

ON GROUPS OF FIBONACCI TYPE

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(Received 24th March 1975)

1. An embedding

Let $w = w(a_0, a_1, \dots, a_{n-1})$ be a word in the free group freely generated by a_0, a_1, \dots, a_{n-1} ; let w_i denote the word $w(a_i, a_{i+1}, \dots, a_{i+n-1})$, where the subscripts j in a_j are reduced modulo n ; and let

$$G(n; w) = \langle a_0, a_1, \dots, a_{n-1} : w_i = 1; \quad i = 0, 1, \dots, n-1 \rangle.$$

Amongst the groups $G(n; w)$ are those said to be of "Fibonacci type" (see (2)). The Fibonacci groups are the groups $F(r, n) = G(n; w)$, where $w = a_0 a_1 \dots a_{r-1} a_r^{-1}$ (see (3)).

Now let

$$E(n; w) = \langle a, b; \quad b^n = 1, w(a_0, a_1, \dots, a_{n-1}) = 1 \rangle,$$

where here a_j denotes the element $b^{-j} a b^j$ ($0 \leq j \leq n-1$). Then, it is a direct consequence of the Reidemeister-Schreier subgroup theorem (see Theorem 2.9 of (4)) that $G(n; w)$ is embedded as the least normal subgroup of $E(n; w)$ containing the element a ; one chooses the elements $1, b, \dots, b^{n-1}$ as Schreier coset representatives.

It is often easier and more instructive to consider $E(n; w)$ rather than $G(n; w)$. Apparently this was the case in R. C. Lyndon's proof that $F(r; n)$ is infinite if $n \geq 11$. As $G(n; w)$ is a normal subgroup of index n in $E(n; w)$ it is infinite if $E(n; w)$ is infinite, finite of order s/n if $E(n; w)$ is finite of order s . Also, $G(n; w)$ inherits subgroup properties. In any case, $E(n; w)$ is a two-generator two-relator group whose relators can often be easily transformed into simple and workable forms.

This approach may be used to give a simple solution* to part of Problem 3 of (3) (another solution is given in (1)). Indeed, the comments above, together with the argument of Theorem 5 of (3), yield immediately that $F(r; n)$ is metacyclic of order $r^n - 1$ when $r \equiv 1$ modulo n .

2. On a conjecture

In (2), the group $H(r, n, s) = G(n; w)$, where

$$w = a_0 a_1 \dots a_{r-1} (a_r a_{r+1} \dots a_{r+s-1})^{-1},$$

* I understand from the referee, that M. J. Dunwoody has preceded me with this observation, which was made at the International Conference of Mathematicians, Vancouver, 1974.

is defined. According to §1, the group $H(r, n, s)$ is embedded as a normal subgroup of index n in the group $E(n; w) = \langle c, b; b^n = 1, b^r c^r = c^s b^s \rangle$, where $c = ab^{-1}$.

C. M. Campbell and E. F. Robertson conjecture (in (2)) that $H(r, 4, 2)$ is metacyclic if r is odd. This is so; in fact, it will follow below that $H(r, 4, 2)$ is metacyclic of order $(r - 2)(2^\xi - 2 \cdot 2^\eta + 2)/2$, where $\xi \equiv 4\alpha$, $\eta \equiv 2\alpha$ reduced modulo $4(r - 2)$, with α given by $4\alpha \equiv 1$ modulo $r - 2$.

Consider $E(4; w) = \langle c, b; b^4 = 1, b^r c^r = c^2 b^2 \rangle$. If r is even then $E(4; w)$ is infinite, as it has the group $\langle b, c; b^2 = c^2 = 1 \rangle$ as an epimorph. Thus we assume in the sequel that r is odd.

In case $r \equiv 3$ modulo 4, place $d = bc^{r-2}$, and in case $r \equiv 1$ modulo 4, place $d = c^{r-2}b^{-1}$. Then,

$$E(4; w) = \langle d, c; dd_{r-2} = d_r, (dc^{-(r-2)})^4 = 1 \rangle,$$

where d_i is used to denote $d^{c^i} = c^{-i}dc^i$ for integers i .

