On Greedy Clique Decompositions and Set Representations of Graphs*

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Abstract

In 1994 S. McGuinness showed that any greedy clique decomposition of an n-vertex graph has at most $\lfloor n^2/4 \rfloor$ cliques (The greedy clique decomposition of a graph, *J. Graph Theory* **18** (1994) 427-430), where a *clique decomposition* means a clique partition of the edge set and a *greedy clique decomposition* of a graph is obtained by removing maximal cliques from a graph one by one until the graph is empty. This result solved a conjecture by P. Winkler. A *multifamily set representation* of a simple graph G is a family of sets, not necessarily distinct, each member of which represents a vertex in G, and the intersection of two sets is non-empty if and only if two corresponding vertices in G are adjacent. It is well known that for a graph G, there is a one-to-one correspondence between multifamily set representations and clique coverings of the edge set. Further for a graph one may have a one-to-one correspondence between particular multifamily set representations with intersection size at most one and clique partitions of the edge set. In this paper, we study for an n-vertex graph the variant of the set representations using a family of distinct sets, including the greedy way to get the corresponding clique partition of the edge set of the graph. Similarly, in this case, we obtain a result that any greedy clique decomposition of an n-vertex graph has at most $\lfloor n^2/4 \rfloor$ cliques.

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1 Background and Introduction

By an *multigraph* $M = (V(M), E(M), q)$ we mean a triple consisting of a set $V(M)$ of *vertices*, a set $E(M)$ of *edges*, and an integer-valued function q defined on $V(M) \times V(M)$ in the following way. For each unordered pair $\{u, v\} \subset V(M)$, let $q(u, v)$ be the number of *parallel edges* joining u with v. If $q(u, v) \neq 0$, then we say that $\{u, v\}$ is an *edge* of M and $q(u, v)$ is called the *multiplicity* of the edge $\{u, v\}$. For the main results in this paper, we consider only *finite, undirected, simple* multigraphs, where *simple* means that $q(u, v) \leq 1$ for every $\{u, v\} \subset V$ and $q(u, u) = 0$ for every $u \in V(M)$. Therefore we simply call such multigraphs to be *graphs* for short throughout this article, unless otherwise stated.

For a vertex subset $S \subseteq V(M)$, $\langle S \rangle_V$ denotes the *subgraph induced by* S. For a vertex v in M, $d_M(v)$ or $d(v)$ denote the *degree* of v in M. Let $\mathcal{F} = \{S_1, ..., S_p\}$ be a *family* of distinct nonempty subsets of a set X. Then $\mathbf{S}(\mathcal{F})$ denotes the union of sets in $\mathcal{F}$. The *intersection multigraph* of $\mathcal{F}$, denoted $\Omega(\mathcal{F})$, is defined by $V(\Omega(\mathcal{F})) = \mathcal{F}$, with $|S_i \cap S_j| = q(S_i, S_j)$ whenever $i \neq j$. Of course, so long as we are involved in this paper, $|S_i \cap S_j|$ always equal either 0 or 1 for all $i \neq j$, as appointed above.

We say that a multigraph M is an intersection multigraph on a family (a multifamily, respectively) $\mathcal{F}$, if there exist a family (a multifamily, respectively) $\mathcal{F}$ such that $M \cong \Omega(\mathcal{F})$. We say that $\mathcal{F}$ is a *representation* (a *multifamily representation* respectively) of the multigraph M. The *intersection number*, denoted $\omega(M)$ (*multifamily intersection number*, denoted $\omega_m(M)$, respectively), of a given multigraph M is the minimum cardinality of a set X such that M is an intersection multigraph (*multifamily intersection multigraph*, respectively) on a family (a multifamily, respectively) $\mathcal{F}$ consisting of distinct (not necessarily distinct, respectively) subsets of X. In this case we also say that $\mathcal{F}$ is a *minimum representation* (*multifamily representation*, respectively) of M.

Note that given a representation $\{S_v \mid v \in V(M)\}$ of M and a vertex subset $S \subseteq V(M)$, then $\{S_v \mid v \in S\}$ form a representation of $\langle S \rangle_V$. Thus we know that $\omega(M)$ is not less than $\omega(\langle S \rangle_V)$ for any $S \subseteq V(M)$. Similarly for $\omega_m(M)$.

In 1966 P. Erdös et al. [1] proved that the edge set of any simple graph G with n vertices, no one of which is isolated vertex, can be partitioned using at most $\lfloor n^2/4 \rfloor$ cliques. In a couple decades S. McGuinness [3] showed that any greedy clique partition is such a partition.

A multifamily representation of a graph G is a family of sets each member of which represent a vertex in G and the intersection relation of two members of which represent the adjacency of the two corresponding vertices in G. P.

