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On gravitomagnetic precession around black holes

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ABSTRACT

We compute exactly the frequency of Lense–Thirring precession for point masses in the Kerr metric, for arbitrary black hole mass and specific angular momentum. We show that this frequency, for point masses at or close to the innermost stable orbit, and for holes with moderate to extreme rotation, is less than, but comparable to, the rotation frequency. Thus, if the quasi-periodic oscillations (QPOs) observed in the modulation of the X-ray flux from some black hole candidates (BHCs) are caused by Lense–Thirring precession of orbiting material, we predict that a separate, distinct QPO ought to be observed in each object.

Key words: accretion, accretion discs – black hole physics – relativity.

1 INTRODUCTION

The large effective area, very high time resolution and excellent telemetry of the *Rossi X-ray Timing Explorer* (*RXTE*) have made possible the discovery of quasi-periodic oscillations (QPOs) in the range ~100–1200 Hz from a variety of accreting collapsed objects, weakly magnetic neutron stars (see van der Klis 1998 for a review) and, more surprisingly, in black hole candidates (BHCs: see Morgan, Remillard & Greiner 1997; Remillard et al. 1997). It has been recently suggested (Cui, Zhang and Chen 1998) that these QPOs in BHCs arise through Lense–Thirring (1918, hereafter LT) precession of matter from the accretion discs.

As the motion of a point mass in a Kerr metric allows an exact treatment, and a detailed comparison of Keplerian and LT frequencies has not been explicitly carried out in the literature up to now, it seems worthwhile to derive these quantities for an arbitrary black hole mass and specific angular momentum. This allows us to clarify the meaning of the precession frequency, about which some confusion seems to be present in the literature. This is the aim of this paper. In the last section we shall also discuss the problems that are raised by this computation, regarding the interpretation of Cui, Zhang & Chen (1998).

2 BOUND ORBITS IN THE KERR METRIC

In what follows we will consider a test particle of unit mass in motion inside a Kerr space–time. The metric, in Boyer–Lindquist coordinates (Boyer & Lindquist 1967) and in units $G = c = 1$ is

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 - \frac{4aMr\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\phi + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{\Lambda\sin^2\theta}{\rho^2}\mathrm{d}\phi^2, \quad (1)$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta,$$
$$\Delta = r^2 + a^2 - 2Mr,$$
$$\Lambda = -\Delta a^2\sin^2\theta + (r^2 + a^2)^2.$$

M and a are, respectively, the mass and the specific angular momentum of the black hole.

As Carter (1968) first demonstrated, the equation of motion can be separated, and the resulting equations become

$$\rho^2\dot{r} = \pm\sqrt{R(r)}, \quad (2)$$
$$\rho^2\dot{\theta} = \pm\sqrt{\Theta(\theta)}, \quad (3)$$
$$\rho^2\dot{\phi} = (L\sin^{-2}\theta - aE) + a\Delta^{-1}P, \quad (4)$$
$$\rho^2\dot{t} = a(L - aE\sin^2\theta) + (r^2 + a^2)\Delta^{-1}P, \quad (5)$$

with

$$\Theta = Q - \cos^2\theta[a^2(1 - E^2) + L^2\sin^{-2}\theta],$$
$$P = E(r^2 + a^2) - La,$$
$$R = P^2 - \Delta[r^2 + Q + (L - aE)^2].$$

The dot denotes differentiation with respect to the proper time τ; signs in (2) and (3) can be chosen independently. E, L and Q are the three constants of the particle motion: E and L are, respectively, the energy and the angular momentum in the azimuthal direction as seen by an observer at rest at infinity; Q is related to Carter's constant of motion (see e.g. Chandrasekhar 1983 and de Felice 1980) and characterizes the θ motion.

