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ON GREEN'S FUNCTION OF AN n-POINT BOUNDARY VALUE PROBLEM

BY

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ABSTRACT. The Green's function $g_n(x, s)$ for an n-point boundary value problem, $y^{(n)}(x)=0$, $y(a_1) = y(a_2) = \cdots = y(a_n) = 0$ is explicitly given. As a tool for discussing $\operatorname{sgn} g_n(x, s)$ on the square $[a_1, a_n] \times [a_1, a_n]$, some results about polynomials with coefficients as symmetric functions of a's are obtained. It is shown that

$$\int_{a_1}^{a_n} |g_n(x, s)|\, ds$$

is a suitable polynomial in x. Applications to n-point boundary value problems and lower bounds for a_m $(m \geq n)$ are included.

1. Introduction. Beesack [1] considered the boundary value problem

$$\begin{aligned} &y^{(n)}(x) = 0, \\ &y(a_i) = y'(a_i) = \cdots = y^{(k_i)}(a_i) = 0 \qquad (1 \leq i \leq r), \end{aligned} \tag{1.1}$$

where $a_1 < a_2 < \cdots < a_r$, $0 \leq k_i$, $k_1 + k_2 + \cdots + k_r = n - r$. For the Green's function $g_n(x, s)$ he proved that

$$|g_n(x, s)| \leq \frac{\prod_{i=1}^r |x - a_i|^{k_i+1}}{(n-1)!(a_r - a_1)}. \tag{1.2}$$

In [2] Nehari gave a short proof of the same when $r = n$. Since in relation to multipoint boundary value problem as in [1], $\int_{a_1}^{a_r} |g_n(x, s)|\, ds$ appears (see 3.5 there), the natural question is to consider alternately this function.

In this paper, we consider (1.1) when $r = n$ first. In §2, we give the Green's function $g_n(x, s)$ explicitly and alternately exhibit it in a form which yields conclusions as to the sign of $g_n(x, s)$. In §3, the results about $\operatorname{sgn} g_n(x, s)$ and the identity

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$$\int_{a_1}^{a_n} |g_n(x, s)|\, ds = \frac{1}{n!}(x - a_1)(a_n - x) \prod_{i=2}^{n-1} |x - a_i| \tag{1.3}$$

are obtained. (There are a few auxiliary results given as lemmas which may be of some independent interest!) Applications to n-point boundary value problems and lower bounds for the mth zero of solutions form the contents of §4.

2. **The Green's function.** Throughout, n denotes a fixed natural number greater than 2. Let k be a natural number such that $2 < k \le n$. Consider the boundary value problem

$$\begin{aligned} &y^{(k)}(x) = 0, \\ &y(a_1) = y(a_2) = \cdots = y(a_{k-1}) = y(a_n) = 0, \end{aligned} \tag{2.1}$$

where $a_1 < a_2 < \cdots < a_{k-1} < a_n$.

Theorem 2.1. *The Green's function* $g_k(x, s)$ *for* (2.1) *is given by*

$(k-1)!g_k(x, s)$

$$= \left(\prod_{i=1}^{k-1} \frac{x - a_i}{a_n - a_i}\right)(a_n - s)^{k-1}, \qquad x \le s,\ a_{k-1} \le s;$$

$$\begin{aligned} = &\left(\prod_{i=1}^{k-1} \frac{x - a_i}{a_n - a_i}\right)(a_n - s)^{k-1} \\ &+ \sum_{j=2}^{r+1} (-1)^j \left(\prod_{\substack{i=1 \\ (i \ne k-j+1)}}^{k-1} \frac{x - a_i}{|a_{k-j+1} - a_i|} \right) \frac{a_n - x}{a_n - a_{k-j+1}} (a_{k-j+1} - s)^{k-1} \end{aligned}$$

$$(\equiv g_k^r(x, s)), \qquad x \le s,\ a_{k-r-1} \le s \le a_{k-r}\ (r = 1, \cdots, k-3); \tag{2.2}$$

$$= \left(\prod_{i=2}^{k-1} \frac{x - a_i}{a_i - a_1}\right) \frac{a_n - x}{a_n - a_1}(s - a_1)^{k-1} + (-1)^{k-1}(s - x)^{k-1}, \qquad x \le s,\ s \le a_2;$$

$$= \left(\prod_{i=1}^{k-1} \frac{x - a_i}{a_n - a_i}\right)(a_n - s)^{k-1} - (x - s)^{k-1}, \qquad a_{k-1} \le s \le x;$$

$$= g_k^r(x, s) - (x - s)^{k-1}, \qquad s \le x,\ a_{k-r-1} \le s \le a_{k-r}\ (r = 1, \cdots, k-3);$$

$$= \left(\prod_{i=2}^{k-1} \frac{x - a_i}{a_i - a_1}\right) \frac{a_n - x}{a_n - a_1}(s - a_1)^{k-1}, \qquad s \le x,\ s \le a_2;$$

where a product for empty set of indices is interpreted as 1.

