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ON HAMILTONIAN CIRCUITS AND SPANNING TREES OF HYPERCUBES

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1. INTRODUCTION

The aim of this paper is to prove that certain trees are spanning trees of the hypercubes Q_n ($n \geqq 1$). Obviously, the simplest spanning tree of Q_n is the path p_{2^n-1} of length $2^n - 1$ (where the length of a path is measured by the number of its edges). Other two spanning trees of Q_n (similar to each other) have been found by Nebeský when solving a different problem in [8]; they arise by means of certain "reduplication" of binary trees.

A complete solution of the problem of spanning trees of hypercubes would be provided by characterizing them; such a characterization seems to be of interest especially in view of the fact that hypercubes have been characterized (cf. e.g. [1], [6] and [7]). Unfortunately, we are not able to solve the above mentioned problem; we present it therefore as an open question (together with some related conjectures) at the end of the paper (Sec. 5).

In Sec. 2 we prove certain assertions concerning the structure and properties of hamiltonian circuits and paths in Q_n. Using them we find in Sec. 4 some spanning trees of Q_n. Sec. 3 describes spanning trees of Q_n obtained in a different way, namely, by modifications of binary trees.

In the whole paper we deal only with finite undirected graphs without loops and multiple edges. $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The maximum degree of vertices in G will be denoted by maxdeg (G).

The hypercube Q_n ($n \geqq 1$) is defined in the usual way (cf. e.g. [2]); its vertices are all the vectors of length n consisting of 0's and 1's. For $u, v \in V(Q_n)$, $\varrho(u, v)$ denotes the Hamming distance of u and v, i.e., the number of coordinates in which u and v differ from each other. $(u, v) \in E(Q_n)$ iff $\varrho(u, v) = 1$. Given i, $1 \leqq i \leqq n$, Q_n can be decomposed into two copies of Q_{n-1} (denoted by Q'_{n-1}, Q''_{n-1}) whose vertices are joined by 2^{n-1} edges of a perfect matching; the vertices of $Q'_{n-1}(Q''_{n-1})$ are those of Q_n with the i-th coordinate equal to 0 (1, respectively). We call this decomposition of Q_n canonical (more precisely, i-canonical).

The notion of the so called C_n-valuation of a graph will be frequently used (cf. [5]); the definition and the basic property, modified for the case of trees, are as follows: a tree T is said to be C_n-valued, if the edges of T are labelled by integers from $\{1, \ldots, n\}$ in such a way that for any path p of T there is $k \in \{1, \ldots, n\}$ such that an odd number of edges of p are assigned k. Then T is isomorphic to a subgraph of Q_n (in other words: T is embeddable in Q_n) if and only if there is a C_n-valuation of T. Given a C_n-valuation of T and a path p in T, we define "the odd set of p" by

$$O(p) = \{k \in \{1, \ldots, n\};\ \text{an odd number of edges of } p \text{ are labelled by } k\} .$$

With a C_n-valuation of T a certain embedding of T in Q_n can be associated, i.e., an injection ε: $V(T) \to V(Q_n)$ such that $(u, v) \in E(T) \Rightarrow (\varepsilon(u), \varepsilon(v)) \in E(Q_n)$. The mapping ε is obviously an isomorphism of T to a subgraph of Q_n (this subgraph not necessarily being an induced one). If p is a path in T with end-vertices u, v and $|O(p)| = l$, then $\varrho(\varepsilon(u), \varepsilon(v)) = l$.

It is clear that every tree can be C_n-valued (for n sufficiently large). By dim T we shall denote the smallest n such that there is a C_n-valuation of T (obviously, dim T is the smallest n with the property that T is isomorphic to a subgraph of Q_n).

We shall frequently need C_n-valuations of paths; let us construct one of them as follows: for $i \geqq 1$ let $i = 2^j \cdot m$, where m is odd. Putting $a_i = j + 1$ we obtain the sequence $\{a_i\}_{i \geqq 1}$ whose members are

$$1\ 2\ 1\ 3\ 1\ 2\ 1\ 4\ 1\ 2\ 1\ 3\ 1\ 2\ 1\ 5\ 1\ \ldots$$

It is not difficult to see that for $k \geqq 1$ the values $\{a_i\}_{i=1}^{2^k-1}$ may be used as the values of a C_k-valuation of the path p_{2^k-1} of length $2^k - 1$. Let us call this valuation the basic C_k-valuation of p_{2^k-1}. We have $O(p_{2^k-1}) = \{k\}$ and the basic C_k-valuation may easily be modified so that e.g. $O(p_{2^k-1}) = \{1\}$.

