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Research Article

On Hadamard-Type Inequalities Involving Several Kinds of Convexity

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We do not only give the extensions of the results given by Gill et al. (1997) for log-convex functions but also obtain some new Hadamard-type inequalities for log-convex m-convex, and (α, m)-convex functions.

1. Introduction

The following inequality is well known in the literature as Hadamard's inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \to R$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. This inequality is one of the most useful inequalities in mathematical analysis. For new proofs, note worthy extension, generalizations, and numerous applications on this inequality; see ([1–6]) where further references are given.

Let I be on interval in R. Then $f : I \to R$ is said to be convex if, for all $x, y \in I$ and $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.2}$$

(see [5], Page 1). Geometrically, this means that if K, L, and M are three distinct points on the graph of f with L between K and M, then L is on or below chord KM.

Recall that a function $f : I \to (0,\infty)$ is said to be log-convex function if, for all $x, y \in I$ and $t \in [0,1]$, one has the inequality (see [5], Page 3)

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{(1-t)}. \tag{1.3}$$

It is said to be log-concave if the inequality in (1.3) is reversed.

In [7], Toader defined m-convexity as follows.

Definition 1.1. The function $f : [0,b] \to R$, $b > 0$ is said to be m-convex, where $m \in [0,1]$, if one has

$$f(tx + m(1-t)y) \leq t f(x) + m(1-t) f(y) \tag{1.4}$$

for all $x, y \in [0,b]$ and $t \in [0,1]$. We say that f is m-concave if $-f$ is m-convex.

Denote by $K_m(b)$ the class of all m-convex functions on $[0,b]$ such that $f(0) \leq 0$ (if $m < 1$). Obviously, if we choose $m = 1$, Definition 1.1 recaptures the concept of standard convex functions on $[0,b]$.

In [8], Miheşan defined (α, m)-convexity as in the following:

Definition 1.2. The function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-convex, where $(\alpha, m) \in [0,1]^2$, if one has

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha) f(y) \tag{1.5}$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

Denote by $K_m^\alpha(b)$ the class of all (α, m)-convex functions on $[0,b]$ for which $f(0) \leq 0$. It can be easily seen that for $(\alpha, m) = (1, m)$, (α, m)-convexity reduces to m-convexity and for $(\alpha, m) = (1,1)$, (α, m)-convexity reduces to the concept of usual convexity defined on $[0,b]$, $b > 0$.

For recent results and generalizations concerning m-convex and (α, m)-convex functions, see ([9–12]).

In the literature, the logarithmic mean of the positive real numbers p, q is defined as the following:

$$L(p,q) = \frac{p-q}{\ln p - \ln q} \quad (p \neq q) \tag{1.6}$$

(for $p = q$, we put $L(p,p) = p$).

In [13], Gill et al. established the following results.

Theorem 1.3. *Let f be a positive,* log*-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq L\big(f(a), f(b)\big), \tag{1.7}$$

where $L(\cdot,\cdot)$ is a logarithmic mean of the positive real numbers as in (1.6).

For f a positive log*-concave function, the inequality is reversed.*

Corollary 1.4. *Let f be positive* log*-convex functions on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \min_{x\in[a,b]} \frac{(x-a)L\big(f(a), f(x)\big) + (b-x)L\big(f(x), f(b)\big)}{b-a}. \tag{1.8}$$

If f is a positive log*-concave function, then*

$$\frac{1}{b-a}\int_a^b f(x)dx \geq \max_{x\in[a,b]} \frac{(x-a)L\big(f(a), f(x)\big) + (b-x)L\big(f(x), f(b)\big)}{b-a}. \tag{1.9}$$

For some recent results related to the Hadamard's inequalities involving two log-convex functions, see [14] and the references cited therein. The main purpose of this paper is to establish the general version of inequalities (1.7) and new Hadamard-type inequalities involving two log-convex functions, two m-convex functions, or two (α, m)-convex functions using elementary analysis.

2. Main Results

We start with the following theorem.

Theorem 2.1. *Let $f_i : I \subset R \to (0,\infty)$ $(i = 1,2,\ldots,n)$ be* log*-convex functions on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b \prod_{i=1}^{n} f_i(x)dx \leq L\left(\prod_{i=1}^{n} f_i(a), \prod_{i=1}^{n} f_i(b)\right), \tag{2.1}$$

where L is a logarithmic mean of positive real numbers.