As Wilkins (1972) showed, bound motion is possible only if $E^2 < 1$ and $Q \geq 0$; moreover, for given Q and L and $|E| < 1$, there may be at most one region of binding. Analysis of the θ effective potential shows that every orbit either remains in the equatorial plane ($Q = 0$) or crosses it repeatedly ($Q > 0$). For every bound

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motion, introducing the angle-action variables, we can define the three fundamental proper frequencies

$$1/\tau_{\phi,\mathrm{p}} = \nu_{\phi,\mathrm{p}},\ 1/\tau_{\theta,\mathrm{p}} = \nu_{\theta,\mathrm{p}},\ 1/\tau_{r,\mathrm{p}} = \nu_{r,\mathrm{p}},$$

where $\tau_{\phi,\mathrm{p}}$, $\tau_{\theta,\mathrm{p}}$ and $\tau_{r,\mathrm{p}}$ are the proper time periods for ϕ, θ and r motions respectively. Unlike the Newtonian case of particle motion around a spherically symmetric central object, where all orbits close and the three fundamental frequencies are equal, in the Kerr field ($a \neq 0$) there is no degeneracy, i.e.

$$\nu_{\phi,\mathrm{p}} \neq \nu_{\theta,\mathrm{p}} \neq \nu_{r,\mathrm{p}}.$$

The same is also true for coordinate frequencies ν_ϕ, ν_θ and ν_r.

Let us first consider a circular geodesic in the equatorial plane ($\theta = \pi/2$). We have, for the coordinate angular velocities measured by an observer static at infinity (Bardeen, Press & Teulkoski 1972)

$$\Omega_r = \Omega_\theta = 0,$$

$$\Omega_\phi = \frac{2\pi}{\tau_\phi} = \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\pm\sqrt{M/r^3}}{1 \pm a\sqrt{M/r^3}}; \qquad (6)$$

the angular velocity Ω_ϕ deviates from its Keplerian value at small radii. The upper sign refers to prograde orbits and the lower one to retrograde ones. If we slightly perturb a circular orbit, introducing velocity components in the r and θ directions, we can compute the coordinate frequencies of the small-amplitude oscillations within the plane (the epicyclic frequency Ω_r) and in the perpendicular direction (the vertical frequency Ω_θ) (Okazaki, Kato & Fukue 1987; Kato 1990; de Felice & Usseglio-Tomasset 1996; Perez et al. 1997):

$$\Omega_\theta^2 = \Omega_\phi^2\left[1 \mp 4\frac{aM^{1/2}}{r^{3/2}} + 3\frac{a^2}{r^2}\right], \qquad (7)$$

$$\Omega_r^2 = \frac{M(r^2 - 6Mr \pm 8aM^{1/2}r^{1/2} - 3a^2)}{r^2(r^{3/2} \pm aM^{1/2})^2}. \qquad (8)$$

In the case of the Schwarzschild metric ($a = 0$), there is a partial degeneracy, as the vertical frequency coincides with the azimuthal one. The epicyclic frequency, instead, is always lower than the other two, reaching a maximum for $r = 8M$ and going to zero at $r = 6M$ (Okazaki et al. 1987). This qualitative behaviour of the epyciclic frequency is preserved in the Kerr field ($a \neq 0$), and is a key feature for the existence of trapped discoseismic g modes (Perez et al 1997).

3 SPHERICAL ORBITS AND FRAME DRAGGING

We now confine ourselves to the study of those orbits with constant r, which are arbitrarily (not infinitesimally) lifted over the equatorial plane, i.e. with a finite value of Q.

The conditions for the stability of a spherical orbit with radius $r = r_0$ are (see equation 2)

$$R(r_0) = 0, \qquad (9)$$

$$\left.\frac{\partial R}{\partial r}\right|_{r=r_0} = 0, \qquad (10)$$

$$\left.\frac{\partial^2 R}{\partial r^2}\right|_{r=r_0} < 0. \qquad (11)$$

Conditions (9) and (10) introduce two relations between r and the constants of motion E, L and Q, reducing the free parameters that characterize the orbit to two; thus, given a specific Kerr black hole (i.e. given the values of M and a), a spherical orbit is completely determined, for example, by specifying its radius and the value of Q, which fixes the amplitude of motion in the θ direction (Wilkins 1972).