2. SOME STRUCTURAL PROPERTIES OF HAMILTONIAN CIRCUITS AND PATHS IN Q_n

In this section we derive certain properties of hamiltonian circuits and paths in Q_n that will be needed in Sec. 4 (some of them seem to be of a certain interest by themselves). If $u, v \in V(Q_n)$, $u \neq v$, and if an arbitrary path p containing u and v or a hamiltonian circuit c in Q_n is given, we use in addition to the well-known Hamming distance $\varrho(u, v)$ of the vertices u, v also the notion of "the distance of u, v along p or along the circuit c" (with the obvious meaning); the vertices u, v have always two distances d_1, d_2 along c (where $d_1 + d_2 = 2^n$ and the equality $d_1 = d_2$ may hold).

2.1. Proposition. *Let $n \geqq 2$, $u, v \in V(Q_n)$, $u \neq v$. Let $r \equiv \varrho(u, v) \pmod 2$, $\varrho(u, v) \leqq$ $\leqq r \leqq 2^n - \varrho(u, v)$. Then there is a hamiltonian circuit c in Q_n such that one of the distances of u, v along c is r (and the other is $2^n - r$).*

Proof. Let s, t, d and n be positive integers. We shall write $\mathrm{HC}(s, t; d, n)$ if the following holds: for any $u, v \in V(Q_n)$ fulfilling $\varrho(u, v) = d$ there is a hamiltonian circuit in Q_n such that one of the distances of u and v along this circuit is s and the other is t. Obviously, $\mathrm{HC}(s, t; d, n)$ iff $\mathrm{HC}(t, s; d, n)$ and if $\mathrm{HC}(s, t; d, n)$, then

(1) $s + t = 2^n$,

(2) $d \leqq \min(s, t)$, $d \leqq n$, and

(3) $d \equiv s \equiv t \pmod 2$.

We shall show now that $\mathrm{HC}(s, t; d, n)$ holds for all quadruples s, t, d, n of positive integers fulfilling (1), (2) and (3); obviously, this will prove the proposition.

First we prove two lemmas using the notion of a canonical decomposition of Q_{n+1} (into Q'_n and Q''_n).

Lemma 1. $\mathrm{HC}(s, t; d, n) \Rightarrow \mathrm{HC}(s, t + 2^n; d, n + 1)$.

The implication easily follows from Fig. 2.1; given $u, v \in V(Q_{n+1})$ with $\varrho(u, v) = d$, it is always possible to find an i-canonical decomposition of Q_{n+1} such that both u and v belong to the same part of it (e.g. to Q'_n). Then, $\mathrm{HC}(s, t; d, n)$ guarantees the existence of a hamiltonian circuit c' in Q'_n such that the distances of u' and v' along c' are s and t. Let c'' be the image of c' in Q''_n. A hamiltonian circuit c of Q_{n+1} with the properties required (i.e., such that the distances of u and v along c are s and $t + 2^n$) can now be easily constructed from c' and c'' according to Fig. 2.1.

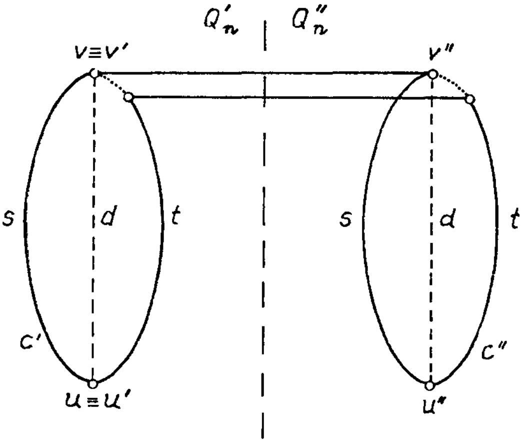

Fig. 2.1.

Lemma 2. $\mathrm{HC}(s, t; d, n)$ *and* $0 \leqq q < t \Rightarrow \mathrm{HC}(s + 2q + 1,\ s + 2t - 2q - 1;\ d + 1, n + 1)$.

