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ON HADAMARD INEQUALITIES FOR k-FRACTIONAL INTEGRALS

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Abstract. In this paper, we prove Hadamard inequalities for k-fractional Riemann-Liouville integrals and Hadamard inequalities for fractional Riemann-Liouville integrals are deduced. Also we extend these results on two coordinates and Hadamard inequalities for fractional Riemann-Liouville integrals on two coordinates are deduced.

1. Introduction

Study of integration or differentiation of fractional order is known as fractional calculus. Its history is as old as the history of calculus. In 1695, Leibniz discussed with L' Hospital the differentiation of products functions of order $\frac{1}{2}$. It is considered as first discussion of fractional calculus. A lot of work has been published by mathematicians over the years. For example Liouville, Riemann, and Weyl made their major contributions to prospers this theory of fractional calculus. This subject continued with contributions of Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov (see, [8, 10, 11] and references there in).

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Fractional integral inequalities are useful in establishing the uniqueness of solutions for certain fractional partial differential equations. They also provide upper as well as lower bounds for solutions of the fractional boundary value problems. These considerations have led various researchers in the field of integral inequalities to explore certain extensions and generalizations by involving fractional calculus operators (see, [1, 8, 10, 13, 18]).

Let $f \in L_1[a,b]$. Then Riemann-Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined as follows:

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{1.1}$$

and

$$I^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b. \tag{1.2}$$

For further details one may see [7, 10, 13].

In [9], there is given definition of k-fractional Riemann-Liouville integrals as follows:

Let $f \in L_1[a,b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as

$$I^{\alpha,k}_{a+} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a \tag{1.3}$$

and

$$I^{\alpha,k}_{b-} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b, \tag{1.4}$$

where $\Gamma_k(\alpha)$ is the k-Gamma function defined as

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt.$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha)$$

and

$$I^{0,1}_{a+} f(x) = I^{0,1}_{b-} f(x) = f(x).$$

For $k = 1$, k-fractional integrals give Riemann-Liouville integrals.

In [17], Sarikaya *et al.* proved following Hadamard and Hadamard-type inequalities for Riemann-Liouville fractional integrals:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities* *for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[I_{a+}^{\alpha} f(b)+I_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.5}$$

with $\alpha > 0$.

Theorem 1.2. *Let* $f : [a,b] \to \mathbb{R}$ *be a differential mapping on* (a,b) *with* $a < b$. *If* $|f'|$ *is convex on* $[a,b]$, *then the following inequality for fractional integral holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}[I_{a+}^{\alpha} f(b)+I_{b-}^{\alpha} f(a)]\right| \\ &\leq \frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(a)|+|f'(b)|\right]. \end{aligned} \tag{1.6}$$

The rest of this paper is organised in the following manner. In Section 2, we give Hadamard and Hadamard-type inequalities for k-fractional integrals and show that inequalities (1.5) and (1.6) are their special cases. In Section 3, we extend results of Section 1 in two coordinates. Also we deduce some results given in [16].

2. Main results

Theorem 2.1. *Let* $f : [a,b] \to \mathbb{R}$ *be a positive function with* $0 \leq a < b$. *If* f *is a convex function on* $[a,b]$, *then the following inequalities for* k*-fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k} f(b)+I_{b-}^{\alpha,k} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2.1}$$

with $\alpha, k > 0$.

Proof. By the convexity of f we have,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \quad \text{for all} \quad x,y \in [a,b]. \tag{2.2}$$

Let $x = ta+(1-t)b, y = (1-t)a+tb$ for $t \in [0,1]$. Then $x,y \in [a,b]$ and (2.2) gives

$$2f\left(\frac{a+b}{2}\right) \leq f(ta+(1-t)b)+f((1-t)a+tb). \tag{2.3}$$

Multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\frac{2k}{\alpha} f\left(\frac{a+b}{2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}dt\\
&\leq \int_0^1 t^{\frac{\alpha}{k}-1} f(ta+(1-t)b)dt + \int_0^1 t^{\frac{\alpha}{k}-1} f((1-t)a+tb)dt\\
&= \frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k} f(b) + I_{b-}^{\alpha,k} f(a)\right].
\end{aligned}$$

It follows

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k} f(b) + I_{b-}^{\alpha,k} f(a)\right]. \tag{2.4}$$