Orbits do not close, since the two fundamental frequencies ν_ϕ and ν_θ (proper or coordinate) are incommensurate; the Fourier spectra of every function of the position of the test particle will then contain a superposition of the two fundamental frequencies and all their harmonics, and will be of the kind

$$\sum_{l=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} C_{lm}\mathrm{e}^{\mathrm{i}(l\nu_\phi + m\nu_\theta)t+\beta},$$

where β is an arbitrary phase.

Therefore, the most natural signals to look for in such a system are the two fundamental coordinate frequencies themselves and the difference between them, which, as we will show, coincide with the unique correct definition of precession frequency of the nodes of a spherical orbit.

In fact we can compute exactly the coordinate period of the θ motion: if we call the two roots of the equation $\Theta(\theta) = 0$ $\theta_\pm$ (with $\theta_- < \theta_+$), we see from (3) that the particle oscillates on the coordinate sphere between the angles θ_- and $\pi/2 + \theta_-$. Dividing (5) by (3) and integrating, we obtain

$$\tau_\theta = 4\left\{[K(k) - E(k)]\left(\frac{z_+}{\beta}\right)^{1/2}Ea + \frac{K(k)}{a\sqrt{\beta z_+}}\left[aL + \frac{P(r^2 + a^2)}{\Delta} - Ea^2\right]\right\}, \qquad (12)$$

where $\beta = 1 - E^2$, $k^2 = z_-/z_+$ (with $z_\pm = \cos^2\theta_\pm$) and $K(k)$ and $E(k)$ are the elliptic integrals of the first and second kind, respectively.

The change of azimuth during one quarter oscillation of latitude is given by

$$\Delta\phi = \frac{1}{a\sqrt{\beta z_+}}\left\{L\Pi(-z_-, k) + \left[\frac{a}{\Delta}(2MrE - aL)K(k)\right]\right\} \qquad (13)$$

where $\Pi(k)$ is the elliptic integral of the third kind.

An orbit is called corevolving (or prograde) if $\Delta\phi > 0$, and counter-revolving (or retrograde) if $\Delta\phi < 0$. If the θ and ϕ frequencies were the same, $\Delta\phi$ would equal $\pi/2$; this means that we can define

$$\frac{\nu_\phi}{\nu_\theta} = |\Delta\phi|/\frac{\pi}{2}. \qquad (14)$$

The angle by which the nodes of a spherical orbit are dragged during each nodal period is therefore

$$\Delta\Omega = 2\pi|\frac{\nu_\phi}{\nu_\theta} - 1| \qquad (15)$$

and, consequently, the coordinate precession frequency of the nodes (or *frame-dragging* frequency) is

$$\nu_{\mathrm{FD}} = \frac{\Delta\Omega}{\tau_\theta} = |\nu_\phi - \nu_\theta|. \qquad (16)$$

We stress here that this definition is different from the one given in equation (2) of Cui, Zhang & Chen (1998) ($\nu_{\mathrm{FD}} = \nu_\phi\Delta\Omega/2\pi$), with which it coincides only far from the source, where $\nu_\phi \simeq \nu_\theta$. In this case, however, it would be sufficient to consider the weak-field limit, the well-known LT equation (Lense & Thirring 1918), and this point clearly frustrates the aim of their work. In fact, approaching the horizon (to which the innermost stable orbit tends in the limit $a \longrightarrow 1$) their definition leads to a divergence of the frame-dragging frequency, while it can be shown that

$$\lim_{r\longrightarrow r_{\mathrm{hor}}} 2\pi\nu_{\mathrm{FD}} = \Omega_{\mathrm{BH}}$$

where Ω_{BH} is the angular velocity of the black hole (Christodolou & Ruffini 1971; Misner, Thorne & Wheeler 1973), i.e. the angular