Using again a canonical decomposition of Q_{n+1} we assume that if $u, v \in V(Q_{n+1})$, $\varrho(u, v) = d + 1$, then $u \in V(Q'_n)$ and $v \in V(Q''_n)$. The construction then easily follows from Fig. 2.2.

We are now ready to prove the main proposition using induction on n: both HC(1, 3; 1, 2) and HC(2, 2; 2, 2) obviously hold. Let $n \geqq 2$, suppose HC$(s', t'; d', n)$ holds whenever $s' + t' = 2^n$, $d' \leqq \min(s', t')$, $d' \leqq n$ and $s' \equiv t' \equiv d' \pmod 2$. Let $s + t = 2^{n+1}$, $d \leqq \min(s, t)$, $d \leqq n + 1$, $s \equiv t \equiv d \pmod 2$. If $d = 1$, then

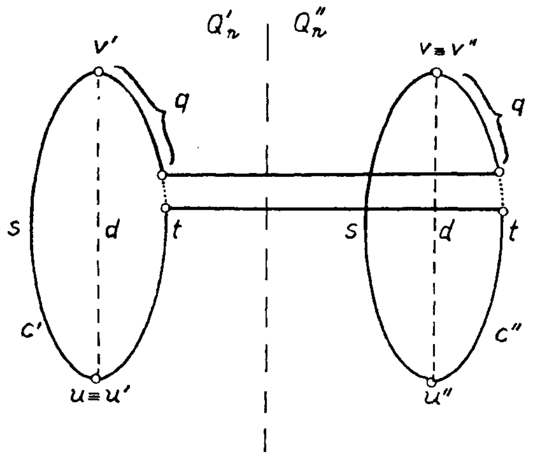

Fig. 2.2.

$s \neq t$; let e.g. $s < t$. Using Lemma 1 we have HC$(s, t - 2^n; 1, n) \Rightarrow$ HC$(s, t; 1, n + 1)$. Let $d > 1$; then $1 \leqq d - 1 \leqq n$ and therefore HC$(d - 1, 2^n - d + 1; d - 1, n)$ holds. Suppose $s \leqq t$ and put $q = (s - d)/2$. Then $0 \leqq q < 2^n - d + 1$ and, using Lemma 2, we obtain HC$(s, t; d, n + 1)$, q.e.d.

The following result is an easy consequence of 2.1.

2.2. Corollary. *Let $n \geqq 1$, consider the path p_{2^n-1} of length $2^n - 1$. Assume $i, j \in \{1, \dots, n\}$, $i \neq j$, $1 \leqq l \leqq 2^n - 1$. Let p_l be the initial part of p_{2^n-1} of length l. Then it is possible to construct a C_n-valuation of p_{2^n-1} such that $O(p_l) = \{i\}$ if l is odd and $O(p_l) = \{i, j\}$ if l is even.*

In fact, if e.g. l is odd, choose $u, v \in V(Q_n)$ differing in the i-th coordinate; then $\varrho(u, v) = 1$ and there is a hamiltonian circuit c in Q_n such that one of the distances of u, v along c is l. Let us delete the edge incident with u from the other chord (of length $2^n - l$) of c. In this way a hamiltonian path p in Q_n is obtained; since p is embedded in Q_n, we obviously can use the corresponding C_n-valuation of p as the desired one and proceed quite similarly also in the case of even l.

For $u \in V(Q_n)$ let $\bar{u}$ denote the vertex opposite to u in Q_n (i.e. such that $\varrho(u, \bar{u}) = n$).

2.3. Proposition. *Let $u, v \in V(Q_n)$, $\varrho(u, v) \equiv 1 \pmod 2$. Then there is a hamiltonian path p in Q_n with end-vertices u and v. Moreover, if $u \neq \bar{v}$ (i.e. if $\varrho(u, v) < n$), then p can be constructed in such a way that the distance of u and v along p equals*

$n - \varrho(u, v)$ (*i.e. "p goes from u first to $\bar{v}$ in the shortest possible way and then from $\bar{v}$ to v"*).

Proof. 1. Assume first $\varrho(u, v) = 1$; according to 2.1 there is a hamiltonian circuit c in Q_n such that one of the distances of v and $\bar{v}$ along c is n. Without loss of generality we may assume that the edge (u, v) belongs to the chord of length n of c joining v and $\bar{v}$. Removing the edge (u, v) from c we obtain the required hamiltonian path.