On the other hand convexity of f gives

$$\begin{aligned}
&f(ta+(1-t)b) + f((1-t)a+tb)\\
&\leq tf(a) + (1-t)f(b) + (1-t)f(a) + tf(b).
\end{aligned}$$

Multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$, and integrating over $[0,1]$ leads us to

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}-1} f(ta+(1-t)b)dt + \int_0^1 t^{\frac{\alpha}{k}-1} f((1-t)a+tb)dt\\
&\leq [f(a)+f(b)]\int_0^1 t^{\frac{\alpha}{k}-1}dt.
\end{aligned}$$

It follows

$$\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k} f(b) + I_{b-}^{\alpha,k} f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{2.5}$$

Combining inequality (2.4) and inequality (2.5), we get inequality (2.1) . □

Remark 2.2. If we take $k=1$, Theorem 2.1 gives inequality (1.5) of Theorem 1.1 and putting $\alpha=1$ along with $k=1$ leads us to classical Hadamard inequality.

For the next result we need the following lemma.

Lemma 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f' \in L[a,b]$, then the following equality for k-fractional integral holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f(b) + I_{b-}^{\alpha,k}f(a)\right]$$
$$= \frac{b-a}{2}\int_0^1 ((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}})f^{'}(ta+(1-t)b)dt. \tag{2.6}$$

Proof. One can note that

$$\frac{b-a}{2}\int_0^1 ((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}})f^{'}(ta+(1-t)b)dt$$
$$= \frac{b-a}{2}\left[\int_0^1 (1-t)^{\frac{\alpha}{k}} f^{'}(ta+(1-t)b)dt - \int_0^1 t^{\frac{\alpha}{k}} f^{'}(ta+(1-t)b)dt\right],$$

where by simple calculation one can get

$$\frac{b-a}{2}\int_0^1 (1-t)^{\frac{\alpha}{k}} f^{'}(ta+(1-t)b)dt$$
$$= \frac{f(b)}{b-a} - \frac{\alpha}{k(b-a)}\int_a^b \left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}-1}\frac{f(x)}{b-a}dx$$
$$= \frac{f(b)}{b-a} - \frac{\Gamma_k\left(\alpha+k\right)}{(b-a)^{\frac{\alpha}{k}+1}} I_{b-}^{\alpha,k}f(a)$$

and

$$-\frac{b-a}{2}\int_0^1 t^{\frac{\alpha}{k}} f^{'}(ta+(1-t)b)dt$$
$$= \frac{f(a)}{b-a} - \frac{\alpha}{k(b-a)}\int_a^b \left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}-1}\frac{f(x)}{b-a}dx$$
$$= \frac{f(a)}{b-a} - \frac{\Gamma_k\left(\alpha+k\right)}{(b-a)^{\frac{\alpha}{k}+1}} I_{a+}^{\alpha,k}f(b).$$

Hence (2.6) can be established. □

Using above lemma we give following k-fractional Hadamard-type inequality.

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for k-fractional integral holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f(b)+I_{b-}^{\alpha,k}f(a)\right]\right|$$
$$\leq \frac{b-a}{2(\frac{\alpha}{k}+1)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left[|f'(a)|+|f'(b)|\right] \tag{2.7}$$

with $\alpha, k>0$.

Proof. From Lemma 2.3 and the convexity of $|f'|$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f(b)+I_{b-}^{\alpha,k}f(a)\right]\right|$$
$$\leq \frac{b-a}{2}\int_0^1\left|(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|\left|f^{'}(ta+(1-t)b)\right|dt$$
$$\leq \frac{b-a}{2}\int_0^1|(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}|[t|f'(a)|+(1-t)|f'(b)|]dt$$
$$=\frac{b-a}{2}\left[\int_0^{\frac{1}{2}}((1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}})(t\left|f'(a)\right|+(1-t)|f'(b)|)\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left[t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}\right][t|f'(a)|+(1-t)|f'(b)|]dt\right]. \tag{2.8}$$

One can note that

$$\int_0^{\frac{1}{2}}[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}][t|f'(a)|+(1-t)|f'(b)|dt$$
$$=|f'(a)|\left[\int_0^{\frac{1}{2}}t(1-t)^{\frac{\alpha}{k}}dt-\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}+1}dt\right]$$
$$+|f'(b)|\left[\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}+1}-\int_0^{\frac{1}{2}}(1-t)t^{\frac{\alpha}{k}}dt\right]$$
$$=|f'(a)|\left[\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]+|f'(b)|\left[\frac{1}{\frac{\alpha}{k}+2}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right].$$