Erdös et al. [1] suggested a one-one correspondence between multifamily representations and clique coverings of a graph G. In fact, we may define a *multifamily representation of a multigraph* M to be a family of sets for which each member represents a vertex in M, and the two vertices are adjacent with q edges in M if and only if the corresponding representation sets have an intersection of cardinality q. Then there is also a one-one correspondence between multifamily representations and clique partitions of M.

In next section we will narrate this correspondence in full detail. If a multifamily representation of a multigraph M has pairwise distinct member sets, then it is called a *representation* of M. And then we turn the correspondence to prove that any n-vertex graph can be represented by at most $\lfloor n^2/4 \rfloor$ elements and we can accomplish such a representation from any greedy clique partition by a straightforward method based on this correspondence. In the end, certain future directions will be mentioned.

2 Partition Edge Set by Cliques

Given a multigraph $M = (V(M), E(M), q)$, $Q \subseteq V(M)$ is said to be a *clique* of M if every pair of distinct vertices u, v in Q has $q(u, v) \neq 0$. A *clique partition* $\mathcal{Q}$ of a multigraph is a set of cliques such that every pair of distinct vertices u, v in $V(M)$ simultaneously appear in exactly $q(u, v)$ cliques in $\mathcal{Q}$ and for each *isolated vertex*, that is, vertex with no edge incident to it, we need to use at least one *trivial clique*, that is, clique with only one vertex, in $\mathcal{Q}$ to cover it. The minimum cardinality of a clique partition of M is called the *clique partition number* of M, and is denoted by $cp(M)$. This number must exist as the edge set of M forms a clique partition for M. We refer to a clique partition of M with the cardinality $cp(M)$ as a *minimum clique partition* of M.

Note that a clique partition $\mathcal{Q}$ of M give rise to a clique partition of $M - v$ by deleting the vertex v from each clique in $\mathcal{Q}$. Thus $cp(M)$ is not less than the clique partition number of any induced subgraph of M.

P. Erdös et al. [1] proved the following theorem.

Theorem 2.1. *The edge set of any simple graph G with n vertices no one of which is isolated vertex can be partitioned using at most $\lfloor n^2/4 \rfloor$ triangles and edges, and that the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ gives equality.*

We somewhat modify their proof to prove the following theorem. We use $G^{(n)}$ to denote a graph G with n vertices.

Theorem 2.2. *Any graph G with $n \geq 4$ vertices (perhaps with isolated vertices) can be partitioned with at most $\lfloor n^2/4 \rfloor$ cliques $Q_1, ..., Q_N$ such that for any two vertices u, v in G, we have*

$$\begin{aligned} &\{Q_i \mid u \in Q_i \in \{Q_1, ..., Q_N\}\} \\ &\qquad \neq \quad \{Q_i \mid v \in Q_i \in \{Q_1, ..., Q_N\}\}. \end{aligned} \tag{1}$$

Note that in such partition we need only use edges and triangles. Furthermore, the upper bound $\lfloor n^2/4 \rfloor$ is optimal.

Proof. For $n = 4$, it is easy to check the theorem holds for the 11 different graphs on 4 vertices. (Please see Figure 1)

