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ON HADAMARD'S INEQUALITY FOR THE DETERMINANTS OF ORDER NON-DIVISIBLE BY 4

BY

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Let $\boldsymbol{B}_n$ be the family of all real $n\times n$ matrices $A=[b_{ij}]$ with $-1\leqslant b_{ij}\leqslant 1$. The determinant $|B_n|$ of a matrix $B_n\epsilon\boldsymbol{B}_n$ will be called *maximal* if $|B_n|\geqslant|A|$ for all $A\epsilon\boldsymbol{B}_n$. Maximal determinants obviously exist. Let us note that all elements of a maximal determinant are equal to ± 1, which simply results from the linearity of $|A|$ in all b_{ij}.

It was proved by J. Hadamard (see e.g. [5], p. 418 and 419) that

$$|B_n|\leqslant\sqrt{n^n}. \tag{1}$$

The matrices B_n for which the equality $|B_n|=\sqrt{n^n}$ holds, are called *Hadamard matrices*; their elements are, on account of the remark made above, equal to ± 1 and their rows are mutually orthogonal, i.e. $\sum_{k=1}^{n} b_{ik}b_{jk}=0$ for all $i\neq j$. It is well known [6] that Hadamard matrices can exist for $n=2$ and $n\equiv 0 (\mathrm{mod}\, 4)$ only.

J. Sylvester conjectured that Hadamard matrices exist for all $n\equiv 0(\mathrm{mod}\, 4)$. However, the existence of such matrices has been proved but for some special cases (see [2] and [3]).

It is clear that inequality (1) can be improved for $n\not\equiv 0(\mathrm{mod}\, 4)$. There has been known only one inequality of this kind, namely the inequality given by Barba [1] for odd n. He proved that for every such n,

$$|B_n|\leqslant\sqrt{2n-1}\sqrt{(n-1)^{n-1}}. \tag{2}$$

It is easy to verify that the right-hand side of the last inequality is asymptotically equal to $\sqrt{2/e}\sqrt{n^n}$, which gives approximately $0{,}858\sqrt{n^n}$. Estimation (2) is, for all odd n, better than (1) and in view of the examples for $n=5$ (see [7]) and $n=13$ (see p. 82 of this paper) this estimation cannot be improved in general.

The aim of this paper is to investigate the case $n \equiv 2 \pmod 4$. For this case we derive an estimation stronger than (1) and we show with the aid of some examples that this estimation is, in general, the best possible. In particular, we get Barba's inequality as a consequence of a theorem used by deriving our estimate. At the end of this paper there are given examples of maximal determinants for $n = 10$, 13 and 26. The construction of determinants of this kind for $n \not\equiv 0 \pmod 4$ is a rather difficult task and maximal determinants have been known [7] for $n = 2, 3, 5, 6$ and 7 only.

In the sequel we shall apply the following generalization of a theorem of J. Hadamard given by E. Fischer (see [4], p. 208 and 209):

Let $A_n = [a_{ij}]$ be a real positive definite (1) *symmetric $n \times n$ matrix. Let us partition A_n into four blocks according to the form*

$$A_n = \begin{bmatrix} C & F \\ F^T & D \end{bmatrix},$$

where both C and D are square matrices and F^T denotes the transposed matrix of F. Then

$$|A_n| \leqslant |C|\,|D|. \tag{3}$$

Hence, in particular, we get

$$|A_n| \leqslant \prod_{i=1}^{n} a_{ii}. \tag{4}$$

The equality

$$|A_n| = |C|\,|D|$$

holds if and only if F is a zero matrix, and the equality

$$|A_n| = \prod_{i=1}^{n} a_{ii}$$

holds if and only if A_n is a diagonal matrix.

We shall frequently use the following simple remark concerning a matrix A_n:

The symmetry and positive definiteness of a matrix A_n is left unchanged under operations described by:

(i) add the i-th row, multiplied by any numbers, to the last row, and then add the i-th column, multiplied by the same number, to the last column,

(1) For the definition see [5], pp. 394 and 395.

(ii) interchange the i-th and j-th row, and then the i-th and j-th column,

(iii) multiply by -1 the i-th row, and then the i-th column [2].

Definition 1. A transformation T of a matrix A_n which is a composition of a number of operations of the form (ii) or (iii) will be called *non--essential*. Two matrices differ *non-essentially* if each of them may be transformed into the other by a non-essential transformation.