2. Assume now $3 \leqq \varrho(u, v) \leqq n$. Let, without loss of generality, $u = (0, \ldots, 0)$, $v = (0, \ldots, 0, 1, \ldots, 1)$, put $w = (0, \ldots, 0, 1, 0)$. Let Q'_{n-1} and Q''_{n-1} be parts of the n-canonical decomposition of Q_n (i.e., Q'_{n-1} is the hypercube induced in Q_n by all the vertices having 0 as its n-th coordinate). It follows from what has been proved above that there is a hamiltonian path p' in Q'_{n-1} joining u and w such that the distance of the vertices $(0, \ldots, 0)$ and $(1, \ldots, 1, 0, 0)$ along p' is $n - 2$; we may assume that the beginning of p' is formed by the vertices $(0, \ldots, 0)$, $(1, 0, \ldots, 0)$, $(1, 1, 0, \ldots \ldots, 0), \ldots, (1, 1, \ldots, 1, 0, 0)$. Let us extend p' by adding the edge joining $w = (0, \ldots \ldots, 0, 1, 0)$ with $w'' = (0, \ldots, 0, 1, 1)$, where w'' belongs to Q''_{n-1}. We have $\varrho(w'', v) = \varrho(u, v) - 2$ and the distance of v and w'' in Q''_{n-1} is again odd, therefore by induction there is a hamiltonian path p'' in Q''_{n-1} with end-vertices v and w''. Joining the paths p' and p'' by the edge (w, w'') we obtain a path p with the desired properties, q.e.d.

2.4. Remark. From 2.3 the following fact can be easily obtained: Let $u, v \in V(Q_n)$, $\varrho(u, v) \equiv 1 \pmod 2$; let l_1, l_2 be integers fulfilling $l_1, l_2 \geqq 1$, $l_1 + l_2 = 2^n - 2$. Then there are two vertex-disjoint paths of lenghts l_1, l_2 in Q_n with end-vertices u and v.

A similar fact can be proved also in the case of even $\varrho(u, v)$:

2.5. Proposition. *Let $u, v \in V(Q_n)$, $u \neq v$, $\varrho(u, v) \equiv 0 \pmod 2$; let l_1, l_2 be odd integers fulfilling $l_1, l_2 \geqq 1$, $l_1 + l_2 = 2^n - 2$. Then there are two vertex-disjoint paths of lenths l_1, l_2 in Q_n with end-vertices u and v.*

Proof. Assume first $l_1 \geqq \varrho(u, v) - 1$, $l_2 \geqq \varrho(u, v) - 1$. It follows from 2.1 that there is a hamiltonian circuit c in Q_n such that the distances of u and v along c are $l_1 + 1$, $l_2 + 1$. Removing two suitably chosen edges from c we obtain the paths required. Suppose now e.g. $l_1 < \varrho(u, v) - 1$. Let $u' \in V(Q_n)$ such that $\varrho(u, u') = l_1$, $\varrho(u, v) = \varrho(u, u') + \varrho(u', v)$. Then $\varrho(u', v)$ is odd and from 2.3 we conclude that there is a hamiltonian path p in Q_n with end-vertices u', v going "in the shortest possible way" from u' to $\bar{v}$. If necessary, we can achieve by a permutation of coordinates (more exactly, by constructing a new hamiltonian path arising from p) that p goes in the shortest way from u' to u. By removing one edge from p (namely that incident with u from the part of p joining u with v) we obtain the paths required.

2.6. Remark. The assumption of 2.5 that l_1, l_2 are odd cannot be omitted. (This may be seen from the example of $u, v \in V(Q_3)$ such that $\varrho(u, v) = 2$. There is no pair of vertex-disjoint paths of lengths 2 and 4 with end-vertices u and v in Q_3).

We shall need one more technical result:

2.7. Proposition. *Let $u, v, u', v' \in V(Q_n)$ be a quadruple of different vertices, let $\varrho(u, u') = \varrho(v, v') = 1$, $\varrho(u, v) = \varrho(u', v')$. Then there is a pair of vertex-disjoint paths p_1, p_2 in Q_n such that p_1 joins u with u', p_2 joins v with v' and both p_1 and p_2 have the same length $2^{n-1} - 1$.*

Proof. We prove that, given u, u', v, v' with the properties described above, there is i $(1 \leqq i \leqq n)$ such that for the i-canonical decomposition of Q_n into Q'_{n-1} and Q''_{n-1} the following holds: $u, u' \in V(Q'_{n-1})$, $v, v' \in V(Q''_{n-1})$. Then the assertion to be proved follows easily from 2.3.