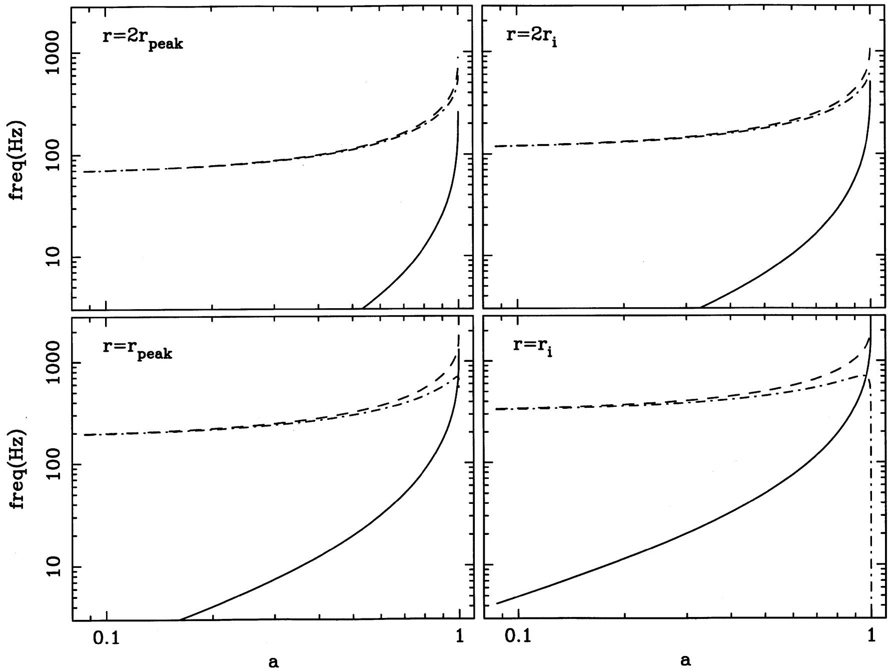

Figure 1. The three coordinate frequencies (ν_ϕ, dashed line; ν_θ, dot–dashed line; $\nu_{\rm FD}$, solid line) of spherical orbits at four different radii as functions of the dimensionless angular moment of an $M = 7\text{-M}_\odot$ black hole. r_i is the radius of the innermost stable orbit and $r_{\rm peak}$ is the radius of maximum surface emissivity of the disc. Q is always set equal to 1.

velocity of the zero angular momentum observers on the horizon. This is a relevant difference between this present work and that of Cui et al. (1998).

We chose to set, in our calculations, $M = 7\mathrm{M}_\odot$, in order to compare our results with the 300-Hz QPO observed from GRO J1655−40 (Remillard et al. 1997), the BHC for which the mass is most accurately measured (Orosz & Bailyn 1997). We considered only direct (i.e. prograde) orbits. In Fig. 1 we plot the three frequencies calculated at selected radii as functions of a for $Q = 1$. The radii are r_i, the radius of the innermost stable circular orbit, which has been calculated solving the quartic equation

$$\left.\frac{\partial^2 R(r)}{\partial r^2}\right|_{r=r_i} = 0,$$

$r_{\rm peak} = r_i/\eta$ (with η slowly varying from 0.62 to 0.76 as a goes from -1 to 1), the radius of maximum surface emissivity of the disc (Page & Thorne 1974), $2r_i$ and $2r_{\rm peak}$. The value of $Q = 1$ (which corresponds to an 'opening angle' of the orbit over the equatorial plane which varies from about 3° for $a = 0.5$ to about 5° for $a = 0.99$) was chosen for simplicity, because, exploring the whole range $Q = 0.01$–10, we found relative changes in the frequencies ranging from ~2 per cent ($a = 0.5$) to a maximum of only ~5 per cent ($a = 0.99$). It is immediately seen that for decreasing radii and increasing values of a the splitting of ν_ϕ and ν_θ increases dramatically. Correspondingly the frame-dragging frequency increases, reaching values that are comparable to ν_θ for $r = 2r_{\rm peak}$ and $2r_i$ or even larger than ν_θ for $r = r_{\rm peak}$ and r_i.