By similar evaluation one can have

$$\int_{\frac{1}{2}}^1\left[t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}\right][t|f'(a)|+(1-t)|f'(b)|]dt$$
$$=|f'(a)|\left[\frac{1}{\frac{\alpha}{k}+2}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]+|f'(b)|\left[\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right].$$

Therefore (2.8) implies

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f(b)+I_{b-}^{\alpha,k}f(a)\right]\right|$$
$$\leq\frac{b-a}{2}\left[\left|f'(a)\right|\left[\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]+\left|f'(b)\right|\left[\frac{1}{\frac{\alpha}{k}+2}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]\right.$$
$$\left.+\left|f'(a)\right|\left[\frac{1}{\frac{\alpha}{k}+2}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]+\left|f'(b)\right|\left[\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]\right].$$

From which after a little computation one can have (2.7). □

If we take $k=1$ in Theorem 2.4, we get inequality (1.6) of Theorem 1.2, and if we take $\alpha=1$ along with $k=1$ in Theorem 2.4 , then inequality (2.7) gives the following result given in [3].

Corollary 2.5. *Consider a differentiable function $f:I\to\mathbb{R}$ on I with $a,b\in I$ and $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{2.9}$$

3. Hadamard-type inequalities for coordinated convex functions via k-fractional integrals

In this section we use coordinate convex functions to give k-fractional Hadamard inequalites on two coordinates. First we give preliminaries for this section.

Definition 3.1. ([5]) Let $a,b,c,d\in\mathbb{R}$ with $a<b$, $c<d$ and $\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2$. A mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the following inequality holds:

$$f(tx+(1-t)z,ty+(1-t)w)\leq tf(x,y)+(1-t)f(z,w),$$

for all $(x,y),(z,w)\in\Delta$ and $t\in[0,1]$.

Definition 3.2. ([5]) Let $a,b,c,d\in\mathbb{R}$ with $a<b$, $c<d$ and $\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2$. A function $f:\Delta\to\mathbb{R}$ is said to be coordinated convex on Δ if the following inequality holds:

$$f(tx+(1-t)y,su+(1-s)w)$$
$$\leq tsf(x,u)+s(1-t)f(y,u)+t(1-s)f(x,w)+(1-t)(1-s)f(y,w), \tag{3.1}$$

for all $(x,y),(u,w)\in\Delta$ and $t,s\in[0,1]$.

In [16], Riemann-Liouville integrals on two coordinates are defined as follows:

Definition 3.3. Let $a, b, c, d \in [0, \infty)$ with $a < b$, $c < d$ and $\Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2$. Consider $f \in L_1(\Delta)$, then the Riemann-Liouville integrals $I^{\alpha,\beta}_{a+,c+}$, $I^{\alpha,\beta}_{a+,d-}$, $I^{\alpha,\beta}_{b-,c+}$, $I^{\alpha,\beta}_{b-,d-}$ of order $\alpha, \beta > 0$ are defined as:

$$I^{\alpha,\beta}_{a+,c+} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1} f(t,s) ds dt,$$

$$I^{\alpha,\beta}_{a+,d-} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1} f(t,s) ds dt,$$

$$I^{\alpha,\beta}_{b-,c+} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1} f(t,s) ds dt$$

and

$$I^{\alpha,\beta}_{b-,d-} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1} f(t,s) ds dt,$$

respectively, where Γ is the Gamma function defined already in Section 1. Also,

$$I^{0,0}_{a+,c+} f(x,y) = I^{0,0}_{a+,d-} f(x,y) = I^{0,0}_{b-,c+} f(x,y) = I^{0,0}_{b-,d-} f(x,y) = f(x,y).$$

There in [16] also defined:

$$\begin{aligned}
I^{\alpha}_{a+} f\left(x, \frac{c+d}{2}\right) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f\left(t, \frac{c+d}{2}\right) dt, \\
I^{\alpha}_{b-} f\left(x, \frac{c+d}{2}\right) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f\left(t, \frac{c+d}{2}\right) dt, \\
I^{\beta}_{c+} f\left(\frac{a+b}{2}, y\right) &= \frac{1}{\Gamma(\beta)} \int_c^y (y-s)^{\beta-1} f\left(\frac{a+b}{2}, s\right) ds, \\
I^{\beta}_{d-} f\left(\frac{a+b}{2}, y\right) &= \frac{1}{\Gamma(\beta)} \int_y^d (s-y)^{\beta-1} f\left(\frac{a+b}{2}, s\right) ds.
\end{aligned}$$