For f a positive log*-concave function, the inequality is reversed.*

Proof. Since $f_i(i = 1, 2, \dots, n)$ are log-convex functions on I, we have

$$f_i(ta + (1-t)b) \le [f_i(a)]^t [f_i(b)]^{(1-t)} \tag{2.2}$$

for all $a, b \in I$ and $t \in [0, 1]$. Writing (2.2) for $i = 1, 2, \dots, n$ and multiplying the resulting inequalities, it is easy to observe that

$$\begin{aligned}\prod_{i=1}^{n} f_i(ta + (1-t)b) &\le \left[\prod_{i=1}^{n} f_i(a)\right]^t \left[\prod_{i=1}^{n} f_i(b)\right]^{(1-t)} \\ &= \prod_{i=1}^{n} f_i(b) \left[\prod_{i=1}^{n} \frac{f_i(a)}{f_i(b)}\right]^t\end{aligned} \tag{2.3}$$

for all $a, b \in I$ and $t \in [0, 1]$.

Integrating inequality (2.3) on $[0, 1]$ over t, we get

$$\int_0^1 \prod_{i=1}^{n} f_i(ta + (1-t)b)dt \le \prod_{i=1}^{n} f_i(b) \int_0^1 \left[\prod_{i=1}^{n} \frac{f_i(a)}{f_i(b)}\right]^t dt. \tag{2.4}$$

As

$$\int_0^1 \prod_{i=1}^{n} f_i(ta + (1-t)b)dt = \frac{1}{b-a} \int_a^b \prod_{i=1}^{n} f_i(x)dx, \tag{2.5}$$

$$\int_0^1 \left[\prod_{i=1}^{n} \frac{f_i(a)}{f_i(b)}\right]^t dt = \frac{1}{\prod_{i=1}^{n} f_i(b)} L\left(\prod_{i=1}^{n} f_i(a), \prod_{i=1}^{n} f_i(b)\right), \tag{2.6}$$

the theorem is proved. □

Remark 2.2. By taking $i = 1$ and $f_1 = f$ in Theorem 2.1, we obtain (1.7).

Corollary 2.3. *Let* $f_i : I \subset R \to (0, \infty)$ $(i = 1, 2, \dots, n)$ *be* log-*convex functions on* I *and* $a, b \in I$ *with* $a < b$. *Then*

$$\begin{aligned}&\frac{1}{b-a} \int_a^b \prod_{i=1}^{n} f_i(x)dx \\ &\le \min_{x \in [a,b]} \frac{(x-a)L\left(\prod_{i=1}^{n} f_i(a), \prod_{i=1}^{n} f_i(x)\right) + (b-x)L\left(\prod_{i=1}^{n} f_i(x), \prod_{i=1}^{n} f_i(b)\right)}{b-a}.\end{aligned} \tag{2.7}$$

If $f_i(i = 1, 2, \dots, n)$ are positive log*-concave functions, then*

$$\frac{1}{b-a}\int_a^b \prod_{i=1}^n f_i(x)dx \geq \max_{x\in[a,b]} \frac{(x-a)L\left(\prod_{i=1}^n f_i(a), \prod_{i=1}^n f_i(x)\right) + (b-x)L\left(\prod_{i=1}^n f_i(x), \prod_{i=1}^n f_i(b)\right)}{b-a}. \tag{2.8}$$

Proof. Let $f_i(i = 1, 2, \dots, n)$ be positive log-convex functions. Then by Theorem 2.1 we have that

$$\int_a^b \prod_{i=1}^n f_i(t)dt = \int_a^x \prod_{i=1}^n f_i(t)dt + \int_x^b \prod_{i=1}^n f_i(t)dt \leq (x-a)L\left(\prod_{i=1}^n f_i(a), \prod_{i=1}^n f_i(x)\right) + (b-x)L\left(\prod_{i=1}^n f_i(x), \prod_{i=1}^n f_i(b)\right), \tag{2.9}$$

for all $x \in [a, b]$, whence (2.7). Similarly we can prove (2.8). □

Remark 2.4. By taking $i = 1$ and $f_1 = f$ in (2.7) and (2.8), we obtain the inequalities of Corollary 1.4.

We will now point out some new results of the Hadamard type for log-convex, m-convex, and (α, m)-convex functions, respectively.