Remark. Here, as well as in the following, r ranging over a vacuous set of indices means the collapse of regions of the form $x \le s$ $(x \ge s)$, $a_{k-r-1} \le s \le a_{k-r}$.

Proof. Starting with the Green's function for $y''(x) = 0$, $y(a_1) = y(a_2) = 0$, namely

$$g_2(x, s) = \begin{cases} (x - a_1)(a_n - s)/(a_n - a_1), & x \le s; \\ (a_n - x)(s - a_1)/(a_n - a_1), & s \le x; \end{cases}$$

and the relation in [1],

$$(2.3) \quad g_{m+1}(x, s) = \frac{1}{m}\left\{(x - s)g_m(x, s) - (a_m - s)g_m(a_m, s)\left(\prod_{i=1}^{m-1} \frac{x - a_i}{a_m - a_i}\right)\frac{a_n - x}{a_n - a_m}\right\},$$

it is easily checked that

$$2!g_3(x, s) = \begin{cases} \dfrac{(x - a_1)(x - a_2)}{(a_n - a_1)(a_n - a_2)}(a_n - s)^2, & x \le s,\ a_2 \le s; \\[2ex] \dfrac{(x - a_2)(a_n - x)}{(a_2 - a_1)(a_n - a_1)}(s - a_1)^2 + (s - x)^2, & x \le s,\ s \le a_2; \\[2ex] \dfrac{(x - a_1)(x - a_2)}{(a_n - a_1)(a_n - a_2)}(a_n - s)^2 - (x - s)^2, & s \le x,\ a_2 \le s; \\[2ex] \dfrac{(x - a_2)(a_n - x)}{(a_2 - a_1)(a_n - a_1)}(s - a_1)^2, & s \le x,\ s \le a_2. \end{cases}$$

Thus (2.2) is valid for $k = 3$. Again, assuming that (2.2) holds when $k = m$, if $x \le s$, $a_m \le s$, then (2.3) gives

$$m!g_{m+1}(x, s) = \left(\prod_{i=1}^{m-1} \frac{x - a_i}{a_n - a_i}\right)\left\{(x - s) - (a_m - s)\frac{a_n - x}{a_n - a_m}\right\}(a_n - s)^{m-1}$$

$$= \left(\prod_{i=1}^{m} \frac{x - a_i}{a_n - a_i}\right)(a_n - s)^m.$$

Similarly, if $x \le s$, $a_{m-r} \le s \le a_{m-r+1}$, where $r = 2, \cdots, (m+1) - 3$, (2.3) yields

$$m!g_{m+1}(x,s)$$
$$= \left((x-s)-(a_m-s)\frac{a_n-x}{a_n-a_m}\right)\left(\prod_{i=1}^{m-1}\frac{x-a_i}{a_n-a_i}\right)(a_n-s)^{m-1} + \left(\prod_{i=1}^{m-1}\frac{x-a_i}{a_m-a_i}\right)\frac{a_n-x}{a_n-a_m}(a_m-s)^m$$
$$+ \sum_{j=2}^{r}(-1)^j\left(\prod_{\substack{i=1\\(i\neq m-j+1)}}^{m-1}\frac{x-a_i}{|a_{m-j+1}-a_i|}\right)\frac{a_n-x}{a_n-a_{m-j+1}}$$
$$\cdot (a_{m-j+1}-s)^{m-1}\left((x-s)-(a_m-s)\frac{x-a_{m-j+1}}{a_m-a_{m-j+1}}\right)$$
$$= \left(\prod_{i=1}^{m}\frac{x-a_i}{a_n-a_i}\right)(a_n-s)^m + \sum_{j=1}^{r}(-1)^{j+1}\left(\prod_{\substack{i=1\\(i\neq m-j+1)}}^{m}\frac{x-a_i}{|a_{m-j+1}-a_i|}\right)\frac{a_n-x}{a_n-a_{m-j+1}}(a_{m-j+1}-s)^m.$$

