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ON GROUPS IN WHICH IDEMPOTENT REDUCTS FORM A CHAIN

BY

J. PŁONKA (WROCŁAW)

For a given algebra $\mathfrak{A} = (A, F)$ we denote by $A(F)$ the family of its all algebraic operations. An algebra $\mathfrak{B} = (A, G)$ is called a *reduct* of $\mathfrak{A} = (A, F)$ if $G \subseteq A(F)$. We write then $\mathfrak{B} \leqslant \mathfrak{A}$. If $\mathfrak{A} \leqslant \mathfrak{B}$ and $\mathfrak{B} \leqslant \mathfrak{A}$, we put $\mathfrak{A} = \mathfrak{B}$. An operation f is generated by a set S of operations if f is a superposition of operations from S and trivial operations.

In [2] we have proved that

(*) In the idempotent reduct of any group $\mathfrak{G} = (G, \cdot, {}^{-1}, 1)$, the two ternary algebraic operations $x^{-1} \cdot y \cdot z$ and $x \cdot y \cdot z^{-1}$ generate every idempotent algebraic operation in $\mathfrak{G}$. If $\mathfrak{G}$ is abelian, then these operations generate each other.

Further, if $\mathfrak{G}$ has the exponent m, then the algebraic operation $x_1^{k_1} \cdot x_2^{k_2} \cdot \ldots \cdot x_n^{k_n}$ is idempotent iff

$$k_1 + k_2 + \ldots + k_n \equiv 1 \pmod{m}.$$

This last operation we shall often denote by

$$x_1 \cdot x_2^{k_2} \cdot \ldots \cdot x_n^{k_n},$$

because k_1 is determined by the idempotency of this operation, i.e.

$$k_1 \equiv 1 - (k_2 + \ldots + k_n) \pmod{m}.$$

In this paper we prove that if $\mathfrak{G}$ is an abelian group with the exponent p^k, where p is a prime and $k \geqslant 0$, then the idempotent reducts of $\mathfrak{G}$ form a $(k+1)$-element chain with respect to the relation $\leqslant$. Moreover, abelian groups with the exponents p^k are the only groups in which idempotent reducts form a chain. The problem of characterizing such groups was suggested to me by G. H. Wenzel during the conference in Mannheim, November 1971.

LEMMA 1. *The idempotent operation* $x_1 \cdot x_2^{q \cdot p^l}$ $(1 \leqslant l \leqslant k)$ *generates every idempotent operation of the form*

$$x_1 \cdot x_2^{q \cdot p^l} \cdot x_3^{q \cdot p^l} \cdot \ldots \cdot x_r^{q \cdot p^l} \quad (r \geqslant 2, 1 \leqslant q < p).$$

Proof. For $r = 2$ the lemma is obvious. Suppose we have already constructed the operation

$$f(x_1, \ldots, x_n) = x_1 \cdot x_2^{q\cdot p^l} \cdot \ldots \cdot x_n^{q\cdot p^l}.$$

Consider the operation

$$\begin{aligned} f_1(x_1, \ldots, x_{n+1}) &= f\big(f(x_1, x_1, \ldots, x_1, x_{n+1}), x_2, \ldots, x_n\big) \\ &= x_1 \cdot x_{n+1}^{s\cdot q\cdot p^l} \cdot x_2^{q\cdot p^l} \cdot \ldots \cdot x_n^{q\cdot p^l} \end{aligned}$$

and, by induction, let

$$\begin{aligned} f_t(x_1, \ldots, x_{n+1}) &= f\big(f_t(x_1, x_1, \ldots, x_1, x_{n+1}), x_2, \ldots, x_n\big) \\ &= x_1 \cdot x_{n+1}^{s^t\cdot q\cdot p^l} \cdot x_2^{q\cdot p^l} \cdot \ldots \cdot x_n^{q\cdot p^l} \quad \text{for } t \geqslant 2. \end{aligned}$$

Since $s = p^k + 1 - (n-1)q\cdot p^l$, s and p^k are relatively prime. Thus, there exists t_0 such that $s^{t_0} \equiv 1 \pmod{p^k}$. Hence

$$f_{t_0}(x_1, \ldots, x_{n+1}) = x_1 (x_{n+1}^{s^{t_0}})^{q\cdot p^l} \cdot x_2^{q\cdot p^l} \cdot \ldots \cdot x_n^{q\cdot p^l} = x_1^{s_0} \cdot x_2^{q\cdot p^l} \cdot \ldots \cdot x_n^{q\cdot p^l} \cdot x_{n+1}^{q\cdot p^l}.$$

LEMMA 2. *The idempotent operations* $x_1 \cdot x_2^{p^l}$ *and* $x_1 \cdot x_2^{q\cdot p^l}$ $(1 < q < p)$ *generate each other.*

Proof. The operation $x_1 \cdot x_2^{p^l}$ generates the other. It is enough to put in Lemma 1 $q = 1$, $r = q+1$ and $x_2 = \ldots = x_r$. On the other hand, since q and p^k are relatively prime, there exists an integer m such that

$$q \cdot m \equiv 1 \pmod{p^k}.$$

Putting in Lemma 1 $r = m+1$ and $x_2 = \ldots = x_r$, we get the first operation.