Fig. 2 shows the three frequencies ν_ϕ, ν_θ and $\nu_{\rm FD}$ versus ν_ϕ for selected values of the angular momentum of the black hole ($a = 0.5$, 0.9, 0.95 and 0.998). These graphs represent the frequency changes that would take place if, for a given black hole, the orbital radius of the precessing matter changed (see Section 4).

These results, obtained in a fully general relativistic framework, admit a simple interpretation in terms of a Newtonian analogy. In the classical gravitational potential resulting from a spherical star, $\propto 1/r$, the frequencies of motion for a bound orbit in the azimuthal (ϕ), radial and latitudinal (θ) directions are all equal; this well-known property assures that all orbits close in this potential. Whenever a small perturbation is introduced, such as that resulting from the oblateness of a star, this property is lost and the θ-frequency ν_θ becomes different from the ϕ-frequency ν_ϕ. Then the spectrum emitted by a source on this orbit will contain all harmonics of type $n\nu_\phi + m\nu_\theta$, with n, m integers; of these, the line with, most likely, the largest amplitude is that at frequency $\nu_\theta - \nu_\phi$. This, in particular, is the classical precession frequency caused by a Newtonian star not being perfectly spherical. As, in classical mechanics, departures from spherical symmetry are always modest, we always find $\nu_\phi - \nu_\theta \ll \nu_\theta, \nu_\phi$. In the gravitational field around a fast-rotating black hole, however, such departures are much more significant, implying that this inequality is no longer satisfied. In other words, as departures from a Newtonian potential increase, either because we are moving to a strong–field limit or because the black hole is rotating faster, we expect to move toward a situation where $\nu_\phi - \nu_\theta \approx \nu_\theta \approx \nu_\phi$. This is exactly what we see happening in Figs 1 and 2.

4 APPLICATION TO BLACK HOLE CANDIDATES

By using the black hole mass and angular momentum that have been measured (or indirectly inferred) for several BHCs, Cui et al. (1998) find a reasonably good agreement of the predicted point-mass precession frequencies, at the radius where the disc emissivity is highest, with the observed QPO frequencies. If disc precession is not confined to such a radius and differential precession takes place at a frequency close to the local frame-dragging frequency, it remains to be demonstrated that a sufficiently narrow QPO peak matching the observations can be generated as a result of the different precession frequencies that might occur at different radii. More crucially, it is well known that in viscous accretion