A similar computation using the first expression of (2.2) with $k=m$ gives the result for $x \le s$, $a_{m-1} \le s \le a_m$. Moreover, if $x \le s \le a_2$, then

$$m!g_{m+1}(x,s)$$
$$= (-1)^m(s-x)^m + \left(\prod_{i=2}^{m-1}\frac{x-a_i}{a_i-a_1}\right)\frac{a_n-x}{a_n-a_1}(s-a_1)^{m-1}\left((x-s)-(a_m-s)\frac{x-a_1}{a_m-a_1}\right)$$
$$= \left(\prod_{i=2}^{m}\frac{x-a_i}{a_i-a_1}\right)\frac{a_n-x}{a_n-a_1}(s-a_1)^m + (-1)^m(s-x)^m.$$

This completes the induction argument for the triangle $x \le s$. On the same lines the region $a_1 \le s \le x \le a_n$ can be handled. Hence the conclusion.

Corollary 2.2. *Alternatively, if* $s \le x$ *and* $a_{k-r-1} \le s \le a_{k-r}$ *where* $r = 1, \cdots, k-3$, *we have*

$$(2.4)\quad (k-1)!g_k(x,s) = (a_n-x)\sum_{l=0}^{k-r-2}(x-s)^l\left(\prod_{i=l+2}^{k-1}(x-a_i)\right)(s-a_{l+1})$$
$$\cdot\left\{\frac{(a_n-s)^{k-l-2}}{\prod_{i=l+1}^{k-1}(a_n-a_i)} + \sum_{j=2}^{r+1}(-1)^{j+1}\frac{(a_{k-j+1}-s)^{k-l-2}}{\prod_{i=l+1\,(i\neq k-j+1)}^{k-1}|a_{k-j+1}-a_i|}\,\frac{1}{a_n-a_{k-j+1}}\right\}.$$

Proof. First observe that $(x-a_i)(a_m-s) = (a_m-a_i)(x-s) + (s-a_i)(a_m-x)$, $i \ne m$. Applying this first with $i=1$ and $m=n$ or $k-j+1$—note that m is always different from 1—, we get

$$g_k^r(x,s)-(x-s)^{k-1}=(a_n-x)\left[\left(\prod_{i=2}^{k-1}(x-a_i)\right)(s-a_1)\right.$$
$$\left.\cdot\left\{\frac{(a_n-s)^{k-2}}{\prod_{i=1}^{k-1}(a_n-a_i)}+\sum_{j=2}^{r+1}(-1)^{j+1}\frac{(a_{k-j+1}-s)^{k-2}}{\prod_{i=1\,(i\neq k-j+1)}^{k-1}|a_{k-j+1}-a_i|}\,\frac{1}{a_n-a_{k-j+1}}\right\}\right]$$
$$+(x-s)\left[\left(\prod_{i=2}^{k-1}\frac{x-a_i}{a_n-a_i}\right)(a_n-s)^{k-2}-(x-s)^{k-2}\right.$$
$$\left.+\sum_{j=2}^{r+1}(-1)^j\left(\prod_{\substack{i=2\\(i\neq k-j+1)}}^{k-1}\frac{x-a_i}{|a_{k-j+1}-a_i|}\right)\frac{a_n-x}{a_n-a_{k-j+1}}(a_{k-j+1}-s)^{k-2}\right].$$

Repeating the above with $i=2,\cdots,k-r-1$ and $m=n$ or $k-j+1$ (once again $i\neq m$ always) on the last term each time, we finally obtain

$$g_k^r(x,s)-(x-s)^{k-1}=(a_n-x)\sum_{l=0}^{k-r-2}(x-s)^l\left(\prod_{i=l+2}^{k-1}(x-a_i)\right)(s-a_{l+1})$$
$$\cdot\left\{\frac{(a_n-s)^{k-l-2}}{\prod_{i=1}^{k-1}(a_n-a_i)}+\sum_{j=2}^{r+1}(-1)^{j+1}\frac{(a_{k-j+1}-s)^{k-l-2}}{\prod_{i=l+1\,(i\neq k-j+1)}^{k-1}|a_{k-j+1}-a_i|}\,\frac{1}{a_n-a_{k-j+1}}\right\}$$
$$+(x-s)^{k-r-1}\left[\left(\prod_{i=k-r}^{k-1}\frac{x-a_i}{a_n-a_i}\right)(a_n-s)^r-(x-s)^r\right.$$
$$\left.+\sum_{j=2}^{r+1}(-1)^j\left(\prod_{\substack{i=k-r\\(i\neq k-j+1)}}^{k-1}\frac{x-a_i}{|a_{k-j+1}-a_i|}\right)\frac{a_n-x}{a_n-a_{k-j+1}}(a_{k-j+1}-s)^r\right].$$

Now (2.4) follows from the above in view of the fact that the factor multiplying $(x-s)^{k-r-1}$ is a polynomial of degree r in x and takes the value zero at $a_{k-r},\cdots,a_{k-1}$ and a_n.