Let $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$, $u' = (u'_1, \ldots, u'_n)$, $v' = (v'_1, \ldots, v'_n)$. Put $J = \{j;\ u_j \neq v_j\}$, let $u_k \neq u'_k$, $v_l \neq v'_l$. From the assumptions we easily derive the following assertion: if $k = l$, then $J - \{k\} \neq \emptyset$; if $k \neq l$, then $J - \{k, l\} \neq \emptyset$ as well. Thus it is in both cases possible to choose an integer i such that $u_i = u'_i \neq v'_i = v_i$, q.e.d.

3. SPANING TREES OF HYPERCUBES OBTAINED BY TRANSFORMATIONS OF BINARY TREES

In this section we describe certain spanning trees of hypercubes arising by simple transformations of binary trees.

For $n \geqq 2$ let B_n denote a complete binary tree on n vertex-levels with one edge added to its root. Fig. 3.1 shows B_2, B_3 and B_4.

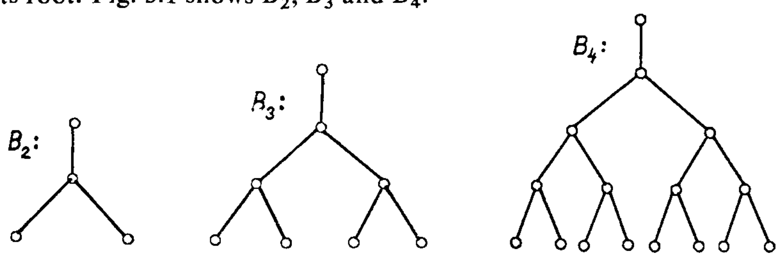

Fig. 3.1.

Obviously, B_n has $2^{n-1} + 1$ leaves and $2^{n-1} - 1$ vertices of degree 3, $|V(B_n)| = 2^n$.

Further, denote B_n by $B_n^{(1)}$ and for $k \geqq 2$ let $B_n^{(k)}$ arise from B_n by splitting each vertex into k new vertices (see examples in Fig. 3.2). Then obviously $|V(B_n^{(k)})| = k \cdot 2^n$.

For $k \geqq 2$ there is a unique path of length $2k - 1$ joining a leaf with a vertex of degree 3 in $B_n^{(k)}$. We call this path the main branch of $B_n^{(k)}$ (and draw it vertically).

3.1. Remark. It follows from [3] that dim $B_n = n + 1$ for $n \geqq 2$. Now we will prove that dim $B_n^{(2)} = n + 1$ as well.

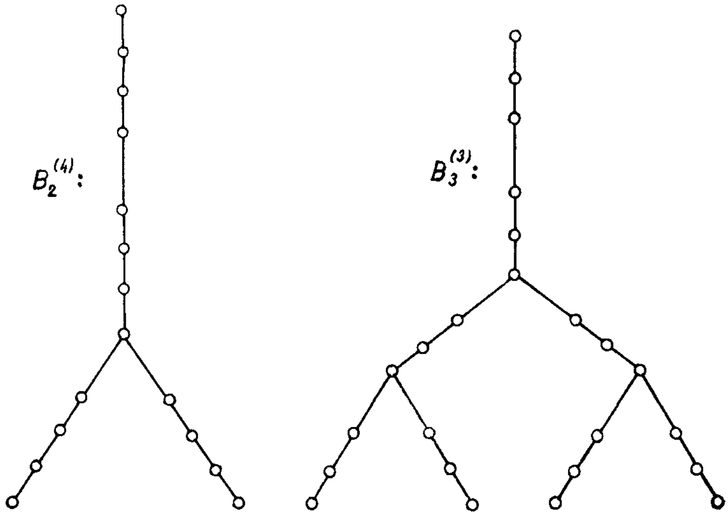

Fig. 3.2.

3.2. Proposition. *For* $n \geqq 2$, dim $B_n^{(2)} = n + 1$ *and since* $|V(B_n^{(2)})| = 2^{n+1}$, $B_n^{(2)}$ *is a spanning tree of* Q_{n+1}.

