_2, \ldots, i_{2^k}$,

$$e(Z_{i_1}(1), Z_{i_2}(2), \ldots, Z_{i_{2^k}}(2^k)) < \frac{d'x_k^2 \binom{2^k}{2} l^{2^k-2}}{(1-\beta)^{2k-2} l^{2^k}} = \frac{d'x_k^2 \binom{2^k}{2}}{l^2(1-\beta)^{2k-2}}.$$

But for each of the 2^k pairs $A'_{i_1}(j) \cup B'_{i_1}(j) = Z_{i_1}(j)$, we have $e(A'_{i_1}(j), B'_{i_2}(j)) < d'x_{k+1}^2$. Thus, the total number T of edges *between* the 2^{k+1} sets

$$A'_{i_1}(j_1), B'_{i_1}(j_1), A'_{i_2}(j_2), B'_{i_2}(j_2), \ldots, A'_{i_{2^k}}(j_{2^k}), B'_{i_{2^k}}(j_{2^k})$$

is bounded above by

$$T \le \frac{d'x_k^2 \binom{2^k}{2}}{l^2(1-\beta)^{2k-2}} + 2^k d' x_{k+1}^2.$$

Since $l \ge \frac{x_k(1-\beta)}{2x_{k+1}}$ then we find that

$$\begin{aligned} T &\le d' \left(\frac{x_k^2 \cdot 4x_{k+1}^2 \binom{2^k}{2}}{x_k^2 (1-\beta)^{2k}} + 2^k x_{k+1}^2 \right) \\ &\le d'x_{k+1}^2 \frac{1}{(1-\beta)^{2k}} \left(4\binom{2^k}{2} + 2^k \right) \\ &= d'x_{k+1}^2 \frac{1}{(1-\beta)^{2k}} \binom{2^{k+1}}{2} \end{aligned}$$

verifying (1) for $t = k+1$.

We continue now until we reach $t = s$. Thus, we have $W_1, W_2, \ldots, W_{2^s}$ with at most $\frac{d}{2}\binom{2^s}{2}\frac{x_s^2}{(1-\beta)^{2s-2}}$ "crossing" edges (between any two $W's$). On the other hand, *within* the W_i there are at most $2^s\binom{x_s}{2}$ edges altogether so that the total number of edges spanned by $Z = \bigcup_i W_i$ is at most

$$\frac{d}{2}\binom{2^s}{2}\frac{x_s^2}{(1-\beta)^{2s-2}} + 2^s\binom{x_s}{2} < d\binom{2^s x_s}{2},$$

since $s \geq \log_2(4/d)$ and $(1-\beta)^{2s} \geq 2/3$. However, $|Z| = 2^s x_s = 2^s \rho^s \beta^{s-1} N$ and so the claim is proved. This proves Lemma 1. ■

We say that a graph H can be embedded into graph G if there is an injection $f : V(H) \to V(G)$, called an embedding, such that for every edge $xy \in E(H)$, we have $f(x)f(y) \in E(G)$. Lemma 2 is a standard embedding result.

Lemma 2. *For all integers $\Delta \geq 1$ and $n \geq \Delta + 1$, and for all positive numbers a, ε, and γ such that*

$$(\gamma^{\Delta - r} - r\varepsilon)a \geq 1 \qquad \text{for} \quad r = 0, \ldots, \Delta \tag{3}$$

the following holds. If graphs H and G satisfy

(i) $H \in \mathcal{H}(n, \Delta)$,
(ii) $|V(G)| = a(\Delta + 1)n$, *and*
(iii) *G is bi-$(\varepsilon/(\Delta + 1), \gamma)$-dense,*

then H can be embedded into G.

When applying this lemma in the proof of Theorem 1, we will choose $a = (16\Delta)^\Delta, \varepsilon = 1/(16\Delta)^\Delta$, and $\gamma = 1/(16\Delta)$. Then condition (3) will be satisfied with room to spare.