Theorem 2.5. *Let $f, g : I \to (0, \infty)$ be* log*-convex functions on I and $a, b \in I$ with $a < b$. Then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left\{\frac{1}{b-a}\int_a^b [f(x)f(a+b-x) + g(x)g(a+b-x)]dx\right\} \leq \frac{f(a)f(b) + g(a)g(b)}{2}. \tag{2.10}$$

Proof. We can write

$$\frac{a+b}{2} = \frac{ta + (1-t)b}{2} + \frac{(1-t)a + tb}{2}. \tag{2.11}$$

Using the elementary inequality $cd \leq 1/2[c^2 + d^2]$ $(c, d \geq 0$ reals) and equality (2.11), we have

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\
&\quad\le \frac{1}{2}\left[f^2\left(\frac{a+b}{2}\right)+g^2\left(\frac{a+b}{2}\right)\right]\\
&\quad= \frac{1}{2}\left[f^2\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)+g^2\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)\right]\\
&\quad\le \frac{1}{2}\left\{\left[(f(ta+(1-t)b))^{1/2}\right]^2\left[(f((1-t)a+tb))^{1/2}\right]^2\right.\\
&\qquad\qquad \left.+\left[(g(ta+(1-t)b))^{1/2}\right]^2\left[(g((1-t)a+tb))^{1/2}\right]^2\right\}\\
&\quad= \frac{1}{2}\left[f(ta+(1-t)b)f((1-t)a+tb)+g(ta+(1-t)b)g((1-t)a+tb)\right].
\end{aligned}
\tag{2.12}
$$

Since f, g are log-convex functions, we obtain

$$
\begin{aligned}
&\frac{1}{2}\left[f(ta+(1-t)b)f((1-t)a+tb)+g(ta+(1-t)b)g((1-t)a+tb)\right]\\
&\quad\le \left\{\frac{1}{2}[f(a)]^t[f(b)]^{(1-t)}[f(a)]^{(1-t)}[f(b)]^t+[g(a)]^t[g(b)]^{(1-t)}[g(a)]^{(1-t)}[g(b)]^t\right\}\\
&\quad= \frac{f(a)f(b)+g(a)g(b)}{2}
\end{aligned}
\tag{2.13}
$$

for all $a, b \in I$ and $t \in [0,1]$.

Rewriting (2.12) and (2.13), we have

$$
f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\le \frac{1}{2}\left[f(ta+(1-t)b)f((1-t)a+tb)+g(ta+(1-t)b)g((1-t)a+tb)\right],
\tag{2.14}
$$

$$
\frac{1}{2}\left[f(ta+(1-t)b)f((1-t)a+tb)+g(ta+(1-t)b)g((1-t)a+tb)\right]\le \frac{f(a)f(b)+g(a)g(b)}{2}.
\tag{2.15}
$$

Integrating both sides of (2.14) and (2.15) on $[0,1]$ over t, respectively, we obtain

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x)f(a+b-x)+g(x)g(a+b-x)]\,dx\right],\\
&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x)f(a+b-x)+g(x)g(a+b-x)]\,dx\right]\le \frac{f(a)f(b)+g(a)g(b)}{2}.
\end{aligned}
\tag{2.16}
$$

Combining (2.16), we get the desired inequalities (2.10). The proof is complete. □

Theorem 2.6. *Let $f, g : I \to (0, \infty)$ be* log-convex *functions on I and $a, b \in I$ with $a < b$. Then the following inequalities hold:*

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \left[f^2(x) + g^2(x)\right]dx \le \frac{f(a)+f(b)}{2}L(f(a), f(b)) + \frac{g(a)+g(b)}{2}L(g(a), g(b)), \tag{2.17}$$

where $L(\cdot,\cdot)$ is a logarithmic mean of positive real numbers.

Proof. From inequality (2.14), we have

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[f(ta+(1-t)b)f((1-t)a+tb) + g(ta+(1-t)b)g((1-t)a+tb)\right]. \tag{2.18}$$

for all $a, b \in I$ and $t \in [0, 1]$.