Now consider the sequence of reducts of $\mathfrak{G}$,

$$(1) \quad \mathfrak{G}_0 = (G, \boldsymbol{I}),\ \mathfrak{G}_1 = (G, x_1^{s_1} \cdot x_2^{p}),\ \mathfrak{G}_2 = (G, x_1^{s_2} \cdot x_2^{p^2}), \ldots, \mathfrak{G}_k = (G, x_1^{s_k} \cdot x_2^{p^k}),$$

where $\boldsymbol{I}$ is the set of all idempotent operations in $\mathfrak{G}$.

LEMMA 3. *The idempotent reducts* G_i *from* (1), $0 \leqslant i \leqslant k$, *are the only idempotent reducts in* $\mathfrak{G}$.

Proof. Let $\hat{\mathfrak{G}} = (G, \boldsymbol{F})$ be an arbitrary idempotent reduct of $\mathfrak{G}$. If there exists an operation

$$f = x_1^{q_1} \cdot x_2^{q_2} \cdot \ldots \cdot x_n^{q_n}$$

in $\mathfrak{G}$ such that two different exponents q_i and q_j are relatively prime to p, then, as in Theorem 1 of [1], we can prove that $\hat{\mathfrak{G}} = \mathfrak{G}_0$; otherwise, any algebraic operation in $\hat{\mathfrak{G}}$ would be of the form

$$x_1 \cdot x_2^{q_2 \cdot p^{l_2}} \cdot \ldots \cdot x_n^{q_n \cdot p^{l_n}}, \quad \text{where } 1 \leqslant q_i < p.$$

Let

$$l_0(f) = \min l_j \quad \text{for any } f$$

and

$$l_0 = \min\{l_0(f) : f \in F\}.$$

Let $f_0(x_1, \ldots, x_n)$ be an algebraic operation belonging to F such that l_0 appears among its exponents. We can assume

$$f_0 = x_1 \cdot x_2^{q_2 \cdot p^{l_2}} \cdot \ldots \cdot x_n^{q_n \cdot p^{l_n}}, \quad \text{where } l_n = l_0.$$

Consider the algebraic operation

$$g(x_1, x_2) = f_0(x_1, x_1, \ldots, x_1, x_2) = x_1 \cdot x_2^{q \cdot p^{l_0}}, \quad \text{where } q_n = q.$$

By Lemma 2, the operation g generates the operation $x_1 \cdot x_2^{p^{l_0}}$. Thus $\mathfrak{G}_{l_0} \leqslant \hat{\mathfrak{G}}$.

We now prove that $\hat{\mathfrak{G}} \leqslant \mathfrak{G}_{l_0}$. Let

$$f(x_1, \ldots, x_n) = x_1 \cdot x_2^{q_2 \cdot p^{l_2}} \cdot \ldots \cdot x_n^{q_n \cdot p^{l_n}}.$$

Write $r_i = q_i \cdot p^{l_i}/p^{l_0}$ $(i = 2, 3, \ldots, n)$. Putting in Lemma 1 $q = 1$ and $l = l_0$ we see that the operation $x_1 \cdot x_2^{p^{l_0}}$ generates the following one:

$$x_1 \cdot x_2^{(1)p^{l_0}} \cdot x_2^{(2)p^{l_0}} \cdot \ldots \cdot x_2^{(r_1)p^{l_0}} \cdot x_3^{(1)p^{l_0}} \cdot \ldots \cdot x_3^{(r_3)p^{l_0}} \cdot \ldots \cdot x_n^{(1)p^{l_0}} \cdot \ldots \cdot x_n^{(r_n)p^{l_0}}.$$

Putting in the last operation $x_i^{(j)} = x_i$ for $i = 2, 3, \ldots, n$, we get $f(x_1, \ldots, x_n)$.