Proof. The proof is adopted from Chvatál et al. [6]. Partition $V(G)$ into $V(G) = A_1 \cup \cdots \cup A_{\Delta+1}$, where $|A_i| = an$. Note that H can be partitioned into $\Delta + 1$ independent sets. Let $V(H) = X_1 \cup \cdots \cup X_{\Delta+1}$ be one such partition. We shall construct an embedding f such that $f(X_k) \subseteq A_k$ for every $k = 1, 2, \ldots, \Delta + 1$. Write $V(H) = \{x_1, x_2, \ldots, x_n\}$ and $L_i = \{x_1, x_2, \ldots, x_i\}$, $i = 1, 2, \ldots, n, L_0 = \emptyset$. We will apply induction on $i = 1, \ldots, n$.

Assume we have already embedded the set L_i, so that for each $k = 1, 2, \ldots, \Delta + 1, f(L_i \cap X_k) \subseteq A_k$. For each $y \in X_k \backslash L_i$, let C_y^i be the set of the vertices of A_k adjacent to the images of all already embedded neighbors of y, i.e., the vertices of G adjacent to every vertex in the set $f(N_{L_i}(y))$. (In particular, we have $C_y^0 = A_k$.) In other words, C_y^i is the set of possible candidates for the image of y, if y were to be embedded in the $(i+1)$st step. This set may shrink each time one of at most Δ neighbors of y is being embedded. Our goal is to show that its unused portion, i.e., $C_y^i \backslash f(L_i)$, survives as nonempty by the time y is to be embedded. But to achieve this goal for all vertices y of H, we will have to keep the sets C_y^i sufficiently large. Namely, we will maintain the following condition at all times. For every $i = 0, 1, \ldots, n$ and every $y \in X_k \backslash L_i$, $|C_y^i| \geq \gamma^{v(i,y)} an$, where $v(i, y) = |N_{L_i}(y)|$.

For $i = 0$, there is nothing to prove. Next, suppose that for some $i \geq 1$, the set L_i of the first i points $x_1, \ldots, x_i$ have been successfully embedded. We must now embed $x = x_{i+1}$ into an appropriate $v \in C_x^i$. Let $Y = \{y_1, \ldots, y_r\}$ be the set of not yet embedded neighbors of x, If $Y = \emptyset$, then we may take any $v \in C_x^i \backslash f(L_i)$ as the image of x. Thus, assume that $1 \leq r \leq \Delta$. We claim that there is a vertex $v \in C_{x_i}^i \backslash f(L_i)$ such that for all $y \in Y$, we have $|N_{C_y^i}(v)| \geq \gamma |C_y^i|$. This will complete the proof, since, setting $f(x) = v$, we have $C_y^{i+1} = C_y^i$ if y is not adjacent to x and $C_y^{i+1} = N_{C_y^i}(v)$ otherwise.

To verify the claim, note that the set C_x^i is disjoint from all sets $C_y^i, y \in Y$. By the induction assumption,

$$|C_x^i| \geq \gamma^{v(i,x)} an \geq \gamma^{\Delta - r} an$$

and, for each $y \in Y$,

$$|C_y^i| \geq \gamma^{\Delta - 1} an > \varepsilon an,$$

where the last inequality follows from condition (3). Let W_y be the set of all vertices in C_x^i with fewer than $\gamma |C_y^i|$ neighbors in C_y^i. Then we have $|W_y| < \varepsilon an$, since otherwise there would be a contradiction with the fact that G is bi-$(\varepsilon/(\Delta+1), \gamma)$-dense. Thus,

$$\left| C_x^i \backslash \bigcup_{y \in Y} W_y \right| \geq \gamma^{\Delta - r} an - r\varepsilon an > n$$

by condition (3). Hence, the set $(C_x^i \backslash \bigcup_{y \in Y} W_y) \backslash f(L_i)$ is nonempty, yielding the existence of the required vertex v. ∎

Before we turn to the proof of Theorem 1 we still need one simple lemma. Given two graphs G and H, with $|V(G)| \geq |V(H)|$, we say that there is *a packing* of G and H if there is an embedding of H into the complement $\overline{G}$ of G.