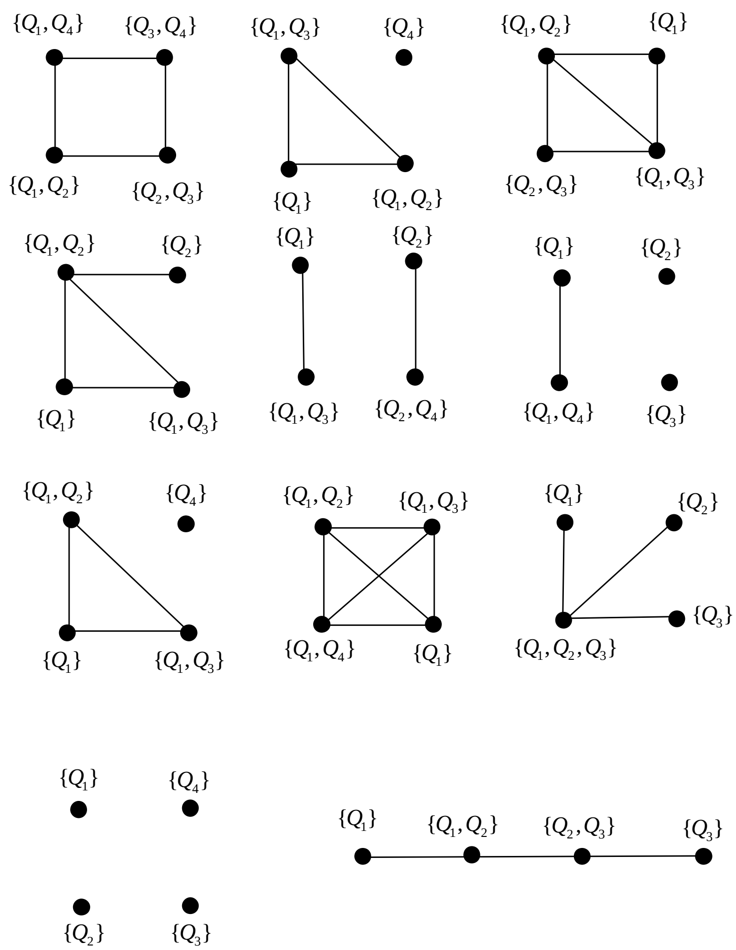

Figure 1: The 11 Non-Isomorphic Graphs on 4 Vertices with Corresponding Family Representations

We proceed by mathematical induction from $n = 4$. First note that given any positive integer n,

$$\lfloor n^2/4 \rfloor = \lfloor (n-1)^2/4 \rfloor + \lfloor n/2 \rfloor.$$

Hence we have to show that from $G^{(n-1)}$ to $G^{(n)}$, at most $\lfloor n/2 \rfloor$ more cliques are needed. We have the following cases:

Case 1: In case $G^{(n)}$ has a vertex v of degree $\leq \lfloor n/2 \rfloor$, then first we delete v and all edges incident with v. Then by induction hypothesis, we partition the resulting graph with at most $\lfloor (n-1)^2/4 \rfloor$ cliques K_2 or K_3. Then from $G^{(n-1)}$ to $G^{(n)}$ we need only to use the edges joining the deleted vertex v to other vertices of $G^{(n)}$, and then give rise to at most $\lfloor n/2 \rfloor$ more cliques as K_2. Clearly the resulting clique partition of $G^{(n)}$ still satisfies (1).

Case 2: On the contrary, every vertex of $G^{(n)}$ is of degree $> \lfloor n/2 \rfloor$. Let x be the vertex with the minimum degree t, and set $t = \lfloor n/2 \rfloor + r$, where $r > 0$. Let x be adjacent to the vertices $y_1, ..., y_t$ and $G^{(t)}$ be the subgraph of $G^{(n)}$ induced by $\{y_1, ..., y_t\}$.

We claim that $G^{(t)}$ has r edges and no two of which have a common vertex. Assume that $G^{(t)}$ has only $r-1$ such edges (note that it is similar to show the case $G^{(t)}$ has less than $r-1$ such edges), say

$$\{y_1, y_2\}, \{y_3, y_4\}, ..., \{y_{2r-3}, y_{2r-2}\}.$$

By $t = \lfloor n/2 \rfloor + r = d(x) \leq n-1$, we know that $r \leq \lfloor n/2 \rfloor$ and thus $t \geq 2r$. Thus we may pick y_{2r-1} from $\{y_1, ..., y_t\}$.

By hypothesis, y_{2r-1} has degree $\geq \lfloor n/2 \rfloor + r$. But it could be adjacent to at most $2r-2$ of the vertices $y_1, ..., y_{2r-2}$ and to at most $n-t$ of the vertices not in $G^{(t)}$, hence the degree of y_{2r-1} is at most

$$\begin{aligned}(2r-2) + (n-t) &= (2r-2) + (n - (\lfloor n/2 \rfloor + r)) \\ &= (n - \lfloor n/2 \rfloor - 2) + r \\ &< \lfloor n/2 \rfloor + r.\end{aligned}$$

However note that $\lfloor n/2 \rfloor + r$ is the minimum degree, hence y_{2r-1} is adjacent to some other vertex, say y_{2r}, in $G^{(t)}$ and

$$\{y_1, y_2\}, \{y_3, y_4\}, ..., \{y_{2r-3}, y_{2r-2}\}, \{y_{2r-1}, y_{2r}\}$$

are r edges in $G^{(t)}$ and no two of which have a common vertex.

We remove these r edges from $G^{(n)} - x$. Partition the resulting graph with at most $\lfloor (n-1)^2/4 \rfloor$ cliques and (1) is satisfied. Then the $\lfloor (n-1)^2/4 \rfloor$ cliques together with the triangles

$$\{x, y_1, y_2\}, \{x, y_3, y_4\}, ..., \{x, y_{2r-1}, y_{2r}\}$$

and the edges

$$\{x, y_k\}, \text{ where } 2r + 1 \le k \le t,$$

form a clique partition, which uses at most

$$\begin{aligned}\lfloor (n-1)^2/4 \rfloor &+ r + (t - 2r) \\ &= \lfloor (n-1)^2/4 \rfloor - r + (\lfloor n/2 \rfloor + r) \\ &= \lfloor n^2/4 \rfloor\end{aligned}$$

cliques.