3. We first give some auxiliary results in the form of lemmas.

Lemma 3.1. *For each* r $(=1,\cdots,k-3)$ *if* ρ*, a natural number, does not exceed* r*; then*

$$(3.1)\quad \frac{(a_n-s)^{\rho-1}}{\prod_{i=k-r}^{k-1}(a_n-a_i)}+\sum_{j=2}^{r+1}(-1)^{j+1}\frac{(a_{k-j+1}-s)^{\rho-1}}{\prod_{i=k-r\,(i\neq k-j+1)}^{k-1}|a_{k-j+1}-a_i|}\,\frac{1}{a_n-a_{k-j+1}}\equiv 0$$

$$(\equiv A_{r,\rho}^{(k)}(s))$$

and hence

$$(3.2)\quad \frac{(a_n-s)^{\rho}}{\prod_{i=k-r-1}^{k-1}(a_n-a_i)}+\sum_{j=2}^{r+1}(-1)^{j+1}\frac{(a_{k-j+1}-s)^{\rho}}{\prod_{i=k-r-1\,(i\neq k-j+1)}^{k-1}|a_{k-j+1}-a_i|}\,\frac{1}{a_n-a_{k-j+1}}$$

$$(\equiv B_{r,\rho}^{(k)}(s))$$

is divisible by $(a_{k-r-1}-s)^{\rho}$.

Proof. That (3.2) has $(a_{k-r-1}-s)^{\rho}$ as a factor is immediate when relations (3.1) are known to be true as seen from the identity

$$B^{(k)}_{r,\rho}(s)=(a_{k-r-1}-s)^{\rho}B^{(k)}_{r,0}+\sum_{\sigma=0}^{\rho-1}(a_{k-r-1}-s)^{\sigma}A^{(k)}_{r,\rho-\sigma}(s)$$

obtained by successive use of

$$B^{(k)}_{r,\rho-\sigma}(s)=(a_{k-r-1}-s)B^{(k)}_{r,\rho-\sigma-1}+A^{(k)}_{r,\rho-\sigma}(s),\qquad \sigma=0,1,\cdots,\rho-1.$$

To establish (3.1) for arbitrary r ($\leq k-3$) and $\rho=1$, it is sufficient to observe that the polynomial

$$1+\sum_{j=2}^{r+1}(-1)^{j+1}\left(\prod_{\substack{i=k-r\\(i\neq k-j+1)}}^{k-1}\frac{t-a_i}{|a_{k-j+1}-a_i|}\right)$$

(of degree $r-1$) vanishes at $a_{k-r},\cdots,a_{k-1}$ and hence identically. (This proves (3.1) when $r=1$.) Now assume that (3.1) holds when r (>1) is replaced by $r-1$ and for $\rho=1,\cdots,\sigma$ ($<r$). Then, if $\rho=\sigma+1$, (3.1) follows from the identity

$$A^{(k)}_{r,\sigma+1}(s)=A^{(k)}_{r-1,\sigma}(s)+(a_{k-r}-s)A^{(k)}_{r,\sigma}(s),\qquad 2\leq r\leq k-3,\ 1\leq\sigma<r.$$

This completes the proof by induction.

Remark. In the special case when $\rho=r$, (3.2) is a polynomial of degree r. In view of the above result, we shall write this polynomial as $C^{(k)}_r(a_{k-r-1}-s)^r$, where k is fixed and $C^{(k)}_r$ are constants for $r=1,\cdots,k-3$. In fact, $C^{(k)}_r=B^{(k)}_{r,0}$ as can be easily checked.

Lemma 3.2. *For each r ($=1,\cdots,k-3$), $C^{(k)}_r(a_{k-r-1}-s)^r$ is positive on (a_{k-r-1},a_{k-r}).*

Proof. It is easy to check that for $r=1$.

$$C^{(k)}_1(a_{k-2}-s)\equiv\frac{(s-a_{k-2})}{(a_n-a_{k-2})(a_{k-1}-a_{k-2})}$$

and thus, in addition to the conclusion, we have $C^{(k)}_1<0$.