Lemma 3. *Let H be an n-vertex graph with $\max_{H' \subseteq H} \delta(H') \leq \delta$ and let G be a graph with $|V(G)| \geq 2n$ and $\Delta(G) \leq |V(G)|/(2\delta)$. Then there exists a packing of H and G.*

Proof. Let us order the vertices of H as $x_1, x_2, \ldots, x_n$ so that for each i, vertex x_i has at most δ neighbors in the set $L_i = \{x_1, \ldots, x_i\}$. Assume we have already packed $H[L_i]$. Since the images of the neighbors of x_{i+1} in L_i have together at most $\delta\Delta(G) \leq |V(G)|/2$ neighbors in G, there is at least one vertex in G not adjacent to any of them. This vertex can be taken as the image of x_{i+1}, yielding a packing of $H[L_{i+1}]$ and G. There is nothing that can stop us from repeating this procedure until the entire graph H is packed with G. ■

Comment. This lemma has a bipartite version as well: if H is an n-vertex bipartite graph with $\max_{H'\subseteq H}\delta(H') \leq \delta$ and $V(G) = X \cup Y$, where $|X| = |Y| \geq 2n$ and $\Delta(G[X, Y]) \leq |X|/(2\delta)$ then there exists a packing of H and G.

Proof of Theorem 1. Set $V = [N]$ and suppose we are given an arbitrary 2-coloring of the edges of K_N, forming the two graphs G_R and G_B on the vertex set V of K_N, consisting of the *Red* and *Blue* edges, respectively.

Case 1. Suppose G_R is *not* $(\rho_1, 1/(8\Delta))$-dense, where ρ_1 will soon be specified. Thus, there exists $U \subset V$ with $|U| = \rho_1 N$ and $d(U) < 1/(8\Delta)$. At most $\frac{1}{2}\rho_1 N$ vertices of U have degree larger than or equal to $2 \cdot (1/(8\Delta))\rho_1 N$. Removing these (and more if necessary), we can find $U' \subset U$ so that $|U'| = \frac{1}{2}\rho_1 N$ and $\Delta(G_R[U']) < 1/(4\Delta)\rho_1 N = |U'|/(2\Delta)$. Thus, by Lemma 3, provided $|U'| \geq 2n$, we can find an edge disjoint packing of $G_R[U']$ and H in (the complete graph on) U'. In such a packing, all the edges of H must be *Blue*, i.e., $H \subset G_B$, and this case is completed. (If we have used a tight packing lemma of Sauer and Spencer [16], we would only need $|U'| \geq n$. This saving of a factor of 2, however, would not have any significant impact on our result.)

Case 2. Suppose G_R is $(\rho_1, 1/(8\Delta))$-dense. It is now time to specify all the constants. Let

$$\gamma = 1/(16\Delta), \quad \varepsilon = 1/(16\Delta)^{\Delta}, \quad a = (16\Delta)^{\Delta},$$
$$s = \lceil \log_2(32\Delta) \rceil, \quad \beta = 1/(4s),$$
$$\rho = \frac{1}{(\Delta + 1)(16\Delta)^{\Delta}}, \quad \rho_1 = (2\rho)^s \beta^{s-1},$$

and

$$N = \left\lceil \frac{a(\Delta + 1)n}{\rho^{s-1}\beta^{s-2}} \right\rceil. \tag{4}$$

Then, by Lemma 1, there is $W \subset V$ with $|W| \geq \rho^{s-1}\beta^{s-2}N$ such that $G_R[W]$ is bi-$(\rho, 1/(16\Delta))$-dense. Since $\rho^{s-1}\beta^{s-2}N = a(\Delta + 1)n$ and $\rho = \varepsilon/(\Delta + 1)$, we can apply Lemma 2 and conclude that H is a subgraph of $G_R[W]$, i.e., there is a *Red* copy of H.

The required constraints on the variables are satisfied, i.e., $s \geq \log_2(32\Delta)$ and $(1 - \beta)^{2s} \geq 2/3$, the latter following from the inequality $(1 - 1/(4x))^x \geq 2/3$, which is valid for $x \geq 1$. Note that with this choice of N and ρ_1, we also have the condition $|U'| \geq 2n$ from Case 1 fulfilled.