Using the elementary inequality $cd \le 1/2[c^2 + d^2]$ ($c, d \ge 0$ reals) on the right side of the above inequality, we have

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{4}\left[f^2(ta+(1-t)b) + f^2((1-t)a+tb) + g^2(ta+(1-t)b) + g^2((1-t)a+tb)\right]. \tag{2.19}$$

Since f, g are log-convex functions, then we get

$$\begin{aligned}
&\left[f^2(ta+(1-t)b) + f^2((1-t)a+tb) + g^2(ta+(1-t)b) + g^2((1-t)a+tb)\right] \\
&\le \left\{[f(a)]^{2t}[f(b)]^{(2-2t)} + [f(a)]^{(2-2t)}[f(b)]^{2t} + [g(a)]^{2t}[g(b)]^{(2-2t)} + [g(a)]^{(2-2t)}[g(b)]^{2t}\right\} \\
&= \left[f^2(b)\left[\frac{f(a)}{f(b)}\right]^{2t} + f^2(a)\left[\frac{f(b)}{f(a)}\right]^{2t} + g^2(b)\left[\frac{g(a)}{g(b)}\right]^{2t} + g^2(a)\left[\frac{g(b)}{g(a)}\right]^{2t}\right].
\end{aligned} \tag{2.20}$$

Integrating both sides of (2.19) and (2.20) on $[0,1]$ over t, respectively, we obtain

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \left[f^2(x)+g^2(x)\right]dx,$$

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \left[f^2(x)+g^2(x)\right]dx \\
&\le \frac{1}{2}\left(f^2(b)\int_0^1 \left[\frac{f(a)}{f(b)}\right]^{2t} dt + f^2(a)\int_0^1 \left[\frac{f(b)}{f(a)}\right]^{2t} dt \right.\\
&\qquad \left. + g^2(b)\int_0^1 \left[\frac{g(a)}{g(b)}\right]^{2t} dt + g^2(a)\int_0^1 \left[\frac{g(b)}{g(a)}\right]^{2t} dt\right) \\
&= \frac{1}{2}\left(f^2(b)\left[\frac{[f(a)/f(b)]^{2t}}{2\log f(a)/f(b)}\right]_0^1 + f^2(a)\left[\frac{[f(b)/f(a)]^{2t}}{2\log f(b)/f(a)}\right]_0^1 \right.\\
&\qquad \left. + g^2(b)\left[\frac{[g(a)/g(b)]^{2t}}{2\log g(a)/g(b)}\right]_0^1 + g^2(a)\left[\frac{[g(b)/g(a)]^{2t}}{2\log g(b)/g(a)}\right]_0^1\right) \\
&= \frac{1}{2}\left(\frac{f^2(a)-f^2(b)}{2(\log f(a)-\log f(b))} + \frac{f^2(b)-f^2(a)}{2(\log f(b)-\log f(a))}\right.\\
&\qquad \left. + \frac{g^2(a)-g^2(b)}{2(\log g(a)-\log g(b))} + \frac{g^2(b)-g^2(a)}{2(\log g(b)-\log g(a))}\right) \\
&= \frac{1}{2}\left(\frac{f(a)+f(b)}{2}L(f(a),f(b)) + \frac{f(a)+f(b)}{2}L(f(b),f(a))\right.\\
&\qquad \left. + \frac{g(a)+g(b)}{2}L(g(a),g(b)) + \frac{g(a)+g(b)}{2}L(g(b),g(a))\right) \\
&= \left\{\frac{f(a)+f(b)}{2}L(f(a),f(b)) + \frac{g(a)+g(b)}{2}L(g(a),g(b))\right\}.
\end{aligned} \tag{2.21}$$

Combining (2.21), we get the required inequalities (2.17). The proof is complete. □

Theorem 2.7. *Let $f, g : [0,\infty) \to [0,\infty)$ be such that fg is in $L^1([a,b])$, where $0 \le a < b < \infty$. If f is nonincreasing m_1-convex function and g is nonincreasing m_2-convex function on $[a,b]$ for some fixed $m_1, m_2 \in (0,1]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \min\{S_1, S_2\}, \tag{2.22}$$

where

$$S_1 = \frac{1}{6}\left[\left(f^2(a) + g^2(a)\right) + m_1 f(a) f\left(\frac{b}{m_1}\right) + m_2 g(a) g\left(\frac{b}{m_2}\right) + m_1^2 f^2\left(\frac{b}{m_1}\right) + m_2^2 g^2\left(\frac{b}{m_2}\right)\right], \tag{2.23}$$

$$S_2 = \frac{1}{6}\left[\left(f^2(b) + g^2(b)\right) + m_1 f(b) f\left(\frac{a}{m_1}\right) + m_2 g(b) g\left(\frac{a}{m_2}\right) + m_1^2 f^2\left(\frac{a}{m_1}\right) + m_2^2 g^2\left(\frac{a}{m_2}\right)\right] \tag{2.24}$$