Note that according to our convention in this paper, we need to use at least one trivial clique, even for each isolated vertex in the clique partition of the graph $G^{(n)} - x$ with the r edges

$$\{y_1, y_2\}, \{y_3, y_4\}, ..., \{y_{2r-3}, y_{2r-2}\}, \{y_{2r-1}, y_{2r}\}$$

removed. Thus the resulting clique partition of $G^{(n)}$, obtained from that of $G^{(n)} - x$ with the r edges removed, must agree with the requirement (1) of our theorem in the respect that for any two vertices u, v in $G^{(n)}$,

$$\begin{aligned}\{Q_i \mid u \in Q_i \in \{Q_1, ..., Q_N\}\} \\ \neq \quad \{Q_i \mid v \in Q_i \in \{Q_1, ..., Q_N\}\}.\end{aligned}$$

Last we show that the number $\lfloor n^2/4 \rfloor$ cannot be replaced by any smaller number by giving the following example. Let $n = 2k$ or $2k+1$, we consider the complete bipartite graphs $K_{k,k}$ and $K_{k,k+1}$, which have $2k$ and $2k+1$ vertices, respectively. Clearly these two graphs have no triangle and their numbers of edges are

$$k^2 = \lfloor (2k)^2/4 \rfloor = \lfloor n^2/4 \rfloor, \text{ if } n = 2k,$$

and

$$k(k+1) = \lfloor (2k+1)^2/4 \rfloor = \lfloor n^2/4 \rfloor, \text{ if } n = 2k+1.$$

Hence $K_{k,k}$ and $K_{k,k+1}$ always require $\lfloor n^2/4 \rfloor$ cliques for a clique partition. □

Now we introduce the one-to-one correspondence between multifamily representations and clique partitions of a multigraph M as following.

Given a multigraph $M^{(n)} = (V(M), E(M), q)$, we first construct a clique partition

$$\mathcal{Q} = \{Q_1, ..., Q_p\}$$

Then with each clique Q_k we associate an element e_k and with each vertex v_α we associate a set $S_\mathcal{Q}(v_\alpha)$ of elements e_k, where

$$e_k \in S_\mathcal{Q}(v_\alpha) \Leftrightarrow v_\alpha \in Q_k,$$

i.e., $S_\mathcal{Q}(v_\alpha)$ is the collection of elements for which the corresponding cliques contain v_α. Thus we obtain

$$\mathcal{F}(\mathcal{Q}) \equiv \{S_\mathcal{Q}(v) : v \in V(M)\}.$$

Then clearly

$$\mathbf{S}(\mathcal{F}(\mathcal{Q})) \equiv \bigcup_{v \in V(M)} S_\mathcal{Q}(v)$$

contains p elements. And

$$|S_\mathcal{Q}(v_\alpha) \cap S_\mathcal{Q}(v_\beta)| = q(v_\alpha, v_\beta),$$

since there is exactly $q(v_\alpha, v_\beta)$ cliques simultaneously containing the two vertices v_α, v_β. Thus we have constructed a multifamily representation

$$\mathcal{F}(\mathcal{Q}) = \{S_\mathcal{Q}(v) : v \in V(M)\}$$

from the clique partition $\mathcal{Q}$ of M, where

$$|\mathbf{S}(\mathcal{F}(\mathcal{Q}))| \equiv |\bigcup_{v \in V(M)} S_\mathcal{Q}(v)| = p = |\mathcal{Q}|.$$

Conversely, given a multifamily representation $\mathcal{F} = \{S_1, ...S_n\}$ of M with vertex set $V(M) = \{v_1, ..., v_n\}$, where S_α correspond to the set attaching to v_α, then we can also construct a clique partition of M by the following way.

Let

$$\mathbf{S}(\mathcal{F}) \equiv \bigcup_{\alpha=1}^{n} S_\alpha = \{e_1, ..., e_p\}.$$

For each fixed e_k in $\mathbf{S}(\mathcal{F})$ we form a clique $Q_\mathcal{F}(e_k)$ using those vertices v_α such that the set S_α attaching to it contains e_k. Clearly each $Q_\mathcal{F}(e_k)$ is indeed a clique of M. Thus we obtain

$$\mathcal{Q}(\mathcal{F}) = \{Q_\mathcal{F}(e_1), ..., Q_\mathcal{F}(e_p)\}.$$

And

$$\begin{aligned} q(v_\alpha, v_\beta) &= |S_\alpha \cap S_\beta| \\ &= \text{the number of cliques in } \mathcal{Q}(\mathcal{F}) \text{ simultaneously containing } v_\alpha, v_\beta, \end{aligned}$$

since each element in S_α exactly represents a clique in $\mathcal{Q}(\mathcal{F})$ containing v_α. Thus we have constructed a clique partition $\mathcal{Q}(\mathcal{F})$ of M from the multifamily representation $\mathcal{F}$ of M, where

$$|\mathcal{Q}(\mathcal{F})| = p = |\bigcup_{\alpha=1}^{n} S_\alpha| \equiv |\mathbf{S}(\mathcal{F})|.$$

Thus we have established a one-one correspondence between multifamily representations and edge clique partitions of the multigraph M.