Plugging everything into (4), we see that N satisfies the inequality

$$N \leq (\Delta+1)(16\Delta)^{\Delta}(4\lceil \log_2(32\Delta)\rceil(\Delta+1)(16\Delta)^{\Delta})^{\lceil \log_2(32\Delta)\rceil} n.$$

Consequently, there is a positive constant c such that $r(H) \leq 2^{c\Delta(\log\Delta)^2} n$ whenever $H \in \mathcal{H}(n, \Delta), \Delta \geq 2$. This proves Theorem 1. ■

Comment. Our method of proof cannot give a better upper bound on $r(H)$ than the one obtained. Indeed, the packing lemma (Lemma 3) forces γ to be of the order $1/\Delta$. Consequently, condition (3) of Lemma 2 requires that ε be of the order $\Delta^{-\Delta}$. This is also true of ρ, which, by Lemma 1, appears in the denominator of N raised to the power s. The constant s, however, must be of the order $\log\Delta$, and is solely responsible for the second logarithmic term in the exponent of our bound.

3. BIPARTITE GRAPHS

It turns out that in the case when H is bipartite, we may avoid Lemma 1 altogether. This way the quantity s disappears, and we obtain a better bound than that of Theorem 1. This improvement relies on using the bipartite version of Lemma 3 given above. For more details see [18].

Theorem 2. *For some positive constant c, and for all integers $\Delta \geq 2$ and all $n \geq \Delta + 1$, if H is a bipartite graph with n vertices and maximum degree at most Δ, then*

$$r(H) < 2^{c\Delta\log\Delta} n.$$

Proof. Let $E(K_N) = G_R \cup G_B$ be an arbitrary 2-coloring of the edges of K_N where, this time, $N = a(\Delta+1)n$ and $a = 4(8\Delta)^{\Delta}$. If G_R is bi-$(\rho, 1/(8\Delta))$-dense, where $\rho = (8\Delta)^{-\Delta}/(\Delta+1)$, then, by Lemma 2, we can embed H into G_R, finding a *Red* copy of H. Otherwise, there is a pair of disjoint sets X and Y such that $|X| = \rho N, |Y| = \rho N$ and $d_{G_R}(X, Y) < 1/(8\Delta)$. As before we can peel these sets off and find subsets $X' \subset X$ and $Y' \subset Y$ such that $|X'| = |Y'| = \frac{1}{2}\rho N = 2n$ and $\Delta(G_R[X', Y']) < 1/(4\Delta)\rho N = |X'|/(2\Delta)$. By the bipartite version of Lemma 3, we can find an edge disjoint packing of $G_R[X', Y']$ and H in the complete bipartite graph induced by X' and Y'. This packing yields a *Blue* copy of H. ■

Let Q_n be the n-cube. Then we have $|V(Q_n)| = 2^n$ and $\Delta = n$.

Corollary 1. *There is a constant $c > 0$ such that for all n, $r(Q_n) < 2^{cn\log n}$.*

This improves an old result of Beck [2], who proved that $r(Q_n) < 2^{cn^2}$.

4. *p*-ARRANGEABLE GRAPHS

Given an ordering of the vertices of a graph, $x_1, \ldots, x_n$, let us set $L_i = \{x_1, \ldots, x_i\}$ and $R_i = \{x_{i+1}, \ldots, x_n\}$. A graph of order n is called *p-arrangeable* if its vertices can be ordered as $x_1, \ldots, x_n$ in such a way that for each $i = 1, \ldots, n-1$,

$$|N_{L_i}(N_{R_i}(x_i))| \leq p.$$

In other words, a graph is p-arrangeable if, in some ordering of the vertices, for any vertex x, its neighbors to the right have together at most p neighbors to the left of x (including x). Clearly, such graphs are $(p+1)$-colorable, so they can be partitioned into $p+1$ independent sets $X_1, \ldots, X_{p+1}$. For a vertex $x = x_i \in X_{p+1}$, denote by $Y_k, k = 1, \ldots, p$, the set of its neighbors to the right which belong to X_k, i.e., $Y_k = N_{R_i}(x) \cap X_k$. Then, because $|N_{L_i}(Y_k) \backslash \{x\}| \leq p-1$, there exists a subset $Z_k \subseteq Y_k$ of size $|Z_k| \leq 2^{p-1}$ such that for every vertex $y \in Y_k$, there is a vertex $z \in Z_k$ with $N_{L_i}(y) = N_{L_i}(z)$. Consequently, referring to the notation from the proof of Lemma 2, $C_y^i = C_z^i$. This crucial observation, made by Chen and Schelp, allows us basically to repeat the proof of Lemma 2 for a p-arrangeable graph H. This time, however, setting $Z = \bigcup_{k=1}^{p} Z_k$, we have