LEMMA 1. *Let*

$$A'_n = \begin{bmatrix} a & a_{12} & \dots & a_{1n} \\ a_{21} & m & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & m \end{bmatrix}$$

be a real positive definite symmetric matrix with elements of first row satisfying the condition

$$0 < a \leqslant \min_{2\leqslant j\leqslant n} |a_{1j}|. \tag{5}$$

Then

$$|A'_n| \leqslant a(m-a)^{n-1}. \tag{6}$$

Proof. For every $i = 2, 3, \dots, n$, subtract from the i-th row of A'_n the first row multiplied by a_{1i}/a, then subtract from the i-th column the first column multiplied by $a_{i1}/a = a_{1i}/a$. The resulting matrix A''_n is obviously symmetric and positive definite. The determinant of A''_n is equal to $|A'_n|$ and the elements of its principal diagonal form the sequence

$$a, \left(m-\frac{a_{12}^2}{a}\right), \dots, \left(m-\frac{a_{1n}^2}{a}\right). \tag{iv}$$

We may therefore apply (4) to yield

$$|A''_n| = |A'_n| \leqslant a\prod_{i=2}^{n}\left(m-\frac{a_{1i}^2}{a}\right). \tag{7}$$

The last inequality, together with (5), gives (6).

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[2] The desired property of (i) follows from the known condition which is sufficient and necessary for a matrix A_n to be positive definite:

$$\begin{vmatrix} a_{11} & \dots & a_{1i} \\ \cdot & \cdot & \cdot \\ a_{i1} & \dots & a_{ii} \end{vmatrix} > 0 \quad \text{for} \quad i = 1, 2, \dots, n.$$

COROLLARY 1. *If A_n' satisfies the conditions of lemma 1, then equality in (6) holds if and only if A_n' differs non-essentially* (see definition 1) *from the matrix*

$$\bar{A}_n' = \begin{bmatrix} a & a & \dots & a \\ a & m & \dots & a \\ \cdot & \cdot & \cdot & \cdot \\ a & a & \dots & m \end{bmatrix}.$$

Proof. Without loss of generality, we may suppose that all elements of the first row and the first column of A_n' are non-negative.

If $|A_n'| = a(m-a)^{n-1}$, then inequality (7) becomes equality. It follows from this and from (5) that $a_{1i} = a$ $(i = 2, 3, \dots, n)$. Hence the elements of the principal diagonal of A_n'' form the sequence

$$a, m-a, \dots, m-a.$$

Since the determinant of A_n'' is equal to the product of the elements of this sequence, it follows from Fischer's theorem that A_n'' is diagonal. Hence all elements of A_n', except those of the principal diagonal, are equal to a.

On the other hand, it is easy to verify that $|\bar{A}_n'| = a(m-a)^{n-1}$.

THEOREM 1. *Let $A_n = [a_{ij}]$ be a real positive definite symmetric matrix with all diagonal elements equal to m. Then*

$$|A_n| \leqslant (m+na-a)(m-a)^{n-1} \tag{8}$$

with any real a subject to the condition

$$0 \leqslant a \leqslant \min_{i,j} |a_{ij}|. \tag{9}$$

Proof. We rewrite $|A_n|$ in the form

$$|A_n| = \begin{vmatrix} a & a_{12} & \dots & a_{1n} \\ a_{21} & m & \dots & a_{21} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & m \end{vmatrix} + \begin{vmatrix} m-a & 0 & \dots & 0 \\ a_{21} & m & \dots & a_{21} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & m \end{vmatrix}.$$

Let us denote by $|A_n'|$ the first determinant of the right-hand side of this equality and by $|A_{n-1}|$ the determinant obtained by deletion of both the first row and the first column of $|A_n|$. In terms of this notation the last equality may be rewritten in the form

$$|A_n| = |A_n'| + (m-a)|A_{n-1}|. \tag{10}$$

Let us remark that A_{n-1} satisfies all conditions of lemma 1 with the same a as in the case of A_n.

Now we shall consider two cases separately: (A) $|A'_n| > 0$ and (B) $|A'_n| \leqslant 0$.

Case (A). From $|A'_n| > 0$ and from the positive definiteness of A_n it results that A'_n is positive definite too (3). Hence A'_n satisfies the conditions of lemma 1 with any a satisfying (9).

On account of inequalities (6) and (10) we get

$$|A_n| \leqslant a(m-a)^{n-1} + (m-a)\cdot|A_{n-1}| \tag{11}$$

Case (B). It follows from (10) and from the obvious inequality $m \geqslant a$ that

$$|A_n| \leqslant (m-a)|A_{n-1}| \leqslant a(m-a)^{n-1} + (m-a)\cdot|A_{n-1}|.$$

Thus inequality (11) holds in both cases for all $n \geqslant 2$. The next part of the proof will be established by induction.

