160. On Green's Functions of Elliptic and Parabolic Boundary Value Problems

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1. Introduction. Let $A(x,D)$ be an elliptic operator of order m defined in a domain Ω of R^n, and $B_j(x,D)$, $j=1,\cdots,m/2$, be operators of order $m_j<m$ defined on $\partial\Omega$. We assume

(i) the system $(A(x,D),\{B_j(x,D)\}_{j=1}^{m/2},\Omega)$ as well as its adjoint system $(A'(x,D),\{B'_j(x,D)\}_{j=1}^{m/2},\Omega)$ formally constructed are both regular systems in the sense of S. Amon [1];

(ii) there is an angle $\theta_0\in(0,\pi/2)$ such that $(e^{i\theta}D_t^m-A(x,D_x)$, $\{B_j(x,D_x)\}_{j=1}^{m/2}$, $\Omega\times(-\infty<t<\infty))$ is an elliptic boundary value problem satisfying the coerciveness condition for any $\theta\in[\theta_0,2\pi-\theta_0]$ (cf. S. Agmon [1]).

Let A be the operator defined by

$$D(A)=\{u\in H_m(\Omega): B_j(x,D)u=0 \text{ on } \partial\Omega, j=1,\cdots,m/2\}$$

and $(Au)(x)=A(x,D)u(x)$ for $u\in D(A)$. It is known that the operator defined analogously by the adjoint system $(A'(x,D),\{B'_j(x,D)\},\Omega)$ coincides with the adjoint of A (F.E. Browder [5], [6]).

In this paper we describe a method of establishing global estimates for the Green's function of the resolvent of A as well as the semigroup $\exp(-tA)$ generated by $-A$. Under the present assumptions the resolvent $(A-\lambda)^{-1}$ exists for λ in the set defined by $\Lambda=\{\lambda: \theta_0\leqq\arg\lambda\leqq 2\pi-\theta_0, |\lambda|>C_0\}$ for some $C_0>0$ ([1]) and $-A$ generates a semigroup which is analytic in the sector $\Sigma=\{t: |\arg t|<\pi/2-\theta_0\}$.

Theorem 1. *Let $K_\lambda(x,y)$ be the kernel of $(A-\lambda)^{-1}$. Then there exist constants C and $\delta>0$ such that*

(a) $|K_\lambda(x,y)|\leqq Ce^{-\delta|\lambda|^{1/m}|x-y|}|\lambda|^{n/m-1}$ *if* $m>n$,

(b) $|K_\lambda(x,y)|\leqq Ce^{-\delta|\lambda|^{1/m}|x-y|}|x-y|^{m-n}$ *if* $m<n$,

(c) $|K_\lambda(x,y)|\leqq Ce^{-\delta|\lambda|^{1/m}|x-y|}\{1+\log^+(|\lambda|^{-1/m}|x-y|^{-1})\}$ *if* $m=n$

for $x,y\in\Omega$ and $\lambda\in\Lambda$.

Theorem 2. *Let $G(x,y,t)$ be the kernel of* $\exp(-tA)$. *Then there exist positive constants C and c such that*

$$|G(x,y,t)|\leqq C|t|^{-n/m}\exp(-c|x-y|^{m/(m-1)}/|t|^{1/(m-1)})e^{C|t|}$$

for $x,y\in\Omega$ and $t\in\Sigma$.

Remark 1. The boundedness of Ω is required in the assumption (i); however, it is not essential. The same results remain valid if Ω is an unbounded domain uniformly regular of class C^m and locally regular of class C^{2m} in the sense of F.E. Browder [5],[6] and the system $(A(x,D), \{B_j(x, D)\})$ as well as its adjoint satisfies the assumptions stated above uniformly in $\bar{\Omega}$.

Remark 2. If the coefficients of $A(x, D)$ are Hölder continuous it would be possible to derive the Theorems with the aid of the result of R. Arima [3].

Remark 3. With the aid of Theorem 2 we may establish a result similar to that of K. Masuda [8] and H.B. Stewart [9] which asserts that $-A$ generates an analytic semigroup in the space of bounded and continuous functions vanishing at $\partial\Omega$ and at infinity if the boundary conditions are of Dirichlet type.

Remark 4. Using Theorem 1 it is possible to derive some global version of L. Hörmander's results ([7]) on the Riesz means of the spectral function of A if A is self-adjoint.

2. Outline of the proof of the theorems.

Lemma 1. *For $u \in H_m(\Omega)$ and $\lambda \in \Lambda$ we have*

$$|\lambda|\,\|u\|+\|u\|_m \leqq C\left\{\|(A(x,D)-\lambda)u\|+\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\|g_j\|+\sum_{j=1}^{m/2}\|g_j\|_{m-m_j}\right\}$$

where g_j is an arbitrary function in $H_{m-m_j}(\Omega)$ satisfying $B_j(x, D)u=g_j$ on $\partial\Omega$. The analogue holds for the adjoint system.