Now assume that for $r=\sigma-1$, the conclusion is true and $\operatorname{sgn} C^{(k)}_{\sigma-1}=(-1)^{\sigma-1}$. Then, if $r=\sigma$, we have

$$C_\sigma^{(k)}(a_{k-\sigma-1}-s)^\sigma \equiv (a_{k-\sigma-1}-s)^{\sigma-1}$$

$$\cdot\left\{\frac{a_n - s}{\prod_{i=k-\sigma-1}^{k-1}(a_n-a_i)} + \sum_{j=2}^{\sigma+1}(-1)^{j+1}\frac{a_{k-j+1}-s}{\prod_{i=k-\sigma-1\,(i\neq k-j+1)}^{k-1}|a_{k-j+1}-a_i|}\,\frac{1}{a_n-a_{k-j+1}}\right\}$$

$$(\equiv (a_{k-\sigma-1}-s)^{\sigma-1}H(s)),$$

in view of

$$B_{\sigma,\sigma}^{(k)}(s) = (a_{k-\sigma-1}-s)^{\sigma-1}B_{\sigma,1}^{(k)}(s).$$

Note that

$$H(a_{k-\sigma-1}) = A_{\sigma,1}^{(k)}(a_{k-\sigma-1}) = 0.$$

Moreover, the sign of $H(s)$ is constant on $(a_{k-\sigma-1}, a_{k-\sigma})$ and is that of $H(a_{k-\sigma})$, namely

$$\left(\prod_{\substack{i=k-\sigma-1\\(i\neq k-\sigma)}}^{k-1}(a_n-a_i)\right)^{-1} + \sum_{j=2}^{\sigma}(-1)^{j+1}\left(\prod_{\substack{i=k-\sigma-1\\(i\neq k-j+1,\,k-\sigma)}}^{k-1}|a_{k-j+1}-a_i|\right)^{-1}(a_n-a_{k-j+1})^{-1}$$

which is $\operatorname{sgn} q(a_n)$, where

$$q(t) = 1 + \sum_{j=2}^{\sigma}(-1)^{j+1}\frac{t-a_{k-\sigma-1}}{a_{k-j+1}-a_{k-\sigma-1}}\left(\prod_{\substack{i=k-\sigma+1\\(i\neq k-j+1)}}^{k-1}\frac{t-a_i}{|a_{k-j+1}-a_i|}\right).$$

Also, $\operatorname{sgn} C_{\sigma-1}^{(k)} = \operatorname{sgn} p(a_n)$, where

$$p(t) = 1 + \sum_{j=2}^{\sigma}(-1)^{j+1}\prod_{\substack{i=k-\sigma\\(i\neq k-j+1)}}^{k-1}\frac{t-a_i}{|a_{k-j+1}-a_i|}.$$

In view of the facts that both polynomials $p(t)$ and $q(t)$ are of degree $\sigma-1$, have the same zeros $a_{k-\sigma+l}$ $(l = 1, 2, \cdots, \sigma-1)$, and $p(a_{k-\sigma}) = q(a_{k-\sigma-1}) = 1$, it follows that $\operatorname{sgn} H(a_{k-\sigma}) = \operatorname{sgn} C_{\sigma-1}^{(k)}$. Thus, $C_\sigma^{(k)}(a_{k-\sigma-1}-s)^\sigma$ is positive in $(a_{k-\sigma-1}, a_{k-\sigma})$ and $\operatorname{sgn} C_\sigma^{(k)} = (-1)^\sigma$. This completes the proof.

Lemma 3.3. *For all integers k, r, m such that $4 \leq k$ $(< n)$, $1 \leq r \leq k-3$, and $r \leq m \leq k-2$,*

$$A(k, r, m, s) \equiv \frac{(a_n-s)^m}{\prod_{i=k-m-1}^{k-1}(a_n-a_i)}$$

(3.3)

$$+\sum_{j=2}^{r+1}(-1)^{j+1}\frac{(a_{k-j+1}-s)^m}{\prod_{i=k-m-1\,(i\neq k-j+1)}^{k-1}|a_{k-j+1}-a_i|}\,\frac{1}{a_n-a_{k-j+1}}$$

is nonnegative on $[a_{k-r-1}, a_{k-r}]$.