Proof. Since f is m_1-convex function and g is m_2-convex function, we have

$$\begin{gathered} f(t\mathrm{a} + (1-t)b) \leq t f(a) + m_1(1-t) f\left(\frac{b}{m_1}\right), \\ g(ta + (1-t)b) \leq t g(a) + m_2(1-t) g\left(\frac{b}{m_2}\right) \end{gathered} \tag{2.25}$$

for all $t \in [0,1]$. It is easy to observe that

$$\int_a^b f(x)g(x)dx = (b-a)\int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt. \tag{2.26}$$

Using the elementary inequality $cd \leq 1/2(c^2 + d^2)$ $(c, d \geq 0$ reals), (2.25) on the right side of (2.26) and making the charge of variable and since f, g is nonincreasing, we have

$$\begin{aligned} &\int_a^b f(x)g(x)dx \\ &\quad \leq \frac{1}{2}(b-a)\int_0^1 \left[\{f(t\mathrm{a} + (1-t)b)\}^2 + \{g(ta + (1-t)b)\}^2\right]dt \\ &\quad \leq \frac{1}{2}(b-a)\int_0^1 \left[\left(t f(a) + m_1(1-t) f\left(\frac{b}{m_1}\right)\right)^2 + \left(t g(a) + m_2(1-t) g\left(\frac{b}{m_2}\right)\right)^2\right]dt \\ &\quad = \frac{1}{2}(b-a)\left[\frac{1}{3}f^2(a) + \frac{1}{3}m_1^2 f^2\left(\frac{b}{m_1}\right) + \frac{1}{3}m_1 f(a) f\left(\frac{b}{m_1}\right) + \frac{1}{3}g^2(a) + \frac{1}{3}m_2^2 g^2\left(\frac{b}{m_2}\right)\right. \\ &\qquad\qquad \left. + \frac{1}{3}m_2 g(a) g\left(\frac{b}{m_2}\right)\right] \\ &\quad = \frac{(b-a)}{6}\left[\left(f^2(a) + g^2(\mathrm{a})\right) + m_1 f(a) f\left(\frac{b}{m_1}\right) + m_2 g(a) g\left(\frac{b}{m_2}\right) + m_1^2 f^2\left(\frac{b}{m_1}\right)\right. \\ &\qquad\qquad \left. + m_2^2 g^2\left(\frac{b}{m_2}\right)\right]. \end{aligned} \tag{2.27}$$

Analogously we obtain

$$\int_a^b f(x)g(x)dx$$
$$\leq \frac{(b-a)}{6}\left[\left(f^2(b)+g^2(b)\right)+m_1 f(b) f\left(\frac{a}{m_1}\right)+m_2 g(b) g\left(\frac{a}{m_2}\right)+m_1^2 f^2\left(\frac{a}{m_1}\right)+m_2^2 g^2\left(\frac{a}{m_2}\right)\right]. \tag{2.28}$$

Rewriting (2.27) and (2.28), we get the required inequality in (2.22). The proof is complete. □

Theorem 2.8. *Let $f, g : [0,\infty) \to [0,\infty)$ be such that fg is in $L^1([a,b])$, where $0 \leq a < b < \infty$. If f is nonincreasing (α_1, m_1)-convex function and g is nonincreasing (α_2, m_2)-convex function on $[a,b]$ for some fixed $\alpha_1, m_1, \alpha_2, m_2 \in (0,1]$. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \min\{E_1, E_2\}, \tag{2.29}$$

where

$$\begin{aligned} E_1 = \frac{1}{2}\Bigg[& \frac{1}{2\alpha_1+1} f^2(a) + \frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)} m_1^2 f^2\left(\frac{b}{m_1}\right) \\ & + \frac{2\alpha_1}{(\alpha_1+1)(2\alpha_1+1)} m_1 f(a) f\left(\frac{b}{m_1}\right) + \frac{1}{2\alpha_2+1} g^2(a) \\ & + \frac{2\alpha_2^2}{(\alpha_2+1)(2\alpha_2+1)} m_2^2 g^2\left(\frac{b}{m_2}\right) + \frac{2\alpha_2}{(\alpha_2+1)(2\alpha_2+1)} m_2 g(a) g\left(\frac{b}{m_2}\right)\Bigg], \end{aligned} \tag{2.30}$$