For $n = 1$ inequality (8) is obviously true. Suppose that it has been proved for $n-1$, where $n \geqslant 2$. Then, in view of (11), we have

$$\begin{aligned}|A_n| &\leqslant a(m-a)^{n-1} + (m-a)\{[m+(n-1)\cdot a - a](m-a)^{n-2}\} = \\ &= (m+na-a)(m-a)^{n-1}.\end{aligned}$$

This completes the proof.

COROLLARY 2. *If A_n satisfies the condition of theorem* 1, *then equality in* (8) *holds if and only if A_n differs non-essentially from the matrix*

$$\bar{A}_n = \begin{bmatrix} m & a & \dots & a \\ a & m & \dots & a \\ \dots \\ a & a & \dots & m \end{bmatrix}.$$

Proof. If $|A_n| = (m+na-a)(m-a)^{n-1}$, then (11) becomes equality; hence $|A'_n| = a(m-a)^{n-1}$. It follows from corollary 1 that $\bar{A}'_n = TA'_n$, where T is a non-essential transformation. Since the matrix A_n is obtained from A'_n by putting m in place of the element a in the left upper corner, $\bar{A}_n = TA_n$. On the other hand, it is easy to verify that $|A_n| = (m+na-a)(m-a)^{n-1}$.

From theorem 1 we can easily deduce estimation (2) of G. Barba for odd n.

It is sufficient to prove (2) for maximal determinants $|B_n|$ only. Then $|B_n| \neq 0$ and $b_{ij} = \pm 1$. Denoting by $A_n = [a_{ij}]$ the product $B_n \cdot B_n^T$ we have

$$a_{ij} = \sum_{k=1}^{n} b_{ik} b_{jk}.$$

(3) This property follows from the theorem mentioned in footnote (2).

The matrix A_n, defined as above, is symmetric and positive definite (see [5], p. 418). Moreover, all elements of its principal diagonal are equal to n. From the assumption that n is odd it follows that all the elements a_{ij} are odd integers, hence $|a_{ij}| \geqslant 1$. According to theorem 1, putting $a = 1$ and $m = n$ we have

$$|A_n| = |B_n|^2 \leqslant (2n-1)(n-1)^{n-1},$$

which gives the desired inequality (2).

LEMMA 2. *Let $A_n = [a_{ij}]$ be a real positive definite symmetric matrix with all diagonal elements equal to m. Let moreover A_n satisfy the following condition:*

(α) *if $a_{ik} = a_{jk} = 0$, then $a_{ij} \neq 0$.*

Then

$$|A_n| \leqslant (m+pa-a)(m-pa-a+na)(m-a)^{n-2}, \tag{12}$$

where a is an arbitrary number satisfying the inequality

$$0 < a \leqslant \min_{a_{ij}\neq 0} |a_{ij}| \tag{β}$$

and $p = \max p_i$, p_i denoting the number of elements of the i-th row of A_n equal to zero.

Proof. Let $p+1 < n$.

We interchange rows and columns in A_n according to (ii) on p. 75 so that

(γ) a row with exactly p elements equal to zero becomes the second row and all the non-vanishing elements of the first row precede all zero-elements of this row.

This new matrix differs non-essentially from the original matrix and it will be denoted equally by A_n. It satisfies all assumptions needed in lemma 2 and its determinant is equal to that of the original matrix.

Let us consider the sequence $A_n, A_{n-1}, \ldots, A_{p+1}$ of matrices where A_k, for k satisfying $p+1 < k < n$, results from A_{k+1} by deletion of the first row and the first column and such a rearrangement of the remaining rows and columns according to (ii) as to satisfy condition (γ). Moreover, A_{p+1} results from A_{p+2} by the deletion of the first row and the first column. It is readily seen that all matrices of this sequence satisfy the conditions of the lemma with the same a and p.

If we use the notation introduced in the proof of theorem 1, we may rewrite $|A_n|$ in form (10) with $|A_n'|$ equal to

$$|A_n'| = \left|\begin{array}{c:c} C & 0\ 0\ \ldots\ 0 \\ \hdashline \begin{matrix}0\\0\\\vdots\\0\end{matrix} & D \end{array}\right|, \tag{13}$$

where C and D are square matrices of order $n-p_1$ and p_1 respectively. Moreover, no element of the first row of C vanishes.

We shall consider two cases separately: (A) $|A_n'| > 0$ and (B) $|A_n'| \leqslant 0$.

Case (A). From $|A_n'| > 0$ and from positive definite ness of A_n it results that A_n' is positive definite and, in consequence, C and D are positive definite too.