We define Riemann-Liouville k-fractional integrals on two coordinates as:

Definition 3.4. Let $a, b, c, d \in [0, \infty)$ with $a < b$, $c < d$ and $\Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2$. Consider $f \in L_1(\Delta)$, then the k-fractional integrals ${}_kI^{\alpha,\beta}_{a+,c+}$, ${}_kI^{\alpha,\beta}_{a+,d-}$,

${}_kI^{\alpha,\beta}_{b-,c+}$, ${}_kI^{\alpha,\beta}_{b-,d-}$ of order $\alpha,\beta,k>0$ are defined as following

$$ {}_kI^{\alpha,\beta}_{a+,c+}f(x,y)=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x\int_c^y(x-t)^{\frac{\alpha}{k}-1}(y-s)^{\frac{\beta}{k}-1}f(t,s)dsdt, $$

$$ {}_kI^{\alpha,\beta}_{a+,d-}f(x,y)=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x\int_y^d(x-t)^{\frac{\alpha}{k}-1}(s-y)^{\frac{\beta}{k}-1}f(t,s)dsdt, $$

$$ {}_kI^{\alpha,\beta}_{b-,c+}f(x,y)=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_x^b\int_c^y(t-x)^{\frac{\alpha}{k}-1}(y-s)^{\frac{\beta}{k}-1}f(t,s)dsdt $$

and

$$ {}_kI^{\alpha,\beta}_{b-,d-}f(x,y)=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_x^b\int_y^d(t-x)^{\frac{\alpha}{k}-1}(s-y)^{\frac{\beta}{k}-1}f(t,s)dsdt, $$

respectively, where Γ_k is the k-Gamma function defined already in Section 1. Also,

$$ {}_1I^{0,0}_{a+,c+}f(x,y)={}_1I^{0,0}_{a+,d-}f(x,y)={}_1I^{0,0}_{b-,c+}f(x,y)={}_1I^{0,0}_{b-,d-}f(x,y)=f(x,y). $$

Also we define:

$$ I^{\alpha,k}_{a+}f\left(x,\frac{c+d}{2}\right)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x(x-t)^{\frac{\alpha}{k}-1}f\left(t,\frac{c+d}{2}\right)dt, $$
$$ I^{\alpha,k}_{b-}f\left(x,\frac{c+d}{2}\right)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b(t-x)^{\frac{\alpha}{k}-1}f\left(t,\frac{c+d}{2}\right)dt, $$
$$ I^{\beta,k}_{c+}f\left(\frac{a+b}{2},y\right)=\frac{1}{k\Gamma_k(\beta)}\int_c^y(y-s)^{\frac{\beta}{k}-1}f\left(\frac{a+b}{2},s\right)ds, $$
$$ I^{\beta,k}_{d-}f\left(\frac{a+b}{2},y\right)=\frac{1}{k\Gamma_k(\beta)}\int_y^d(s-y)^{\frac{\beta}{k}-1}f\left(\frac{a+b}{2},s\right)ds. $$

In the following we give Hadamard inequality for k-fractional integrals in two coordinates.

Theorem 3.5. *Let $a,b,c,d\in[0,\infty)$ with $a<b$, $c<d$ and $\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2$. Let $f:\Delta\to\mathbb{R}$ be coordinated convex on Δ. Then the following inequalities for k-fractional integrals hold:*

$$ \begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{4(b-a)^{\frac{\alpha}{k}}(d-c)^{\frac{\beta}{k}}}\Big[{}_kI^{\alpha,\beta}_{a+,c+}f(b,d)+{}_kI^{\alpha,\beta}_{a+,d-}f(b,c)\\ &\quad+{}_kI^{\alpha,\beta}_{b-,c+}f(a,d)+{}_kI^{\alpha,\beta}_{b-,d-}f(a,c)\Big]\\ &\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{3.2} $$