Proof. The Lemma is a slight modification of Theorem 2.1 of [1].

For $\eta \in R^n$ let A_η be the operator defined by the system $(A(x, D+i\eta), \{B_j(x, D+i\eta)\}, \Omega)$. Applying Lemma 1 to a function $u \in D(A_\eta)$ we get

Lemma 2. *There exist positive constants C and δ such that*

$$\|(A_\eta-\lambda)^{-1}\|_{L^2\to L^2}\leqq C/|\lambda|,$$
$$\|(A_\eta-\lambda)^{-1}\|_{L^2\to H_m}\leqq C,$$
$$\|((A_\eta-\lambda)^{-1})^*\|_{L^2\to H_m}\leqq C$$

for $\lambda \in \Lambda$ and $|\eta|\leqq\delta|\lambda|^{1/m}$.

If $K_\lambda^\eta(x, y)$ is the kernel of $(A_\eta-\lambda)^{-1}$, then $K_\lambda^\eta(x, y)=e^{\langle x-y,\eta\rangle}K_\lambda(x, y)$. Hence (*a*) of Theorem 1 is a simple consequence of Lemma 2 and Theorem 3.1 of S. Agmon [2].

In what follows we assume $C_0=0$ adding some positive constant to A if necessary (recall the definition of Λ).

Lemma 3 (R. Beals [4]). *If S and T are bounded operators from $L^2(\Omega)$ to itself such that the ranges of S and T^* are contained in $L^\infty(\Omega)$. Then the operator ST has a bounded kernel $k(x, y)$ satisfying*

$$|k(x, y)|\leqq\|S\|_{L^2\to L^\infty}\|T^*\|_{L^2\to L^\infty}.$$

Next we assume $m>n/2$. For $t \in \Sigma$ we have

$$\exp(-2tA)=(\exp(-tA))^2$$
$$(1)\qquad =\frac{1}{(2\pi i)^2}\int_\Gamma\int_\Gamma e^{-t\lambda}e^{-t\mu}(A-\lambda)^{-1}(A-\mu)^{-1}d\lambda d\mu.$$

In view of Lemma 2 and Sobolev's inequality we have if $|\eta|\leqq\delta\min(|\lambda|^{1/m},|\mu|^{1/m})$

$$\|(A_\eta-\lambda)^{-1}\|_{L^2\to L^\infty}\leqq C|\lambda|^{n/2m-1},$$
$$\|((A_\eta-\mu)^{-1})^*\|_{L^2\to L^\infty}\leqq C|\mu|^{n/2m-1}.$$

Hence by Lemma 3 we see that the kernel $K^\eta_{\lambda,\mu}(x,y)$ of $(A_\eta-\lambda)^{-1}(A_\eta-\mu)^{-1}$ satisfies

$$|K^\eta_{\lambda,\mu}(x,y)|\leqq C|\lambda|^{n/2m-1}|\mu|^{n/2m-1}. \tag{2}$$

If $K_{\lambda,\mu}(x,y)$ is the kernel of $(A-\lambda)^{-1}(A-\mu)^{-1}$, it is readily seen that $K^\eta_{\lambda,\mu}(x,y)=e^{\langle x-y,\eta\rangle}K_{\lambda,\mu}(x,y)$. Hence in view of (2) we get

$$\begin{aligned}&|K_{\lambda,\mu}(x,y)|\\&\quad\leqq C|\lambda|^{n/2m-1}|\mu|^{n/2m-1}\exp\{-\delta\min(|\lambda|^{1/m},|\mu|^{1/m})|x-y|\}\\&\quad\leqq C|\lambda|^{n/2m-1}|\mu|^{n/2m-1}\{e^{-\delta|\lambda|^{1/m}|x-y|}+e^{-\delta|\mu|^{1/m}|x-y|}\}.\end{aligned}$$

Comparing the kernels of the members of (1) and then deforming Γ to

$$\{\lambda:\lambda=re^{\pm i\theta_0},r\geqq a\}\cup\{\lambda:\lambda=ae^{i\phi},\theta_0\leqq\phi\leqq2\pi-\theta_0\}$$

where $a=\varepsilon(|x-y|/|t|)^{m/(m-1)}$ we get without difficulty

$$\begin{aligned}&|G(x,y,2t)|\\&\quad\leqq C|t|^{-n/m}\exp\{-(\delta\varepsilon^{1/m}-4\varepsilon)|x-y|^{m/(m-1)}/|t|^{1/(m-1)}\}.\end{aligned}$$

Taking ε sufficiently small we get Theorem 2 for the case $m>n/2$. The case $m\leqq n/2$ can be dealt with following the method of R. Beals [4]. The assertions (b) and (c) of Theorem 1 can be established by Theorem 2 and

$$(A-\lambda)^{-1}=\int e^{\lambda t}\exp(-tA)dt$$

where we integrate along the ray $\{t=|t|e^{\pm i\theta_0}\}$ according as $\operatorname{Im}\lambda\gtrless0$.

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