From above we know that $\omega_m(M) = cp(M)$. In particular we may consider the simple graphs as special classes of multigraphs:

Theorem 2.3. *Let G be a graph. Then we have $\omega_m(G) = cp(G)$.*

If we are given a graph $G^{(n)}$, then by Theorem 2.2 we may obtain a clique partition $\mathcal{Q}$ with cardinality less or equal to $\lfloor n^2/4 \rfloor$, agreeing with the requirement (1). Then by the above method we may obtain a representation $\mathcal{F}(\mathcal{Q}) = \{S_{\mathcal{Q}}(v): \ v \in V(G)\}$ of G consisting of distinct sets. Thus we have the following theorem.

Theorem 2.4. *Let G be a graph. Then $\omega(G^{(n)}) \leq \lfloor n^2/4 \rfloor$.*

Again considering the two complete bipartite graphs $K_{k,k}$ and $K_{k,k+1}$, one can easily see that the bound $\lfloor n^2/4 \rfloor$ in Theorem 2.4 is sharp.

3 Greedy Clique Decomposition of Graphs

One may not be satisfied with the above theorem and would rather ask that how to obtain a representation of $G^{(n)}$ using at most $\lfloor n^2/4 \rfloor$ elements. S. McGuinness [3] proved the following theorem, which solved a conjecture by P. Winkler [9]:

Theorem 3.1. *Every greedy clique decomposition of an n-vertex graph uses at most $\lfloor n^2/4 \rfloor$ cliques.*

In the theorem, the so-called *clique decomposition* is a clique partition of the edge set, and *greedy clique decomposition* of a graph $G^{(n)}$ means an ordered set $\mathbf{Q} = \{Q_1, ..., Q_m\}$ such that each Q_i is a maximal clique in $G - \bigcup_{j<i} E(Q_j)$, where $G - \bigcup_{j<i} E(Q_j)$ is the subgraph of G obtained by deleting all edges in the edge subset $\bigcup_{j<i} E(Q_j)$ while leaving all vertices in G preserved.

For a representation $\mathcal{F}$ of G, we referred as *monopolized elements* to those elements in $\mathbf{S}(\mathcal{F})$ which appear in only one member of $\mathcal{F}$. Here we prove the following main theorem, as a variant of S. McGuinness's result:

Theorem 3.2. *Every representation $\mathcal{F}$ of $G^{(n)}$ with $n \geq 4$ obtained from $\mathcal{F}(\boldsymbol{Q})$, where $\boldsymbol{Q}$ is any greedy clique decomposition of $G^{(n)}$ by successively attaching monopolized elements to the sets which repetitiously occur in $\mathcal{F}(\boldsymbol{Q})$, uses at most $\lfloor n^2/4 \rfloor$ elements.*

Before proving the theorem, we need the following lemma:

Lemma 3.3. *Let $\mathcal{Q}$ be an edge clique partition of a graph G, then we have that if $\mathcal{F}(\mathcal{Q}) = \{S_{\mathcal{Q}}(v) : v \in V(G)\}$ has two identical sets, say $S_{\mathcal{Q}}(u)$ and $S_{\mathcal{Q}}(v)$, then the clique Q_{uv} in $\mathcal{Q}$ simultaneously containing u, v is a maximal clique in G. Note that Q_{uv} has u and v as its monopolized elements, that is, u, v are in no clique of $\mathcal{Q}$ except Q_{uv}.*

Proof. If there is a clique Q' properly containing Q_{uv} in G, say vertex w being in Q' but not in Q_{uv}, then no clique in $\mathcal{Q}$ can simultaneously contain the three vertices u, v, w. Thus the clique in $\mathcal{Q}$ simultaneously containing u, w doesn't contain v and the clique in $\mathcal{Q}$ simultaneously containing v, w doesn't contain u, and therefore we must have $S_{\mathcal{Q}}(u) \neq S_{\mathcal{Q}}(v)$.

If u, say, belongs to one clique Q'' in $\mathcal{Q}$ other than Q_{uv}, then there is a vertex, say u', adjacent to u and not in Q_{uv}. In case that u' is not adjacent to v we must have $S_{\mathcal{Q}}(u) \neq S_{\mathcal{Q}}(v)$. In case that u' is adjacent to v, then no clique in $\mathcal{Q}$ can simultaneously contain u, v, u'. Thus the clique in $\mathcal{Q}$ simultaneously containing u, u' doesn't contain v and the clique in $\mathcal{Q}$ simultaneously containing u', v doesn't contain u, and therefore we must have $S_{\mathcal{Q}}(u) \neq S_{\mathcal{Q}}(v)$. □

Then we are in a position to proceed the proof of Theorem 3.2:

Proof. We use induction on n.