Proof. From inequality (3.1) with $x = t_1a + (1 - t_1)b$, $y = (1 - t_1)a + t_1b$, $u = s_1c + (1 - s_1)d$, $w = (1 - s_1)c + s_1d$, $t = s = \frac{1}{2}$, we get,

$$\begin{aligned} &f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\leq \frac{1}{4}\left[f(t_1a + (1-t_1)b, s_1c + (1-s_1)d) + f(t_1a + (1-t_1)b, (1-s_1)c + s_1d)\right. \\ &\quad \left.+f((1-t_1)a+t_1b,\ s_1c+(1-s_1)d)+f((1-t_1)a+t_1b,\ (1-s_1)c+s_1d)\right] \quad (3.3) \end{aligned}$$

Multiplying both sides of inequality (3.3) with $t_1^{\frac{\alpha}{k}-1}s_1^{\frac{\beta}{k}-1}$ and integrating the resulting inequality over $[0,1] \times [0,1]$, we get

$$\begin{aligned} &\frac{k^2}{\alpha\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\leq \frac{1}{4}\left[\int_0^1 \int_0^1 t_1^{\frac{\alpha}{k}-1} s_1^{\frac{\beta}{k}-1} f(t_1a + (1-t_1)b, s_1c + (1-s_1)d) ds_1 dt_1\right. \\ &\quad + \int_0^1 \int_0^1 t_1^{\frac{\alpha}{k}-1} s_1^{\frac{\beta}{k}-1} f(t_1a + (1-t_1)b, (1-s_1)c + s_1d) ds_1 dt_1 \\ &\quad + \int_0^1 \int_0^1 t_1^{\frac{\alpha}{k}-1} s_1^{\frac{\beta}{k}-1} f((1-t_1)a + t_1b, s_1c + (1-s_1)d) ds_1 dt_1 \\ &\quad \left.+ \int_0^1 \int_0^1 t_1^{\frac{\alpha}{k}-1} s_1^{\frac{\beta}{k}-1} f((1-t_1)a + t_1b, (1-s_1)c + s_1d) ds_1 dt_1\right]. \end{aligned}$$

Using the change of variables we have

$$\begin{aligned} &\frac{4k^2}{\alpha\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\leq \frac{1}{(b-a)^{\frac{\alpha}{k}}(d-c)^{\frac{\beta}{k}}}\left[\int_a^b \int_c^d (b-x)^{\frac{\alpha}{k}-1}(d-y)^{\frac{\beta}{k}-1} f(x,y) dy dx\right. \\ &\quad + \int_a^b \int_c^d (b-x)^{\frac{\alpha}{k}-1}(y-c)^{\frac{\beta}{k}-1} f(x,y) dy dx \\ &\quad + \int_a^b \int_c^d (x-a)^{\frac{\alpha}{k}-1}(d-y)^{\frac{\beta}{k}-1} f(x,y) dy dx \\ &\quad \left.+ \int_a^b \int_c^d (x-a)^{\frac{\alpha}{k}-1}(y-c)^{\frac{\beta}{k}-1} f(x,y) dy dx\right]. \end{aligned}$$

From which one can have first inequality of (3.2).

On the other hand from (3.1) for $x = a$, $y = b$, $u = c$, $w = d$ we have

$$\begin{aligned} &f(ta + (1-t)b, sc + (1-s)d) \\ &\leq tsf(a,c) + s(1-t)f(b,c) + t(1-s)f(a,d) + (1-t)(1-s)f(b,d), \end{aligned}$$

$$f(ta+(1-t)b,(1-s)c+sd)$$
$$\leq t(1-s)f(a,c)+(1-t)(1-s)f(b,c)+tsf(a,d)+s(1-t)f(b,d),$$

$$f((1-t)a+tb,sc+(1-s)d)$$
$$\leq s(1-t)f(a,c)+tsf(b,c)+(1-t)(1-s)f(a,d)+t(1-s)f(b,d),$$

$$f((1-t)a+tb,(1-s)c+sd)$$
$$\leq (1-t)(1-s)f(a,c)+t(1-s)f(b,c)+(1-t)sf(a,d)+tsf(b,d).$$

Adding the above four inequalities we get,

$$\begin{aligned} &f(ta+(1-t)b,sc+(1-s)d)+f(ta+(1-t)b,(1-s)c+sd)\\ &\quad+f((1-t)a+tb,sc+(1-s)d)+f((1-t)a+tb,(1-s)c+sd)\\ &\leq f(a,c)+f(b,c)+f(a,d)+f(b,d). \end{aligned} \tag{3.4}$$