$$\left| C_x^i \backslash \bigcup_{z \in Z} W_y \right| \geq \gamma^p an - p2^{p-1} \varepsilon an$$

and so our condition (3) in Lemma 2 should be changed to

$$(\gamma^p - p2^{p-1}\varepsilon)a \geq 1.$$

Lemma 3, with $\delta = p$, applies to p-arrangeable graphs H. Of course, Lemma 1 is not affected by the structure of H. Altogether, setting

$$\gamma = 1/(16p), \quad \varepsilon = \gamma^p/(p2^p), \quad a = 2\gamma^{-p}, \quad \rho = \varepsilon/(p+1),$$

while s and β remain as in the proof of Theorem 1, we obtain the following result:

Theorem 3. *For some positive constant c and all integers $p \geq 2$ and all $n \geq p+1$, if H is a p-arrangeable graph with n vertices then*

$$r(H) < 2^{cp(\log p)^2} n.$$

This is a two-level improvement over the triple exponential bound in [7] (cf. Theorem 2.3.1 and Proposition 2.1. there).

Planar graphs are p-arrangeable with $8 \leq p \leq 10$, cf. [11]. Also, since every graph with no K_p-subdivision is p^8-arrangeable [15], then every graph of bounded genus is p-arrangeable with p being a function of the genus. Several classes of p-arrangeable graphs were found in [5]. In particular every graph with at most $p(p-1)+1$ vertices of degree exceeding p, or where every pair of such vertices is at distance at least 3, is $(p(p-1)+1)$-arrangeable. However, already in [3], the graphs in the last class were shown to have Ramsey numbers at most $18n$.

For any given p, there are graphs with density as low as at most 4 that are not p-arrangeable. The example Chen and Schelp gave was obtained from an arbitrary graph of minimum degree at least $2p$ by subdividing each edge with exactly one point. However, for these graphs Alon [1] proved that $r(H) \leq 12n$, where n is the number of vertices in the resulting graph (equal to the sum of the number of vertices and edges in the original graph). Again, the special case where the initial graph is complete was already proved in [3]. (In fact, both results in [1] and [3] had treated a broader family of graphs, where every edge is subdivided with *at least* one vertex.) Very recently, it has been shown [14] that certain classes of bipartite graphs which naturally generalize the subdivision of the complete graph with exactly one vertex on each edge also have linearly growing Ramsey numbers.

5. LOWER BOUND

The main goal of this section is to prove the following lower bound result.

Theorem 4. *There exists a constant $c > 1$ such that for all $\Delta \geq 1$ and all $n \geq \Delta + 1$ (except for $\Delta = 1$ and $n = 2, 3, 5$), there exists a graph $H \in \mathcal{H}(n, \Delta)$ which satisfies $r(H) > c^{\Delta} n$.*

In the three exceptional cases, for all graphs $H \in \mathcal{H}(n, 1)$ we have $r(H) = n$, and clearly, the conclusion of Theorem 4 could not be true.

In this section, $c, c_0, c_1, \ldots$ will denote suitable absolute constants which we usually will not specify. Also, various expressions which do not look like integers should (usually) be rounded to the nearest corresponding integer.

The proof rests on two lemmas, both proved by the probabilistic method with a random graph as a probability space. The random graph $G(n, M)$ is drawn uniformly from all graphs on n labeled vertices and with M edges. The random graph $G(n, 1/2)$ is a result of $\binom{n}{2}$ independent tosses of a fair coin, so its number of edges is a random variable with the binomial distribution $Bi(n, 1/2)$. In this section partitions are allowed to have empty classes. Recall, finally, that $e(X, Y) = e_G(X, Y)$ is the number of edges of G with one end in X and the other in Y.