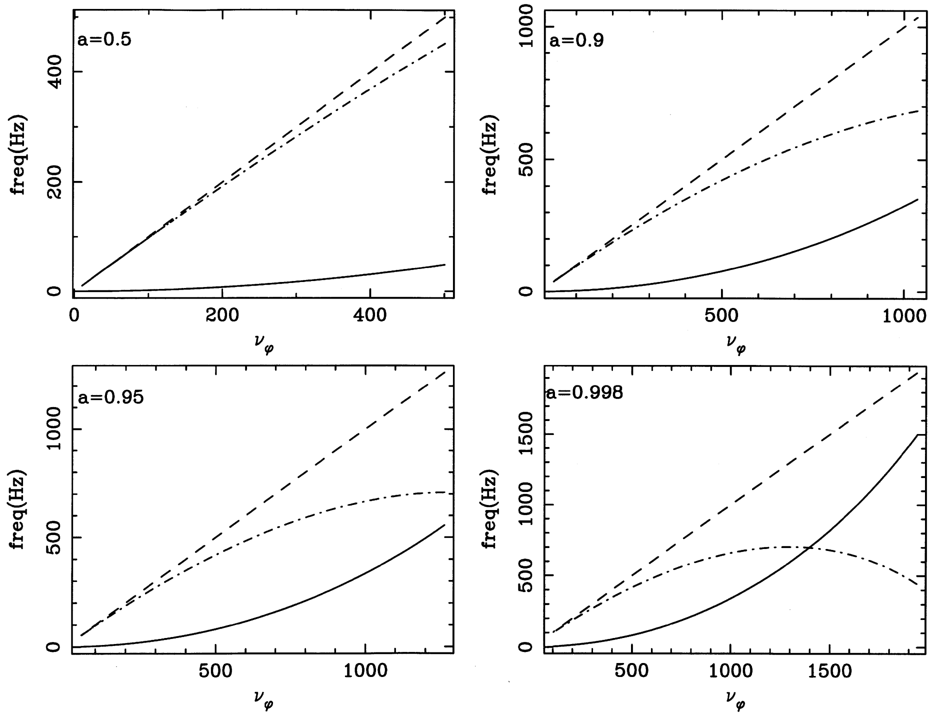

Figure 2. The three coordinate frequencies (ν_ϕ, dashed line; ν_θ, dot–dashed line; $\nu_{\rm FD}$, solid line) of spherical orbits with $Q = 1$ for selected values of the angular momentum of the black hole (*a* = 0.5, 0.9, 0.95 and 0.998).

discs LT precession of the whole disc is strongly damped (Bardeen & Petterson 1975; Pringle 1992 and references therein). However, there appear to be precession modes that are strongly confined to the innermost disc regions and only weakly damped (Markovic & Lamb 1998); these are currently being investigated in greater detail. The mechanism responsible for the excitation of these modes remains an open question. The prospects for some kind of resonant excitation driven from an azimuthal asymmetry do not appear promising in consideration of the black hole 'no hair theorem'.

An alternative possibility is that in the innermost disc region there are individual blobs moving like test particles in the BHC field, also executing LT precession, and modulating the observed X-ray flux either through occultation or because they are self-luminous. The existence of discrete blobs is of course not an embarrassment for this argument, because their existence is required in all scenarios trying to explain QPOs, in particular those involving weakly magnetic neutron stars in low-mass X-ray binaries (LMXRBs). This tendency of the disc to form discrete blobs seems to be independent of the nature of the accreting source; for instance, Krolik (1998) suggests that it may be the result of local instabilities of the disc, irrespective of the properties of the accreting source. This tendency might therefore be present in accretion discs surrounding both neutron stars and black holes. By pushing further the analogy with neutron star LMXRBs, where high-frequency QPOs are very often observed and successfully interpreted in terms of the Keplerian frequency (note that in LMXRBs $\nu_\phi \simeq \nu_\theta$, see below), one would conclude that if the precession of blobs is responsible for the QPOs observed in BHCs, there is no obvious reason why QPOs reflecting the ϕ and θ components of the orbital motion should not be there. Indeed, if, according to the model of Cui et al. (1998), the ~300-Hz QPOs of GRS J1655−40 originate from the frame-dragging frequency of blobs off the equatorial plane in the innermost disc regions, then the ϕ and θ frequencies of the orbital motion are $\nu_\phi \simeq 970$ and 950 Hz and $\nu_\theta \simeq 670$ and 650 Hz, while $a \simeq 0.88$ and 0.96, respectively, in the cases in which frame-dragging QPOs are produced at the innermost stable orbit or the radius of highest disc emissivity. The difference between ν_ϕ and ν_θ is large and two well-separated QPO peaks might be expected. These signals, however, have not been detected yet. Similar considerations would apply to the case of the ~67-Hz QPOs from GRS 1915 + 105[1] (Morgan et al. 1997).