Proof. First observe that the assertion follows from Lemma 3.2 if $m = r$ and k arbitrary, admissible. Also, if $r = 1$ and m, k admissible, then the identity

$$\left(\frac{a_n - s}{a_n - a_{k-m-2}} - \frac{a_{k-1} - s}{a_{k-1} - a_{k-m-2}}\right)\frac{(a_n - s)^m}{\prod_{i=k-m-1}^{k-1}(a_n - a_i)} \equiv \frac{(a_n - s)^m}{\prod_{i=k-m-2}^{k-2}(a_n - a_i)} \frac{s - a_{k-m-2}}{a_{k-1} - a_{k-m-2}}, \tag{3.4}$$

in view of $a_{k-1} \geq s \geq a_{k-2} > a_{k-m-2}$, implies

$$\frac{(a_n - s)^{m+1}}{\prod_{i=k-m-2}^{k-1}(a_n - a_i)} \geq \frac{a_{k-1} - s}{a_{k-1} - a_{k-m-2}} \frac{(a_n - s)^m}{\prod_{i=k-m-1}^{k-1}(a_n - a_i)}.$$

Thus $A(k, 1, m, s)$ is nonnegative by using induction on m.

Now we may assume $r \geq 2$ and thus admissible $k \geq 5$. Let the conclusion be true about $A(k, r, m, s)$ for admissible r, k. Note that in addition to (3.4) we have the identities

$$\left(\frac{a_{k-1} - s}{a_{k-1} - a_{k-m-2}} - \frac{a_{k-j+1} - s}{a_{k-j+1} - a_{k-m-2}}\right)\frac{(a_{k-j+1} - s)^m}{\prod_{i=k-m-1\,(i\neq k-j+1)}^{k-1}|a_{k-j+1} - a_i|} \equiv \frac{(a_{k-j+1} - s)^m}{\prod_{i=k-m-2\,(i\neq k-j+1)}^{k-2}|a_{k-j+1} - a_i|} \frac{s - a_{k-m-2}}{a_{k-1} - a_{k-m-2}}, \tag{3.5$_j$}$$

$$j = 3, \cdots, r+1.$$

Multiplying each (3.5$_j$) by $(-1)^j(a_n - a_{k-j+1})^{-1}$ and adding all to (3.4) we get

$$A(k, r, m+1, s) = \frac{a_{k-1} - s}{a_{k-1} - a_{k-m-2}} A(k, r, m, s) + \frac{s - a_{k-m-2}}{a_{k-1} - a_{k-m-2}} B,$$

$$B = A(k-1, r-1, m, s).$$

By induction hypothesis $A(k, r, m, s)$ as well as $A(k-1, r-1, m, s)$ are nonnegative on $[a_{k-r-1}, a_{k-r}]$ in view of the admissibility of $k-1$ and $r-1$ in addition to that of k and r.

The following theorem is the main result which leads to (1.3).

Theorem 3.4. *For $g_k(x, s)$ the following holds:*

$$\operatorname{sgn} g_k(x, s) = \begin{cases} 1, & (x, s) \in [a_{k-1}, a_n] \times [a_1, a_n], \\ (-1)^r, & (x, s) \in [a_{k-r-1}, a_{k-r}] \times [a_1, a_n], \quad r = 1, \cdots, k-2. \end{cases}$$

Proof. First we consider the triangle $a_1 \le s \le x \le a_n$. The conclusion about $\operatorname{sgn} g_k(x, s)$ in this triangle is obvious from (2.2) when $s \le a_2$, and immediate when $a_{k-1} \le s \le a_n$ since $(x - a_i)(a_n - s) \ge (a_n - a_i)(x - s)$ for $i = 1, \cdots, k-1$. Also, if for $r = 1, \cdots, k-3$, $s \in [a_{k-r-1}, a_{k-r}]$, then the assertion about $\operatorname{sgn} g_k(x, s)$ follows from (2.4) in view of Lemma 3.3, noting that $l + 1 = k - m - 1$ and that

$$\operatorname{sgn}\left(\prod_{i=k-m}^{k-1} (x - a_i)\right) = \begin{cases} 1, & x \in [a_{k-1}, a_n], \\ (-1)^r, & x \in [a_{k-r-1}, a_{k-r}]. \end{cases}$$

To discuss the triangle $a_1 \le x \le s \le a_n$, we begin by observing that if $s \ge a_{k-1}$, then (2.2) at once gives the conclusion. For $s \le a_{k-1}$, we use induction. First note that $g_3(x, s)$ has the asserted signs. Now assume that $g_m(x, s)$ has the asserted signs. Then (2.3) shows that if $x \in [a_{m-1}, a_m]$, $-\operatorname{sgn} g_{m+1}(x, s) \ge 0$. Also, noting that if $x \in [a_{m-r}, a_{m-r+1}]$ where $r = 2, \cdots, m-1$, then $\operatorname{sgn} g_m(x, s) = \operatorname{sgn} (\prod_{i=1}^{m-1} (x - a_i))$, we have the desired conclusion for $k = m + 1$.

This completes the proof.