THEOREM 1. *If $\mathfrak{G}$ is an abelian group with the exponent p^k, $k \geqslant 0$, p — prime, then the idempotent reducts of $\mathfrak{G}$ form a $(k+1)$-element chain with respect to the relation $\leqslant$.*

Proof. For $k = 0$ the proof is trivial. For $k = 1$ our theorem follows from Theorem 1 of [1]. If $k > 1$, then it follows from Lemmas 1, 2 and 3.

LEMMA 4. *If $\mathfrak{G} = (G, \cdot, 1)$ is an abelian group with the exponent $m = p \cdot q \cdot s$, p, q — primes, $p \neq q$, then the idempotent reducts*

$$\mathfrak{G}_1 = (G; x_1 \cdot x_2^p \cdot \ldots \cdot x_{q \cdot s}^p) \quad \text{and} \quad \mathfrak{G}_2 = (G; x_1 \cdot x_2^q \cdot \ldots \cdot x_{p \cdot s}^q)$$

are uncomparable.

Proof. Observe that any operation generated by $x_1 \cdot x_2^p \cdot \ldots \cdot x_{q \cdot s}^p$ has the property that only one exponent is not divisible by p. Thus, it cannot generate $x_1 \cdot x_2^q \cdot \ldots \cdot x_{p \cdot s}^q$. Since the roles of p and q are symmetric, the lemma follows.

LEMMA 5. *If $\mathfrak{G}$ is an abelian group which satisfies no identity $x^m = 1$, then the reducts*

$$\mathfrak{G}_1 = (G, x_1^3 \cdot x_2^{-2}) \quad \text{and} \quad \mathfrak{G}_2 = (G, x_1^5 \cdot x_2^{-4})$$

are uncomparable.

Proof. Observe that any operation generated by $x_1^3 \cdot x_2^{-2}$ has the property that only one exponent is not divisible by 3. So $x_1^3 \cdot x_2^{-2}$ does not generate the operation $x_1^5 \cdot x_2^{-4}$. Analogously, the second operation does not generate the first one.

LEMMA 6. *If $\mathfrak{G}$ is a non-abelian group, then the idempotent reducts of $\mathfrak{G}$,*

$$\mathfrak{G}_1 = (G, x_1^{-1} \cdot x_2 \cdot x_1) \quad \text{and} \quad \mathfrak{G}_2 = (G, x_1 \cdot x_2^{-1} \cdot x_1),$$

are uncomparable.

Proof. Observe that every operation $f(x_1, \ldots, x_n)$ which is generated by the operation $x_1^{-1} \cdot x_2 \cdot x_1$ has the property that $f(x, 1, 1, \ldots, 1) \in \{x, 1\}$. If $x_1 \cdot x_2^{-1} \cdot x_1$ is generated by $x_1^{-1} \cdot x_2 \cdot x_1$, then $x^2 = 1$ for all $x \in G$, which implies that G is abelian and we get a contradiction. Further, observe that every binary operation in $\mathfrak{G}_2$ is of the form

$$x_1 \cdot x_2^{-1} \cdot x_1 \cdot x_2^{-1} \cdot \ldots \cdot x_1 \cdot x_2^{-1} \cdot x_1 = f(x_1, x_2) \text{ or } f(x_2, x_1),$$

where

$$f(x_1, x_2) = x_1^{-1} \cdot x_2 \cdot x_1. \tag{2}$$

Putting $x_2 = 1$, we get $x^k = 1$, where k is the number of occurrences of x_1 in $f(x_1, x_2)$. Multiplying (2) by x_2^{-1} from the right-hand side, we get

$$x_1^{-1} \cdot x_2 \cdot x_1 \cdot x_2^{-1} = (x_1 \cdot x_2^{-1})^k,$$

whence $x_1^{-1} \cdot x_2 \cdot x_1 \cdot x_2^{-1} = 1$. This implies commutativity and so we arrive at a contradiction. If

$$x_1^{-1} \cdot x_2 \cdot x_1 = f(x_2, x_1), \tag{3}$$

then, putting $x_1 = 1$, we get $x_2^k = x_2$, whence $x_2^{k-1} = 1$, so

$$x_1^{-1} \cdot x_2 \cdot x_1 = (x_2 \cdot x_1^{-1})^{k-1} \cdot x_2$$

and, finally,

$$x_1^{-1} \cdot x_2 \cdot x_1 = x_2$$

which means commutativity.

By Lemmas 4, 5 and 6, we have

THEOREM 2. *Abelian groups with the exponents p^k are the only groups in which idempotent reducts form a chain.*

REFERENCES

[1] J. Płonka, *R-prime idempotent reducts of groups,* Archiv der Mathematik 24 (1973), p. 129-132.

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