From lemma 1 we get

$$|C| \leqslant a(m-a)^{n-p_1-1}. \tag{14}$$

From (α) and from the form of A_n' it follows that no element of D vanishes. This matrix and the number a satisfy the conditions of Theorem 1. Therefore we can write

$$|D| \leqslant (m+p_1 a-a)(m-a)^{p_1-1}. \tag{15}$$

From the last inequality and from (3), (13) and (14) we obtain

$$|A_n'| \leqslant |C|\cdot|D| \leqslant a(m+p_1 a-a)(m-a)^{n-2} \tag{16}$$

and in virtue of $p_1 \leqslant p$ we have

$$|A_n'| \leqslant a(m+pa-a)(m-a)^{n-2}. \tag{17}$$

Finally from (10) and (17) we get

$$|A_n| \leqslant a(m+pa-a)(m-a)^{n-2}+(m-a)\,|A_{n-1}|. \tag{18}$$

Case (B). It follows from the obvious inequality $m \geqslant a$ and from (10) that

$$|A_n| \leqslant (m-a)\cdot|A_{n-1}| \leqslant a(m+pa-a)(m-a)^{n-2}+(m-a)\cdot|A_{n-1}|.$$

Thus in both cases inequality (18) holds for all $n \geqslant 2$.

Since the matrices $A_{n-1}, \ldots, A_{p+1}$ satisfy the assumptions of lemma 2 with the same a and p, the inequality

$$|A_k| \leqslant a(m+pa-a)(m-a)^{k-2}+(m-a)\cdot|A_{k-1}|,$$

similar to (18), is valid for $k = n-1, n-2, \ldots, p+1$.

From the last inequalities and from (18) we obtain

$$|A_n| \leqslant a(n-p-1)(m+pa-a)(m-a)^{n-2}+(m-a)^{n-p-1}\cdot|A_{p+1}|, \tag{19}$$

where

$$|A_{p+1}| = \left|\begin{array}{c|c} m & 0 \ \ldots \ 0 \\ \hline 0 & \\ \vdots & A_p \\ 0 & \end{array}\right|.$$

It follows from condition (α) of the lemma and from the form of A_{p+1} that no element of A_p vanishes. On account of theorem 1 we get

$$|A_{p+1}| = m|A_p| \leqslant m(m+pa-a)(m-a)^{p-1}.$$

Applying the last inequality to (19) we obtain the desired inequality (12).

It may be easily verified that (19) is valid for $n = p+1$ too, which gives (12) as before.

This completes the proof.

COROLLARY 3. *If $A_n = [a_{ij}]$ satisfies the conditions of lemma 2, then*

$$|A_n| \leqslant (m+\tfrac{1}{2}na-a)^2(m-a)^{n-2} \tag{20}$$

This corollary follows from the remark that the right-hand side in (12), regarded as a function of p, attains its maximum value only in one point $p = n/2$.

COROLLARY 4. *If $A_n = [a_{ij}]$ satisfies the conditions of lemma 2, then the equality*

$$|A_n| = (m+\tfrac{1}{2}na-a)^2(m-a)^{n-2}$$

holds if and only if n is even and A_n differs non-essentially from the matrix

$$M_n = \begin{bmatrix} \bar{A}_{n/2} & 0 \\ 0 & \bar{A}_{n/2} \end{bmatrix}, \quad \textit{where} \quad \bar{A}_{n/2} = \begin{bmatrix} m & a & \ldots & a \\ a & m & \ldots & a \\ \cdot & \cdot & \cdot & \cdot \\ a & a & \ldots & m \end{bmatrix}.$$

Proof. Let $\bar{A}'_{n/2}$ be a matrix which is obtained from $\bar{A}_{n/2}$ by putting a in place of m in the left upper corner.

If $|A_n| = (m+\frac{1}{2}na-a)^2(m-a)^{n-2}$, then, from the remark in the proof of corollary 3, it follows that $p = n/2$. Therefore n is even. It is obvious that in this case the inequalities (14)-(17) become equalities.

It follows from equality in (17) that $p_1 = p = n/2$. Therefore the matrices C and D are of the same order $n/2$. It follows from equality in (16) and from Fischer's theorem that A'_n (see (13)) is of the form

$$\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}.$$

By equality in (14) and by corollary 1, $C = T_1\bar{A}'_{n/2}$. Similarly in view of the equality in (15) and of corollary 2, $D = T_2\bar{A}_{n/2}$. In both cases T_1 and T_2 are non-essential transformations. Therefore A'_n differs non-essentially from the matrix

$$N_n = \begin{bmatrix} \bar{A}'_{n/2} & 0 \\ 0 & \bar{A}_{n/2} \end{bmatrix}.$$

We see further that A_n differs from A'_n by one element which lies in the left upper corner and which is equal to m in A_n and to a in A'_n. The same is true for matrices M_n and N_n. Thus the matrix A_n required in our corollary differs non-essentially from M_n.