Lemma 4. *There are fixed constants $c_0 > c_1 > 1$ and Δ_0 such that for each $\Delta \geq \Delta_0$ and $n \geq k^2$, where $k = c_0^{\Delta}$, there exists a graph $H \in \mathcal{H}(n, \Delta)$ with the following property. For all partitions $V(H) = V_1 \cup \cdots \cup V_k$ with $|V_i| \leq N/k, i = 1, \ldots, k$, and $N = c_1^{\Delta} n$, we have*

$$\sum_{i<j: e_H(V_i, V_j) > 0} |V_i||V_j| > 0.55 \binom{n}{2}. \tag{5}$$

Proof. Take $1 < c_0 < (10/7)^{1/202}$ and any c_1 such that $1 < c_1^2 < c_0$. Then choose Δ_0 so that $(c_1^2/c_0)^{\Delta_0} < 0.1$ and $((0.7)^{1/202}c_0)^{\Delta_0} < 0.25$.

Let $\Delta \geq \Delta_0$ and $d = \Delta/202$. Consider the random graph $G(m, dm)$ (with m vertices and dm edges), where $m = 1.01n$. Clearly, the number of vertices of degree larger than Δ in any graph with m vertices and dm edges is at most

$$\frac{2dm}{\Delta+1} < \frac{m}{101}.$$

Thus, whatever we prove about $G(m, dm)$, we will form the graph H by deleting from $G(m, dm)$ the $n/100$ largest degree vertices so that $|V(H)| = n$ and $\Delta(H) \leq \Delta$.

Keeping this in mind, we claim that, with positive probability, $G(m, dm)$ satisfies the following property: for every partition $[m] = V_1 \cup \cdots \cup V_k \cup D$, $k = c_0^{\Delta}$, with $|V_i| \leq c_1^{\Delta} n/k$ for all i, and with $|D| = n/100$, the inequality (5) holds for $G = G(m, dm)$, that is

$$\sum_{i<j: e_G(V_i, V_j)>0} |V_i||V_j| > 0.55\binom{n}{2}. \tag{6}$$

Indeed, if a partition $[m] = V_1 \cup \cdots \cup V_k \cup D$ violates (6), then because the total number of pairs within the sets V_i is at most

$$\sum_i \binom{|V_i|}{2} \leq (c_1^2/c_0)^{\Delta_0}\binom{n}{2} < 0.1\binom{n}{2},$$

the partition must satisfy the inequality

$$\sum_{i<j: e_G(V_i, V_j)=0} |V_i||V_j| \geq 0.35\binom{n}{2} \geq 0.3\binom{m}{2}. \tag{7}$$

However, the expected number of partitions $(V_1, \ldots, V_k, D)$ of the vertex set of the random graph $G(m, dm)$ satisfying (7) is smaller than

$$(k+1)^m 2^{\binom{k}{2}} \frac{\binom{0.7\binom{m}{2}}{dm}}{\binom{\binom{m}{2}}{dm}} < 2^{\binom{k}{2}}(2k(0.7)^d)^m < 4^m\left((0.7)^{1/202}c_0\right)^{\Delta_0 m} < 1. \tag{8}$$

Above, the term $(k+1)^m$ bounds the number of partitions, $2^{\binom{k}{2}}$ bounds the number of choices of the pairs (V_i, V_j) to have $e_G(V_i, V_j) = 0$, while the fraction is an upper bound on the probability of no edge of $G(m, dm)$ falling between these pairs.

Hence, there exists a graph $G \in G(m, dm)$ with every partition satisfying (6). Setting D for the set of the $n/100$ largest degree vertices in G, the graph $H = G - D$ fulfils the hypothesis of Lemma 4. ■

The next lemma follows from a weighted version of the well-known fact that with high probability every sufficiently large subset of vertices of a random graph $G(k, 1/2)$ spans about the expected number of edges.