When $n = 4$, it is easy to draw all the eleven different graphs on four vertices, and to check that every representation of each of them uses at most $\lfloor n^2/4 \rfloor$ elements.

For the case $n = 5$, note that $\lfloor 5^2/4 \rfloor - \lfloor 4^2/4 \rfloor = 6 - 4 = 2$ and therefore we have two new elements in proceeding from $n = 4$ to $n = 5$. We have the following four cases:

Case 1: If $G^{(5)}$ has one vertex with degree 2 or less, then we reduce $G^{(5)}$ to $G^{(4)}$ by deleting this vertex and all edges incident to it. Note that this vertex form a maximal clique in $G^{(5)}$ along with some edge in $G^{(4)}$ only if $G^{(5)}$ is one of 13 non-isomorphic graphs in Figure 2, where hollow circle denote this vertex and dashed lines denote the edges incident to it. It is easy to check that every representation of each one of them uses at most $\lfloor 5^2/4 \rfloor = 6$ elements.

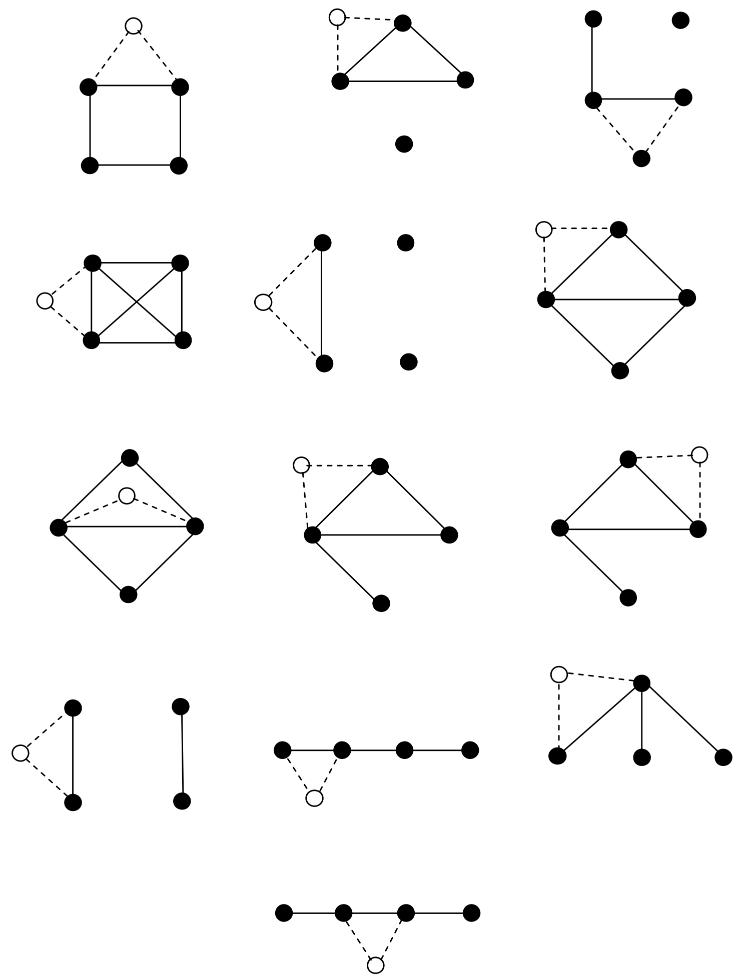

Figure 2: One Case for Representations of Graphs on 5 Vertices

Case 2: As for the case that there is no maximal clique in $G^{(5)}$ simultaneously containing this vertex and some edge in $G^{(4)}$, then in any greedy clique partition of $G^{(5)}$ we must use all the edges incident to this vertex as members of this greedy clique partition. Thus in this case, we may at first take a representation of $G^{(4)}$, and then go back to $G^{(5)}$ using the available two new elements to represent at most two edges incident to this vertex. Then we may confirm that in this case all representations of $G^{(5)}$ use at most $\lfloor 5^2/4 \rfloor = 6$ elements.

Case 3: As for the case that there is no edge in $G^{(5)}$ incident to this vertex, we may at first take a representation of $G^{(4)}$ and then go back to $G^{(5)}$ using one new monopolized element.

Case 4: Due to above, now we need to consider only those graphs on 5 vertices for which every vertex has degree greater than or equal to 3. There are only three such graphs and they are easy to be checked. (Please see Figure 3) Thus the case $n = 5$ is done, and we have proved the theorem for $n = 4$ and $n = 5$.