Our first observation is that $(dc^{-(r-2)})^4 = 1$ can be written in the form $c^{4(r-2)} = dd_{r-2}(dd_{r-2})^{c^{2(r-2)}}$; and therefore, since $d_r = dd_{r-2}$, in the form

$$c^{4(r-2)} = dd_{2(r-2)} \tag{i}$$

The relation $dd_{r-2} = d_r$ is the same as $d = d_{r-2}^c d_{r-2}^{-1}$; hence, using (i), we have $d_{r-2} = d_{2(r-2)}^c d_{2(r-2)}^{-1} = d_2^{-1} c^{4(r-2)} c^{-4(r-2)} d$, and so

$$d_{r-2} = d_2^{-1} d. \tag{ii}$$

From (i), we have $c^{4(r-2)} = d_r d_{r-2}^{-1} d_{2(r-2)} = d_2 d^{-1} d_{r-2}$. In particular, $d_2 d^{-1} d_{r-2} = d_4 d_2^{-1} d_r = d_4 d_2^{-1} dd_{r-2}$, so that $d_4 = d_2 d^{-2} d_2$. Thus, on the one hand, by (ii) we have $d_r = d_4^{-1} d_2 = d_2^{-1} d^2$, and on the other, $d_r = dd_{r-2} = dd_2^{-1} d$. We conclude that $d_2 d = dd_2$.

Relation (ii) implies $d_{2(r-2)} = d_r^{-1} d_{r-2} = d_2^{-1} d_4 d_2^{-1} d = d^{-1}$, since $d_4 = d_2 d^{-2} d_2$; whence, by (i) we have $c^{4(r-2)} = 1$. Moreover, since $d_r = d_2^{-1} d^2$, we obtain $d_{r+2} = d_4^{-1} d_2^2 = d^2$.

Conversely, the relations $d_{r+2} = d^2$, $d_4 = d_2^2 d^{-2}$ and $c^{4(r-2)} = 1$ imply the original two. Firstly, since $(4(r - 2), (r + 2)) = 1$ we obtain, using $c^{4(r-2)} = 1$ and $d_{r+2} = d^2$, that $d_1 = d^\mu$ for some integer μ , and hence that $d_i d_j = d_j d_i$ for each integer i and j . Also, $d_{r+2} = d^2$ implies $d_r = d_{-2}^2$, $d_{r-2} = d_{-4}^2$; and as $d = d_{-2}^2 d_{-4}^{-2}$, we conclude that $d_r = dd_{r-2}$. Further, the relation $d_4 = d^2 d_2^{-2}$ implies that $d_6 = d_2^2 d^{-4}$ and $d_8 = d^{-4}$; so, as $d_{r-2} = d_{-4}^2$, we have that $d_{2(r-2)} = d_{-8}^2 = d^{-1}$, and this verifies relation (i). It follows that

$$E(4; w) = \langle d, c; d_{r+2} = d^2, d_4 = d_2^2 d^{-2}, c^{4(r-2)} = 1 \rangle.$$

Now let α be given by $\alpha(r + 2) \equiv 1$ modulo $4(r - 2)$ (equivalently by

$4\alpha \equiv 1$ modulo $r-2$). Then, if $e = c^{r+2}$, we have $c = e^\alpha$; consequently $d^e = d^2$, and so $d_4 = d_2^2 d^{-2}$ becomes $d^{2^6 - 2 \cdot 2^{\eta+2}} = 1$, where ξ and η are given by 4α and 2α , respectively, when these are reduced modulo $4(r-2)$. Since $d^e = d^2$, we conclude that $d^k = 1$. Thus

$$E(4; w) = \langle d, e; d^e = d^2, d^k = 1, e^{4(r-2)} = 1 \rangle.$$

REFERENCES

- (1) C. M. CAMPBELL and E. F. ROBERTSON, On metacyclic Fibonacci groups, *Proc. Edinburgh Math. Soc.* **19** (1975), 253–256.
- (2) C. M. CAMPBELL and E. F. ROBERTSON, On a class of finitely presented groups of Fibonacci type, *J. London Math. Soc.* **11** (1975), 249–255.
- (3) D. L. JOHNSON, J. W. WAMSLEY and D. WRIGHT, The Fibonacci groups, *Proc. London Math. Soc.* (3) **29** (1974), 577–592.
- (4) W. MAGNUS, A. KARRASS and D. SOLITAR, *Combinational group theory* (Interscience (John Wiley & Sons), New York, London, Sydney, 1966).

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