Multiplying both sides of inequality (3.4) with $t^{\frac{\alpha}{k}-1}s^{\frac{\beta}{k}-1}$ and integrating the resulting inequality over $[0,1]\times[0,1]$, we get

$$\begin{aligned} &\int_0^1\int_0^1 t^{\frac{\alpha}{k}-1}s^{\frac{\beta}{k}-1}f(ta+(1-t)b,sc+(1-s)d)dsdt\\ &\quad+\int_0^1\int_0^1 t^{\frac{\alpha}{k}-1}s^{\frac{\beta}{k}-1}f(ta+(1-t)b,(1-s)c+sd)dsdt\\ &\quad+\int_0^1\int_0^1 t^{\frac{\alpha}{k}-1}s^{\frac{\beta}{k}-1}f((1-t)a+tb,sc+(1-s)d)dsdt\\ &\quad+\int_0^1\int_0^1 t^{\frac{\alpha}{k}-1}s^{\frac{\beta}{k}-1}f((1-t)a+tb,(1-s)c+sd)dsdt\\ &\leq \int_0^1\int_0^1 t^{\frac{\alpha}{k}-1}s^{\frac{\beta}{k}-1}[f(a,c)+f(b,c)+f(a,d)+f(b,d)]dsdt. \end{aligned}$$

Using change of variables we have,

$$\begin{aligned} &\frac{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}{(b-a)^{\frac{\alpha}{k}}(d-c)^{\frac{\beta}{k}}}[{}_kI^{\alpha,\beta}_{a+,c+}f(b,d)+_k I^{\alpha,\beta}_{a+,d-}f(b,c)\\ &\quad+_k I^{\alpha,\beta}_{b-,c+}f(a,d)+_k I^{\alpha,\beta}_{b-,d-}f(a,c)]\\ &\leq \frac{k^2}{\alpha\beta}f(a,c)+f(a,d)+f(b,c)+f(b,d). \end{aligned} \tag{3.5}$$

From which one can have second inequality of (3.2). □

Remark 3.6. In Theorem 3.5, if we put $k=1$, then we get [16, Theorem 3]. If we put $\alpha=\beta=1$ along with $k=1$ in Theorem 3.5, we get Hadamard inequality in two coordinates.

Theorem 3.7. *Let $a,b,c,d\in[0,\infty)$ with $a<b$, $c<d$ and $\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2$. Let $f:\Delta\to\mathbb{R}$ be coordinated convex on Δ. Then the following inequalities for k-fractional integrals hold:*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\leq \frac{\Gamma_k(\alpha+k)}{4(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f\left(b,\frac{c+d}{2}\right)+I_{b-}^{\alpha,k}f\left(a,\frac{c+d}{2}\right)\right]\\
&\quad+\frac{\Gamma_k(\beta+k)}{4(d-c)^{\frac{\beta}{k}}}\left[I_{c+}^{\beta,k}f\left(\frac{a+b}{2},d\right)+I_{d-}^{\beta,k}f\left(\frac{a+b}{2},c\right)\right]\\
&\leq \frac{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{4(b-a)^{\frac{\alpha}{k}}(d-c)^{\frac{\beta}{k}}}\Big[{}_kI_{a+,c+}^{\alpha,\beta}f(b,d)+{}_kI_{a+,d-}^{\alpha,\beta}f(b,c)\\
&\quad+{}_kI_{b-,c+}^{\alpha,\beta}f(a,d)+{}_kI_{b-,d-}^{\alpha,\beta}f(a,c)\Big]\\
&\leq \frac{\Gamma_k(\alpha+k)}{8(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f(b,c)+I_{a+}^{\alpha,k}f(b,d)+I_{b-}^{\alpha,k}f(a,c)+I_{b-}^{\alpha,k}f(a,d)\right]\\
&\quad+\frac{\Gamma_k(\beta+k)}{8(d-c)^{\frac{\beta}{k}}}\left[I_{c+}^{\beta,k}f(a,d)+I_{c+}^{\beta,k}f(b,d)+I_{d-}^{\beta,k}f(a,c)+I_{d-}^{\beta,k}f(b,c)\right]\\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \qquad (3.6)
\end{aligned}$$