Theorem 3.5. *For any k ($\le n$) the following holds*:

$$\int_{a_1}^{a_n} |g_k(x, s)|\, ds = \frac{1}{k!} (x - a_1)(a_n - x)\left(\prod_{i=2}^{k-1} |x - a_i|\right). \tag{3.6}$$

Proof. In view of Theorem 3.4,

$$\int_{a_1}^{a_n} |g_k(x, s)|\, ds = \left|\int_{a_1}^{a_n} g_k(x, s)\, ds\right|. \tag{3.7}$$

The value of the integral on the right-hand side by (2.2) is

$$\frac{1}{k!}\left[\left(\prod_{i=2}^{k-1} \frac{x - a_i}{a_i - a_1}\right)\frac{a_n - x}{a_n - a_1}(a_2 - a_1)^k + \left(\prod_{i=1}^{k-1} \frac{x - a_i}{a_n - a_i}\right)(a_n - a_2)^k + (-1)^{k-1}(a_2 - x)^k\right]$$
$$+ \frac{1}{(k-1)!} \sum_{r=1}^{k-3} \int_{a_{k-r-1}}^{a_{k-r}} \left\{\sum_{j=2}^{r+1} (-1)^j \left(\prod_{\substack{i=1 \\ (i \ne k-j+1)}}^{k-1} \frac{x - a_i}{|a_{k-j+1} - a_i|}\right)\frac{a_n - x}{a_n - a_{k-j+1}} (a_{k-j+1} - s)^{k-1}\right\} ds,$$
$$x \in [a_1, a_2];$$

$$\frac{1}{k!}\left[\left(\prod_{i=2}^{k-1} \frac{x - a_i}{a_i - a_1}\right)\frac{a_n - x}{a_n - a_1}(a_2 - a_1)^k + \left(\prod_{i=1}^{k-1} \frac{x - a_i}{a_n - a_i}\right)(a_n - a_2)^k - (x - a_2)^k\right]$$
$$+ \frac{1}{(k-1)!} \sum_{r=1}^{k-3} \int_{a_{k-r-1}}^{a_{k-r}} \left\{\sum_{j=2}^{r+1} (-1)^j \left(\prod_{\substack{i=1 \\ (i \ne k-j+1)}}^{k-1} \frac{x - a_i}{|a_{k-j+1} - a_i|}\right)\frac{a_n - x}{a_n - a_{k-j+1}} (a_{k-j+1} - s)^{k-1}\right\} ds,$$
$$x \in [a_{k-1}, a_n] \text{ or } x \in [a_{k-l-1}, a_{k-l}],\ l = 1, \cdots, k-3.$$

Thus, whatever $x \in [a_1, a_n]$,

$$\int_{a_1}^{a_n} g_k(x, s)\,ds$$

$$(3.8)\quad = \frac{1}{k!}\left[\left(\prod_{i=2}^{k-1} \frac{x-a_i}{a_i - a_1}\right)\frac{a_n - x}{a_n - a_1}(a_2 - a_1)^k + \left(\prod_{i=2}^{k-1}\frac{x-a_i}{a_n - a_i}\right)(a_n - a_2)^k - (x - a_2)^k\right.$$
$$\left. + \sum_{j=2}^{k-2}(-1)^j\left(\prod_{\substack{i=1\\(i\neq k-j+1)}}^{k-1}\frac{x-a_i}{|a_{k-j+1} - a_i|}\right)\frac{a_n - x}{a_n - a_{k-j+1}}(a_{k-j+1} - a_2)^k\right].$$

It is easily seen that the expression in (3.8) is a polynomial (in x) of degree k which has zeros $a_1, \cdots, a_{k-1}$ and a_n. Moreover, the coefficient of x^k is $-1/k$, hence the conclusion in (3.6).

4. **Applications.** In this section $k = n$. Thus, consider the ordinary differential equation

$$(4.1)\qquad y^{(n)} + f(x, y, y', \cdots, y^{(n-1)}) = 0,$$

where f is continuous on $[a_1, a_n] \times R^n$ and satisfies

$$(4.2)\qquad |f(x, y, y', \cdots, y^{(n-1)})| \leq K|y|.$$

(The above hypothesis is evidently no more restrictive than that of Beesack—see (3.2) in [1].)

The following lemma gives a bound which is better than (2.13) of [1] in situations which are not "highly pathological" (see Remark below).

Lemma 4.1. *Let* $x \in [a_1, a_n]$. *Then,*

$$(4.3)\qquad \prod_{i=1}^{n}|x - a_i| \leq (n-1)^{n-1}\left(\frac{\delta}{2}\right)^n,$$

where $a_1 \leq a_2 \leq \cdots \leq a_n$, $\delta = \max_{2\leq i\leq n}(a_i - a_{i-1})$.