The application of beat frequency models (BFMs) to those neutron star systems that show twin kHz QPO peaks, allows us to identify the higher frequency kHz QPO (~800–1200 Hz) as arising directly from the Keplerian motion of blobs at the inner edge of the disc (moreover the neutron star spin frequency is inferred from the difference frequency of the twin kHz QPOs). Stella & Vietri (1998) noticed that the precession frequency of these blobs, as derived from the neutron star parameters inferred from BFMs, agrees well with a broad peak around 20–35 Hz that is apparent in the power spectra of three sources. In this model, therefore, $\nu_\phi - \nu_\theta \sim 20$–35 Hz, a separation that is comparable to or smaller than the width of the higher frequency kHz QPO peak. Therefore it is not surprising that the signals at ν_ϕ and ν_θ are difficult to disentangle in the case of neutron star LMXRBs. It should also be noticed that, around neutron stars with weak magnetic fields, a natural mechanism exists to lift matter off the equatorial plane, through the interaction with a spinning, tilted magnetic dipole moment (Vietri & Stella 1998).

In the frame-dragging interpretation of BHC QPOs, very specific predictions are also made in relation to the changes in ν_ϕ and ν_θ that result from changes in the frame-dragging frequency $\nu_\phi - \nu_\theta$ (cf. Fig. 2). This would provide a sensitive diagnostic with which to

[1]The harmonic content of the three fundamental frequencies in the problem at hand will depend on the mechanism responsible for the generation of the signal(s) (e.g. self-luminous versus occulting blobs) and on geometry, and is beyond the scope of this paper.

confirm the interpretation and study the motion of matter close to the event horizon of a Kerr black hole with unprecedented detail.

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REFERENCES

Bardeen J.M., Petterson J.A., 1975, ApJ, 195, L65
Bardeen J.M., Press W.H., Teukolski S.A., 1972, ApJ, 178, 347
Boyer P.H., Lindquist D.W., 1967, J. Math. Phys., 8, 265
Carter B., 1968, Phys. Rev., 174, 1559
Chandrasekhar S., 1983, The Mathematical Theory of Black Holes. Oxford University Press, New York
Christodoulou D., Ruffini R., 1971, Gravity research foundation, unpublished
Cui W., Zhang S.N., Chen W., 1998, ApJ, 492, L53
de Felice F., 1980, J. Phys. A, 13, 1701
de Felice F., Usseglio-Tomasset S., 1996, Gen. Rel. Grav., 28, 179
Lense J., Thirring H., 1918, Phys. Z., 19, 156
Kato S., 1990, PASJ, 42, 99
Krolik J.H., 1998, ApJ, 498, L13
Markovic D., Lamb F. K., 1998, ApJ, 507, 316
Misner C.W., Thorne K.S., Wheeler J.A., 1973, Gravitation. Freeman, San Francisco
Morgan E., Remillard R., Greiner J., 1997, ApJ, 482, 993
Orosz J.A., Bailyn C.D., 1997, ApJ, 477, 876
Okazaki T., Kato S., Fukue J., 1987, PASJ, 39, 457
Page D.N., Thorne K.S., 1974, ApJ, 191, 499
Perez C.A., Silbergleit A.S., Wagoner P.V., Lehr D.E., 1997, ApJ, 476, 589
Pringle J.E., 1992, MNRAS, 258, 811
Remillard R.A., Morgan E.H., McClintock J.E., Bailyn C.D., Orosz J., Greiner J., 1997, in Olinto A., Frieman J. , Schramm D., eds, Proc. 18th Texas Symp. on Relativistic Astrophysics. Chicago Univ. Press, Chicago, in press
Stella L., Vietri M., 1998, ApJ, 492, L59
van der Klis M., 1998, in Alpar A., Buccheri R., van Paradijs J., eds, Proc. NATO ASI , The Many Faces of Neutron Stars, Kluwer, Dordrecht, in press (astro-ph/9710016)
Vietri M., Stella L., 1998, ApJ, 503, 350
Wilkins D., 1972, Phys. Rev. D, 5, 814

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