Proof. Since $f:\Delta\to\mathbb{R}$ is coordinated convex, so the mapping $g_x:[c,d]\to\mathbb{R}$, $g_x(y)=f(x,y)$, is convex on $[c,d]$ for all $x\in[a,b]$. By using inequality (2.1), we can write

$$g_x\left(\frac{c+d}{2}\right)\leq\frac{\Gamma_k(\beta+k)}{2(d-c)^{\frac{\beta}{k}}}[I_{c+}^{\beta,k}g_x(d)+I_{d-}^{\beta,k}g_x(c)]\leq\frac{g_x(c)+g_x(d)}{2},$$

which implies

$$\begin{aligned}
&f\left(x,\frac{c+d}{2}\right)\\
&\leq\frac{\beta}{2k(d-c)^{\frac{\beta}{k}}}\left[\int_c^d(d-y)^{\frac{\beta}{k}-1}f(x,y)dy+\int_c^d(y-c)^{\frac{\beta}{k}-1}f(x,y)dy\right]\\
&\leq\frac{f(x,c)+f(x,d)}{2}. \qquad (3.7)
\end{aligned}$$

Multiplying both sides of inequality (3.7) with $\frac{\alpha(b-x)^{\frac{\alpha}{k}-1}}{2k(b-a)^{\frac{\alpha}{k}}}$ and integrating the resulted inequality over [a,b], we get

$$\begin{aligned}
&\frac{\alpha}{2k(b-a)^{\frac{\alpha}{k}}}\int_a^b (b-x)^{\frac{\alpha}{k}-1} f\left(x,\frac{c+d}{2}\right)dx \\
&\leq \frac{\alpha\beta}{4k^2(d-c)^{\frac{\beta}{k}}(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b\int_c^d (b-x)^{\frac{\alpha}{k}-1}(d-y)^{\frac{\beta}{k}-1} f(x,y)dydx\right. \\
&\quad \left.+\int_a^b\int_c^d (b-x)^{\frac{\alpha}{k}-1}(y-c)^{\frac{\beta}{k}-1} f(x,y)dydx\right] \\
&\leq \frac{\alpha}{4k(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b (b-x)^{\frac{\alpha}{k}-1} f(x,c)dx+\int_a^b (b-x)^{\frac{\alpha}{k}-1} f(x,d)dx\right]. \qquad (3.8)
\end{aligned}$$

Multiplying both sides of inequality (3.7) with $\frac{\alpha(x-a)^{\frac{\alpha}{k}-1}}{2k(b-a)^{\frac{\alpha}{k}}}$ and integrating the resulted inequality over [a,b], we get,

$$\begin{aligned}
&\frac{\alpha}{2k(b-a)^{\frac{\alpha}{k}}}\int_a^b (x-a)^{\frac{\alpha}{k}-1} f\left(x,\frac{c+d}{2}\right)dx \\
&\leq \frac{\alpha\beta}{4k^2(d-c)^{\frac{\beta}{k}}(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b\int_c^d (x-a)^{\frac{\alpha}{k}-1}(d-y)^{\frac{\beta}{k}-1} f(x,y)dydx\right. \\
&\quad \left.+\int_a^b\int_c^d (x-a)^{\frac{\alpha}{k}-1}(y-c)^{\frac{\beta}{k}-1} f(x,y)dydx\right] \\
&\leq \frac{\alpha}{4k(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b (x-a)^{\frac{\alpha}{k}-1} f(x,c)dx+\int_a^b (x-a)^{\frac{\alpha}{k}-1} f(x,d)dx\right]. \qquad (3.9)
\end{aligned}$$

Similarly for the mapping $g_y : [a,b] \to \mathbb{R}, g_y(x) = f(x,y)$, we get

$$\begin{aligned}
&\frac{\beta}{2k(d-c)^{\frac{\beta}{k}}}\int_c^d (d-y)^{\frac{\beta}{k}-1} f\left(\frac{a+b}{2},y\right)dy \\
&\leq \frac{\alpha\beta}{4k^2(d-c)^{\frac{\beta}{k}}(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b\int_c^d (b-x)^{\frac{\alpha}{k}-1}(d-y)^{\frac{\beta}{k}-1} f(x,y)dydx\right. \\
&\quad \left.+\int_a^b\int_c^d (x-a)^{\frac{\alpha}{k}-1}(d-y)^{\frac{\beta}{k}-1} f(x,y)dydx\right] \\
&\leq \frac{\beta}{4k(b-a)^{\frac{\beta}{k}}}\left[\int_c^d (d-y)^{\frac{\beta}{k}-1} f(a,y)dy+\int_c^d (d-y)^{\frac{\beta}{k}-1} f(b,y)dy\right] \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\beta}{2k(d-c)^{\frac{\beta}{k}}}\int_c^d (y-c)^{\frac{\beta}{k}-1} f\left(\frac{a+b}{2},y\right)dy\\
&\le \frac{\alpha\beta}{4k^2(d-c)^{\frac{\beta}{k}}(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b\int_c^d (b-x)^{\frac{\alpha}{k}-1}(y-c)^{\frac{\beta}{k}-1} f(x,y)dydx\right.\\
&\quad\left.+\int_a^b\int_c^d (x-a)^{\frac{\alpha}{k}-1}(y-c)^{\frac{\beta}{k}-1} f(x,y)dydx\right]\\
&\le \frac{\beta}{4k(d-c)^{\frac{\beta}{k}}}\left[\int_c^d (y-c)^{\frac{\beta}{k}-1} f(a,y)dy+\int_c^d (y-c)^{\frac{\beta}{k}-1} f(b,y)dy\right]. \qquad (3.11)
\end{aligned}$$