Proof. Let $x \in (a_r, a_{r+1})$, where $r\ (\geq 1) < [(n+1)/2]$, the integral part of $(n+1)/2$. Then,

$$\left[\{(n-2r+1)(x-a_1)\}(x-a_2)\cdots(x-a_r)\prod_{i=r+1}^{n}(a_i-x)\right]^{1/n}$$

$$\le \frac{1}{n}\left(\sum_{i=r+1}^{n}(a_i-a_1)-\sum_{i=1}^{r}(a_i-a_1)\right)$$

$$=\frac{1}{n}\left(\sum_{i=1}^{r-1}(n-2r+i)(a_{i+1}-a_i)+\sum_{i=r}^{n-1}(n-i)(a_{i+1}-a_i)\right)$$

$$\le \frac{n^2-n-2r(r-1)}{2n}\delta,$$

that is

$$\prod_{i=1}^{n}|x-a_i| \le \frac{1}{n-2r+1}\left(\frac{\delta}{2n}\right)^n (n^2-n-2r(r-1))^n.$$

Similarly, if $[(n+1)/2] \le r < n$, we have

$$\prod_{i=1}^{n}|x-a_i| \le \frac{1}{n-2(n-r)+1}\left(\frac{\delta}{2n}\right)^n \{n(n-1)-2(n-r)(n-r-1)\}^n.$$

It is easy to check that

$$f(r)=\{n(n-1)-2r(r-1)\}^n/(n-2r+1)$$

is nonincreasing for $(1 \le)\ r < [(n+1)/2]$ and $f(n-r)$ is nondecreasing for $([(n+1)/2] \le)\ r < n$. The estimate (4.3) follows in view of $f(1)=f(n-1)=n^n(n-1)^{n-1}$.

Remark. The bound in (4.3) is better than Beesack's if and only if

$$\delta < 2(a_n-a_1)/n.$$

If $n \ge 3$, this is always the case when the a_i's are equally spaced. In general, however, (2.13) of [1] gives a sort of best possible bound.

Theorem 4.2. *Let the boundary value problem* (4.1) *and*

$$y(a_1)=\cdots=y(a_n)=0, \quad a_1<a_2<\cdots<a_n, \tag{4.4}$$

have a solution. Then,

$$K^{-1} < \begin{cases} \dfrac{(n-1)^{n-1}}{n!}\left(\dfrac{\delta}{2}\right)^n, & \text{if } \delta \le \dfrac{2}{n}(a_n-a_1), \\[2ex] \dfrac{(n-1)^{n-1}}{n^n}\dfrac{(a_n-a_1)^n}{n!}, & \text{otherwise}, \end{cases} \tag{4.5}$$

where K and δ are as above.

Proof. (4.5) follows from the fact that $y(x)$ satisfies the integral equation

$$y(x) = \int_{a_1}^{a_n} g_n(x, s) f(s, y(s), \cdots, y^{(n-1)}(s))\, ds, \qquad x \in [a_1, a_n], \tag{4.6}$$

and thus identifying x with a point where $|y(x)|$ attains its maximum, we have

$$1 < K \int_{a_1}^{a_n} |g_n(x, s)|\, ds, \tag{4.7}$$

in view of (4.2).

Remark. The above result is an improvement on Beesack's necessary condition whenever the function $h(x)$ in his (3.2) is constant (of course, multiple zeros are not allowed). Apart from the case $h(x) \equiv K$, the two results are not comparable.

Next turning to the question of obtaining a lower bound for the mth zero of solutions of the linear differential equation

$$y^{(n)} + p(x)y = 0, \tag{4.8}$$

we state the following result:

Theorem 4.3. *Let $p(x)$ in (4.8) be continuous and bounded on $[a, \infty)$. If a_1 $(\geq a) < a_2 < \cdots < a_m$ are consecutive simple zeros of a solution of (4.8), then for $m > n$*

$$a_m > a_1 + \left(\frac{(m - n + 1)n!}{K}\left(\frac{n}{n - 1}\right)^{n-1}\right)^{1/n}, \tag{4.9}$$

where $|p(x)| \leq K$.

We omit the proof which is a straightforward adaptation of the above proof and of the proof of (3.15) in [1].

Remark. As in [1], if $m > 2n - 1$, in place of (4.9) we have the estimate

$$a_m > a_1 + (n/(n - 1))((m - n)n!/K)^{1/n}. \tag{4.10}$$

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