Lemma 5. *For every $k \geq 4$ there exists a graph R on the vertex set $[k] = \{1, 2, \ldots, k\}$ such that for all functions $w : [k] \to [0, 1]$ with $\sum_{i=1}^{k} w(i) = x > (10^7 + 2)\log k$, we have*

$$W = \sum_{ij \in R} w(i)w(j) < 0.51\binom{x}{2} \quad \textit{and} \quad \overline{W} = \sum_{ij \notin R} w(i)w(j) < 0.51\binom{x}{2}.$$

Proof. First observe that for any graph R and any fixed x the quantity W is maximized by an assignment such that the set $K = \{i : 0 < w(i) < 1\}$ is a *clique* in R or $K = \emptyset$. For suppose there exists $ij \notin R$ with $0 < w(i), w(j) < 1$. Without loss of generality we may assume that the sum of weights assigned to the neighbors of i is not smaller than the sum of weights assigned to the neighbors of j. Then by changing $w'(i) = w(i) + \varepsilon$ and $w'(j) = w(j) - \varepsilon$, where $\varepsilon = \min\{1 - w(i), w(j)\}$, we can maintain $W' \geq W$, and end up with at least one fewer vertex in K. Continuing this argument shows that we can assume K is a clique or empty. Similarly, if the quantity $\overline{W}$ is maximal then K is an independent set in R or empty.

Now we need two basic facts from the theory of random graphs:

(i) The probability of the existence of a clique of order $s = 2\log_2 k + 1 \geq 5$ in $G(k, 1/2)$ is smaller than

$$\binom{k}{s} 2^{-\binom{s}{2}} < \left(\frac{e}{s}\right)^s < \frac{1}{4}.$$

Since the same is true for independent sets of order s, with probability strictly greater than $1/2$, the largest cliques and largest independent sets in $G(k, 1/2)$ have size at most $2\log_2 k$;

(ii) By Chernoff' inequality (cf. [10]), the probability that there is a set $S \subseteq [k]$ with $s = |S| \geq 10^7 \log_2 k$, for which

$$\left|G(k, 1/2) \cap [S]^2\right| > 0.501\binom{s}{2} \ \textit{or}\ \left|G(k, 1/2) \cap [S]^2\right| < 0.499\binom{s}{2}$$

is smaller than

$$2\sum_{s=10^7\log_2 k}^{k} \binom{k}{s} \exp\left\{-10^{-6}\binom{s}{2}\Big/3\right\} < 2\sum_{s}\left(\frac{ek}{s} e^{-10^{-7}s}\right)^s = 2\sum_{s}(e/s)^s < \frac{1}{2}.$$

Thus, there exists a graph R on $[k]$ such that

(a) the largest clique has size at most $2\log_2 k$,
(b) for every set $S \subseteq [k]$ with $s = |S| \geq 10^7 \log_2 k$, we have $|R \cap [S]^2| < 0.501\binom{s}{2}$,
(c) properties (a) and (b) hold for $\overline{R}$, the complement of R.

Let $T = \{i : w(i) = 1\}$. Then, by (a), $x \geq t = |T| \geq x - 2\log_2 k$. Thus, $t > 10^7 \log_2 k$, and by (b),

$$W \leq \sum_{ij \in R \cap [T]^2} 1 + (2 \log_2 k)x < 0.501\binom{t}{2} + (2 \log_2 k)x < 0.51\binom{x}{2}.$$

A similar argument establishes the second part of the claim in the lemma (for $\overline{W}$). This completes the proof of Lemma 5. ■

Now, to complete the proof of Theorem 4, we proceed as follows. Let c_0, c_1, and Δ_0 be as in the proof of Lemma 4 but with the additional requirement that $(c_0/c_1)^{\Delta_0} > (10^7 + 2)\Delta_0 \log_2 c_0$. Choose also $c_2 > 1$ such that $c_2^{\Delta_0} < 1.1$, and set $c_3 = \sqrt{2}/c_0^2$. We will show that Theorem 4 holds with $c = \min\{c_1, c_2, c_3\}$.