$$\begin{aligned} E_2 = \frac{1}{2}\Bigg[& \frac{1}{2\alpha_1+1} f^2(b) + \frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)} m_1^2 f^2\left(\frac{a}{m_1}\right) \\ & + \frac{2\alpha_1}{(\alpha_1+1)(2\alpha_1+1)} m_1 f(b) f\left(\frac{a}{m_1}\right) + \frac{1}{2\alpha_2+1} g^2(b) \\ & + \frac{2\alpha_2^2}{(\alpha_2+1)(2\alpha_2+1)} m_2^2 g^2\left(\frac{a}{m_2}\right) + \frac{2\alpha_2}{(\alpha_2+1)(2\alpha_2+1)} m_2 g(b) g\left(\frac{a}{m_2}\right)\Bigg]. \end{aligned} \tag{2.31}$$

Proof. Since f is (α_1, m_1)-convex function and g is (α_2, m_2)-convex function, then we have

$$\begin{aligned} f(ta+(1-t)b) &\leq t^{\alpha_1} f(a) + m_1(1-t^{\alpha_1}) f\left(\frac{b}{m_1}\right), \\ g(ta+(1-t)b) &\leq t^{\alpha_2} g(a) + m_2(1-t^{\alpha_2}) g\left(\frac{b}{m_2}\right) \end{aligned} \tag{2.32}$$

for all $t \in [0,1]$. It is easy to observe that

$$\int_a^b f(x)g(x)dx = (b-a)\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt. \tag{2.33}$$

Using the elementary inequality $cd \le 1/2(c^2+d^2)$ $(c,d \ge 0$ reals), (2.32) on the right side of (2.33) and making the charge of variable and since f, g is nonincreasing, we have

$$\begin{aligned}
\int_a^b f(x)g(x)dx &\le \frac{1}{2}(b-a)\int_0^1\left[\{f(ta+(1-t)b)\}^2+\{g(ta+(1-t)b)\}^2\right]dt \\
&\le \frac{1}{2}(b-a)\int_0^1\Bigg[\left(t^{\alpha_1}f(a)+m_1(1-t^{\alpha_1})f\left(\frac{b}{m_1}\right)\right)^2 \\
&\qquad +\left(t^{\alpha_2}g(a)+m_2(1-t^{\alpha_2})g\left(\frac{b}{m_2}\right)\right)^2\Bigg]dt \\
&= \frac{1}{2}(b-a)\Bigg[\frac{1}{2\alpha_1+1}f^2(a)+\frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)}m_1^2f^2\left(\frac{b}{m_1}\right) \\
&\qquad +\frac{2\alpha_1}{(\alpha_1+1)(2\alpha_1+1)}m_1f(a)f\left(\frac{b}{m_1}\right)+\frac{1}{2\alpha_2+1}g^2(a) \\
&\qquad +\frac{2\alpha_2^2}{(\alpha_2+1)(2\alpha_2+1)}m_2^2g^2\left(\frac{b}{m_2}\right)+\frac{2\alpha_2}{(\alpha_2+1)(2\alpha_2+1)}m_2g(a)g\left(\frac{b}{m_2}\right)\Bigg]
\end{aligned} \tag{2.34}$$

Analogously we obtain

$$\begin{aligned}
&\int_a^b f(x)g(x)dx \\
&\quad\le \frac{1}{2}(b-a)\Bigg[\frac{1}{2\alpha_1+1}f^2(b)+\frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)}m_1^2f^2\left(\frac{a}{m_1}\right) \\
&\qquad +\frac{2\alpha_1}{(\alpha_1+1)(2\alpha_1+1)}m_1f(b)f\left(\frac{a}{m_1}\right)+\frac{1}{2\alpha_2+1}g^2(b) \\
&\qquad +\frac{2\alpha_2^2}{(\alpha_2+1)(2\alpha_2+1)}m_2^2g^2\left(\frac{a}{m_2}\right)+\frac{2\alpha_2}{(\alpha_2+1)(2\alpha_2+1)}m_2g(b)g\left(\frac{a}{m_2}\right)\Bigg].
\end{aligned} \tag{2.35}$$

Rewriting (2.34) and (2.35), we get the required inequality in (2.29). The proof is complete. □

Remark 2.9. In Theorem 2.8, if we choose $\alpha_1 = \alpha_2 = 1$, we obtain the inequality of Theorem 2.7.

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