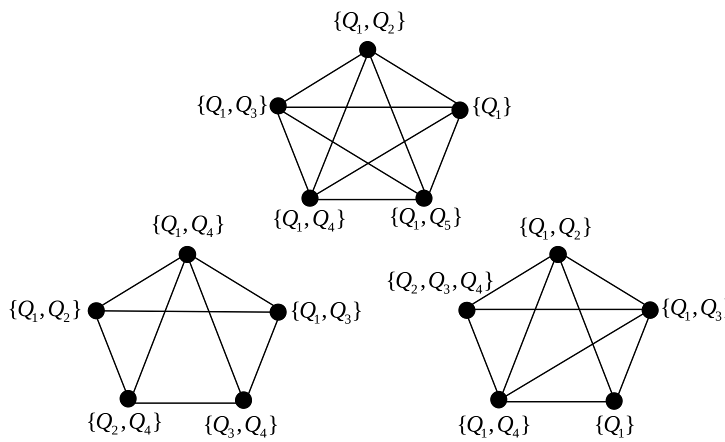

Figure 3: Graphs on 5 Vertices Whose Vertices Have Degree ≥ 3 with Corresponding Representations

Now let $\mathcal{F}$ be a representation of $G^{(n)}$ with $n \geq 6$ derived from $\mathcal{F}(\mathbf{Q})$, where $\mathbf{Q} = \{Q_1, ..., Q_m\}$ is a greedy clique partition of $G^{(n)}$. Note that deleting Q_j from the set $\mathbf{Q}$ leaves a greedy clique partition of $G - E(Q_j)$.

In case that each Q_j has at least three edges, we have $m \leq \binom{n}{2}/3 < n^2/6$. Assume for the time being that every Q_i has exactly three edges, that is, is exactly a triangle. Now if every triangle in $\mathbf{Q}$ has at most one of its three vertices of degree 2, then by Lemma 3.3 we do not need to use any monopolized element for this greedy clique partition. If there is a triangle in $\mathbf{Q}$ with at least two of its three vertices of degree 2, then recall that $G^{(n)}$ have at least six vertices, two vertices of degree 2 in this triangle make m to be less than or equal to $\left(\binom{n}{2}/3\right) - 2 < (n^2/6) - 2$. Thus although we might need two more monopolized elements for this triangle, yet in the same time we also have two less cliques (as K_3) in $\mathbf{Q}$. Besides, if there is a clique of cardinality $3 + r$ where $r > 0$ in $\mathbf{Q}$, then despite that maybe we need r more monopolized elements for this clique, yet in the same time by the fact that $\binom{3+r}{2} \geq 3(r+1)$ we also have r less cliques (as K_3)in $\mathbf{Q}$. Note that $\binom{3+r}{2}$ is the number of edges in a clique of cardinality $3 + r$ and $3(r + 1)$ is the total number of edges in $r + 1$ triangles. In fact, we may need rather $r+1$ or $r+2$ than r more monopolized elements for this clique of cardinality $3 + r$. By Lemma 3.3, we need to use $r + 2$ more monopolized elements for this clique only when either this clique is an isolated clique or $G^{(n)}$ is itself a clique. For the latter case, we use n elements to represent $G^{(n)}$ and note that $n^2/6 \geq n$ for $n \geq 6$. As for the former case, we lose all the edges joining this isolated clique to all the vertices not on this isolated clique, therefore we lose at least 5 edges from the calculated $\binom{n}{2}$ edges and hence further lose at least two cliques from the calculated $n^2/6$ cliques (as K_3). By Lemma 3.3, we need to use $r + 1$ more monopolized elements for this clique only when this clique has exactly $r + 2$ vertices of degree $(3 + r) - 1$. In this case, this clique has a vertex v adjacent to one vertex, say v', not in this clique, and all vertices in this cliques other than v are not adjacent to v'. Therefore in $G^{(n)}$ we have $r + 2 \geq 3$ less edges than complete graph K_n, and thus we have still one less triangle in $\mathbf{Q}$. Now we have brought to the conclusion that in case that each Q_j has at least three edges, we never use more than $n^2/6$ elements to form a representation of $G^{(n)}$. Now we have justified assuming some Q_j is an edge xy. In case that $d(x) = d(y) = 1$, we may first take a representation of $G^{(n)} - x - y$ by the method of Theorem 2.2 using at most $\lfloor (n-2)^2/4 \rfloor$ elements, and then use two new elements for the isolated edge xy to form a representation for $G^{(n)}$ with at most $\lfloor n^2/4 \rfloor$ elements. Thus in this case every representation of $G^{(n)}$ derived from $\mathcal{F}(\mathbf{Q})$, where $\mathbf{Q}$ is any greedy clique partition of $G^{(n)}$, uses at most $\lfloor n^2/4 \rfloor$ elements.