If $1 \leq \Delta < \Delta_0$ and n is even, simply take as H a matching. Then

$$r(H) = \frac{3}{2}n - 1 > 1.1n > c_2^{\Delta_0} n > c^{\Delta} n \quad \text{for } n > 2.$$

In the same case but with odd n, take as H a matching plus one isolated vertex obtaining

$$r(H) = \frac{3n-5}{2} > 1.1n > c_2^{\Delta_0} n > c^{\Delta} n \quad \text{for } n > 6.$$

When $\Delta \geq 2$ and $n = 3$ or $n = 5$, take as H the triangle K_3, or K_3 plus two isolated vertices, respectively. Then $r(H) = 6 > 1.1n > c^{\Delta} n$.

If $2 \leq \Delta + 1 \leq n < c_0^{2\Delta}$, take as H the complete graph $K_{\Delta+1}$ plus $n - \Delta + 1$ isolated vertices. Then

$$r(H) \geq r(K_{\Delta+1}) > 2^{\Delta/2} = c_2^{\Delta} c_0^{2\Delta} \geq c^{\Delta} n.$$

Finally, let us consider the main case when $\Delta \geq \Delta_0$ and $n \geq c_0^{2\Delta}$. Choose H as in Lemma 4 and R as in Lemma 5, and use R to 2-color the edges of $K_N, N = c_1^{\Delta} n$, as follows. Partition $[N] = V(K_N) = U_1 \cup \cdots \cup U_k$, $|U_i| = N/k$, $k = c_0^{\Delta}$. Then for all $e \in [N]^2$, assign the color

$$\chi(e) = \begin{cases} \mathit{Red}, & \text{if } e \in (U_i, U_j), ij \in R,\ i \neq j \\ \mathit{Blue}, & \text{if } e \in (U_i, U_j), ij \notin R, i \neq j \\ \text{arbitrary}, & \text{otherwise.} \end{cases}$$

We claim this coloring does not have a monochromatic copy of H. For suppose there is a *Red* copy H_0 of H formed. Setting $V_i = V(H_0) \cap U_i$, we have by

Lemma 4 that

$$\sum_{ij\in R}|V_i||V_j| \geq \sum_{i<j: e_{H_0}(V_i,V_j)>0} |V_i||V_j| > 0.55\binom{n}{2}. \tag{9}$$

On the other hand, expressing

$$|V_i| = w(i)\cdot N/k, \quad i = 1,2,\ldots,k,$$

we have $0 \leq w(i) \leq 1$, and

$$n = \sum_i |V_i| = N/k \sum_i w(i),$$

so that

$$x = \sum_i w(i) = kn/N = (c_0/c_1)^{\Delta} > (10^7+2)\log k$$

by our choice of c_0, c_1 and Δ_0, and by the monotonicity of $(c_0/c_1)^{\Delta}/\Delta$ as a function of Δ. Hence, by Lemma 5,

$$\sum_{ij\in R}|V_i||V_j| = \frac{N^2}{k^2}\sum_{ij\in R} w(i)w(j) < \frac{N^2}{k^2}(0.51)\binom{x}{2} \leq 0.51\binom{n}{2}.$$

This is a contradiction with (9), and the proof of Theorem 4 is complete. ■

Comment. Answering a question of Alon raised during the workshop in Princeton in November 1998, let us note that a suitable adjustment of the above proof leads to a random construction of a *bipartite* graph H with $r(H) > c^{\Delta}n$. Not surprisingly, in Lemma 4 one has to utilize the bipartite random graph $G(m,m,M)$, while the relevant change in Lemma 5 is to consider two weight functions f and g, both defined on the vertex set of a random graph $G(k,1/2)$, and such that $f+g \leq 1$. In the main proof these two weights will be determined by the intersections of each of the two vertex classes of a presumably monochromatic copy of H with the partition sets $U_i, i = 1,\ldots,k$. More precisely, if V' and V'' are the vertex classes of a *Red* copy H_0 of H then, setting $V_i' = V' \cap U_i$ and $V_i'' = V'' \cap U_i, i = 1,2,\ldots,k$, we have

$$\sum_{ij\in R}(|V_i'||V_j''| + |V_i''||V_j'|) \geq \sum_{i\neq j: e_{H_0}(V_i',V_j'')>0} |V_i'||V_j''|.$$

The two weight functions are now defined by $|V_i'| = f(i)\cdot N/k$ and $|V_i''| = g(i)\cdot N/k, i = 1,2,\ldots,k$. For details see [18].

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