On the other hand, it is easy to verify that

$$|M_n| = (m+\tfrac{1}{2}na-a)^2(m-a)^{n-2}.$$

THEOREM 2. *If* $n \equiv 2 \pmod 4$, *then*

$$|B_n| \leqslant 2(n-1)\sqrt{(n-2)^{n-2}}. \tag{21}$$

Proof. We assume $|B_n| \neq 0$ for otherwise (21) is obviously true. The matrix $A_n = [a_{ij}] = B_nB_n^T$, where $a_{ij} = \sum_{k=1}^{n} b_{ik}b_{jk}$, is symmetric, positive definite and all its elements of the principal diagonal are equal to n.

Since n is even, all a_{ij} are even too, therefore $\min_{a_{ij}\neq 0} |a_{ij}| \geqslant 2$ and the number $a = 2$ satisfies condition ($\boldsymbol{\beta}$) of lemma 2.

To verify that the remaining assumption ($\boldsymbol{\alpha}$) of lemma 2 is satisfied, it is sufficient to remark that for $n \equiv 2 \pmod 4$ no three mutually orthogonal rows do exist in B_n (see e.g. [6]).

From corollary 3 we get

$$|A_n| = |B_n|^2 \leqslant 4(n-1)^2(n-2)^{n-2}$$

from which inequality (21) follows.

Estimation (21) is, for all $n \equiv 2 \pmod 4$, better than (1). It is easy to verify that the right-hand side of (21) is asymptotically equal to $2/e\sqrt{n^n}$, which gives approximately $0.736\sqrt{n^n}$.

We shall finally prove, by using some examples, that inequalities (2) and (21) cannot be, in general, improved.

Let

$$B_2 = \begin{bmatrix} + & + \\ + & - \end{bmatrix}, \quad B_5 = \begin{bmatrix} - & + & + & + & + \\ + & - & + & + & + \\ + & + & - & + & + \\ + & + & + & - & + \\ + & + & + & + & - \end{bmatrix}, \quad B_6 = \begin{bmatrix} + & + & + & + & + & + \\ + & - & + & + & + & - \\ - & + & + & + & + & - \\ - & - & - & + & + & + \\ - & - & + & - & + & + \\ - & - & + & + & - & + \end{bmatrix},$$

$$B_{13} = \begin{bmatrix} - & + & + & + & + & + & + & + & + & + & + & + & + \\ - & - & - & - & + & + & + & + & + & + & - & - & - \\ - & - & - & - & - & - & - & + & + & + & + & + & + \\ - & - & - & - & + & + & + & - & - & - & + & + & + \\ + & - & - & + & + & + & - & + & + & - & + & - & + \\ + & - & - & + & - & + & + & - & + & + & + & + & - \\ + & - & - & + & + & - & + & + & - & + & - & + & + \\ + & + & - & - & + & - & + & + & + & - & + & + & - \\ + & + & - & - & + & + & - & - & + & + & - & + & + \\ + & + & - & - & - & + & + & + & - & + & + & - & + \\ + & - & + & - & + & + & - & + & - & + & + & + & - \\ + & - & + & - & - & + & + & + & + & - & - & + & + \\ + & - & + & - & + & - & + & - & + & + & + & - & + \end{bmatrix},$$

where the sign $+$ denotes the number 1 and the sign $-$ denotes the number -1. The determinants of all these matrices are maximal; the matrices B_2, B_5 and B_6 were given in [7]; the construction of B_{13} is complicated and we do not go into details of it. The estimate given in Barba's inequality (2) is reached for matrices B_5 and B_{13}, and that given in inequality (21) for B_2 and B_6.

To get other examples with analogous property we construct the direct product ([4]) of matrices B_2 and B_5, which leads to a matrix B_{10} of order 10, and the direct product of B_2 and B_{13}, which leads to a matrix B_{26} of order 26, both B_{10} and B_{26} with maximal determinants ([5]).

([4]) If we replace each element of a matrix C_n of order n by a matrix D_m of order m multiplied by the replaced element, we get a matrix which is called a *direct* (or *Kronecker's*) product of the matrices C_n and D_m and is denoted by $C_n \cdot D_m$. It is known that $|C_n \cdot D_m| = |C_n|^m \cdot |D_m|^n$.

([5]) We conjecture that for all $n \equiv 2 \pmod 4$ there exist matrices $B_n = [b_{ij}]$ with elements $b_{ij} = \pm 1$ for which equality in (21) holds.

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