Adding (3.8), (3.9), (3.10) and (3.11) we get second and third inequality of 3.6.

Now from the first inequality of (2.1) we can write

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{\alpha}{2k(b-a)^{\frac{\alpha}{k}}}&\left[\int_a^b (b-x)^{\frac{\alpha}{k}-1} f\left(x,\frac{c+d}{2}\right)dx\right.\\
&\left.+\int_a^b (x-a)^{\frac{\alpha}{k}-1} f\left(x,\frac{c+d}{2}\right)dx\right]
\end{aligned}$$

and

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{\beta}{2k(d-c)^{\frac{\beta}{k}}}&\left[\int_c^d (d-y)^{\frac{\beta}{k}-1} f\left(\frac{a+b}{2},y\right)dy\right.\\
&\left.+\int_c^d (y-c)^{\frac{\beta}{k}-1} f\left(\frac{a+b}{2},y\right)dy\right].
\end{aligned}$$

Adding above two inequalities we get first inequality of 3.6. Now for last inequality of (3.6) we proceed as follows:

Using the second inequality in (2.1) we can write,

$$\begin{aligned}
&\frac{\alpha}{2k(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b (b-x)^{\frac{\alpha}{k}-1} f(x,c)dx+\int_a^b (x-a)^{\frac{\alpha}{k}-1} f(x,c)dx\right]\\
&\le \frac{f(a,c)+f(b,c)}{2},
\end{aligned}$$

$$\begin{aligned}
&\frac{\alpha}{2k(b-a)^{\frac{\alpha}{k}}}\left[\int_a^b (b-x)^{\frac{\alpha}{k}-1} f(x,d)dx+\int_a^b (x-a)^{\frac{\alpha}{k}-1} f(x,d)dx\right]\\
&\le \frac{f(a,d)+f(b,d)}{2},
\end{aligned}$$

$$\frac{\beta}{2k(d-c)^{\frac{\beta}{k}}}\left[\int_c^d (d-y)^{\frac{\beta}{k}-1} f(a,y)dy + \int_c^d (y-c)^{\frac{\beta}{k}-1} f(a,y)dy\right] \le \frac{f(a,c)+f(a,d)}{2}$$

and

$$\frac{\beta}{2k(d-c)^{\frac{\beta}{k}}}\left[\int_c^d (d-y)^{\frac{\beta}{k}-1} f(b,y)dy + \int_c^d (y-c)^{\frac{\beta}{k}-1} f(b,y)dy\right] \le \frac{f(b,c)+f(b,d)}{2}.$$

Finally adding the above four inequalities we get,

$$\frac{\Gamma_k(\alpha+k)}{8(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k} f(b,c) + I_{a+}^{\alpha,k} f(b,d) + I_{b-}^{\alpha,k} f(a,c) + I_{b-}^{\alpha,k} f(a,d)\right] + \frac{\Gamma_k(\beta+k)}{8(d-c)^{\frac{\beta}{k}}}\left[I_{c+}^{\beta,k} f(a,d) + I_{c+}^{\beta,k} f(b,d) + I_{d-}^{\beta,k} f(a,c) + I_{d-}^{\beta,k} f(b,c)\right] \le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

□

Remark 3.8. If we put $k = 1$ in Theorem 3.7, we get [16, Theorem 4] and if we take $\alpha = \beta = 1$ along with $k = 1$ in Theorem 3.7 we get Hadamard inequality in two coordinates.

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