As for the case that one of x, y has degree more than one, in any representation of $G^{(n)}$ we can not use any monopolized element on x or y. Now let R consist of the members of $\mathbf{Q} - \{Q_j\}$ that are incident to x, and S consist of those incident to y. Then the set

$$\mathbf{Q}' = \mathbf{Q} - (R \cup S \cup \{Q_j\})$$

is a greedy clique partition of

$$G' \equiv (G^{(n)} - x - y) - \bigcup_{Q_i \in R \text{ or } S} E(Q_i),$$

except possibly leaving some isolated vertices in G' uncovered by any members of $\mathbf{Q}'$. Recall that for the present case, in $\mathcal{F}$ we never use any monopolized element on x, y. Now if we can prove that every monopolized element in $\mathbf{S}(\mathcal{F})$ is always necessary for deriving a representation of G' from $\mathcal{F}(\mathbf{Q}')$, then by induction hypothesis we prove that

$$|\mathcal{Q}(\mathcal{F}) - (R \cup S \cup \{Q_j\})| \leq \lfloor (n-2)^2/4 \rfloor. \tag{2}$$

If in $\mathcal{F}$ we used one monopolized element on some vertex v not belonging to any member of $R \cup S$, then in $\mathcal{F}(\mathbf{Q})$, the set $S_{\mathbf{Q}}(v)$ must be identical with some $S_{\mathbf{Q}}(u)$ where u is also a vertex not belonging to any member of $R \cup S$. Since both u and v do not belong to any member of $R \cup S$, then $S_{\mathbf{Q}'}(u) = S_{\mathbf{Q}'}(v)$ in $\mathcal{F}(\mathcal{Q}')$. Thus this monopolized element is necessary for deriving a representation of G' from $\mathcal{F}(\mathbf{Q}')$.

If in $\mathcal{F}$ we used one monopolized element on some vertex v belonging to one member, say Q_v, of $R \cup S$. Then in $\mathcal{F}(\mathbf{Q})$, the set $S_{\mathbf{Q}}(v)$ must be identical with some $S_{\mathbf{Q}}(u)$ where u is also a vertex belonging to Q_v. Now by Lemma 3.3 v must have all its neighbors in Q_v. Thus v is an isolated vertex in G'. Thus this monopolized element is necessary for deriving a representation of G' from $\mathcal{F}(\mathbf{Q}')$. Thus we have proved the statement (2).

Now it suffices to prove that

$$|R \cup S| \leq n - 2,$$

since

$$n - 2 \leq \lfloor n^2/4 \rfloor - \lfloor (n-2)^2/4 \rfloor - 1.$$

We prove this by choosing distinct vertices in $V(G) - \{x, y\}$ from the vertex sets of the members of $R \cup S$. Note that since each edge is covered exactly once in a clique partition, each $v \notin \{x, y\}$ appears once in R if v is adjacent to x and once in S if v is adjacent to y. Consider $Q_1 \in R$. If Q_1 contains a vertex v not adjacent to y, then we choose such a v for Q_1. If all vertices in Q_1 are adjacent to y, then we choose a vertex $v \in Q_1$ such that vy belongs to the first member of $\mathbf{Q}$, say Q_2, which contains both y and some vertex of Q_1. Note that Q_2 is the only member of S containing v.

Now we have two cases, that is, either that Q_1 precedes xy in $\mathbf{Q}$ or that xy precedes Q_1 in $\mathbf{Q}$. For the first case, since Q_1 and xy are maximal while chosen, Q_2 must precedes Q_1 in $\mathbf{Q}$ for otherwise from the aforementioned hypothesis that all vertices in Q_1 are adjacent to y and Q_1 precedes xy in $\mathbf{Q}$, Q_1 should have contained y and hence xy. For the second case, since xy is maximal while chosen, one of Q_1, Q_2 precedes xy or otherwise xy should have contained v. Thus in this case Q_2 precedes Q_1 in $\mathbf{Q}$. Note that in both cases, we have that Q_2 precedes both of Q_1, xy in $\mathbf{Q}$.

For the members of S, similarly as above choose vertices by reversing the roles of x and y.

In above we have shown that if v belongs to some $Q_1 \in R$ and to some $Q_2 \in S$, and v is chosen for one of them, then the one for which it is chosen occurs after the other one in the ordered set $\mathbf{Q}$. Hence no vertex is chosen twice. Thus we conclude that

$$|R \cup S| \leq n - 2.$$

$\square$

4 Conclusion Remarks

The edge clique partitions, as a special case of edge clique covers, are served as great classifying and clustering tools in many practical applications, therefore it is interesting to explore the concept in more details.

One may work on the cases besides multifamily and family, say antichain, uniform family etc. The greedy way to obtain these variants also naturally gives the optimal upper bounds for the corresponding intersection numbers.

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