Proof. dim $B_n^{(2)} \geqq n + 1$ follows from $|V(B_n^{(2)})| = 2^{n+1}$. In order to prove dim $B_n^{(2)} \leqq n + 1$ we construct by induction a C_{n+1}-valuation of $B_n^{(2)}$:

a) C_3-valuation of $B_2^{(2)}$ is shown in Fig. 3.3.

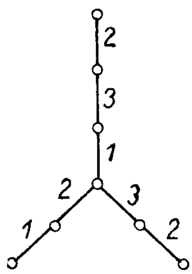

Fig. 3.3.

b) Assume such a C_{n+1}-valuation φ of $B_n^{(2)}$ to be given that the two upper edges of the main branch of $B_n^{(2)}$ have values n and $n + 1$ (top-down, cf. Fig. 3.4). Denote this C_{n+1}-valued tree $B_n^{(2)}$ by T. Take another copy of $B_n^{(2)}$ and construct its C_{n+1}-valuation φ' from φ by interchanging the values n and $n + 1$, i.e. define $\varphi' : E(B_n^{(2)}) \to \to \{1, \dots, n + 1\}$ by putting

$$\varphi'(e) = \begin{cases} \varphi(e) & \text{if } \varphi(e) < n\,, \\ n + 1 & \text{if } \varphi(e) = n\,, \\ n & \text{if } \varphi(e) = n + 1\,. \end{cases}$$

Denote the C_{n+1}-valued tree $B_n^{(2)}$ obtained in this way by T' (cf. Fig. 3.4).

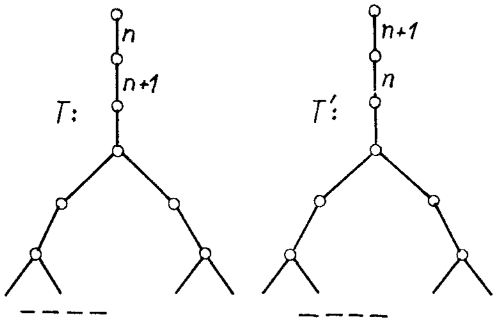

Fig. 3.4.

Now, we construct a C_{n+2}-valued tree $B_{n+1}^{(2)}$ from T and T' as follows (cf. Fig. 3.5): delete two upper edges of the main branch of T', join the leaf obtained by a new edge to the second from above vertex of the main branch of T, assign $n + 2$ to this new edge and finally add the path of length 2 (with values $n + 1$ and $n + 2$ on its edges) to the upper-most leaf (in Fig. 3.5 the new edges are drawn by thick lines).

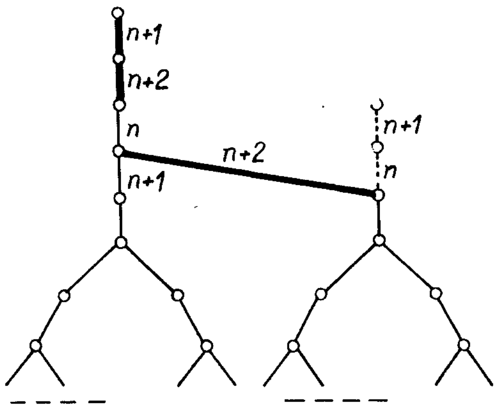

Fig. 3.5.

It may be easily verified that the valuation of $B_{n+1}^{(2)}$ constructed as described is a C_{n+2}-valuation, q.e.d.

3.3. Proposition. *For $n \geqq 2$ and $s \geqq 1$,* dim $B_n^{(2^s)} = n + s$ *and since* $|V(B_n^{(2^s)})| = 2^{n+s}$, $B_n^{(2^s)}$ *is a spanning tree of* Q_{n+s}.

Proof. The case $s = 1$ is solved by 3.2, assume therefore $s \geqq 1$ and let a C_{n+s}-valuation φ of $B_n^{(2^s)}$ be given. Note that $B_n^{(2^{s+1})}$ arises from $B_n^{(2^s)}$ by replacing each vertex by an edge; call these edges "new" and define a valuation $\varphi': E(B_n^{(2^{s+1})}) \to \{1, \ldots, n + s + 1\}$ by putting $\varphi'(e) = \varphi(e)$ if e is not a new edge and $\varphi'(e) = n + s + 1$ for all the new edges. Obviously, φ' is a C_{n+s+1}-valuation of $B_n^{(2^{s+1})}$.

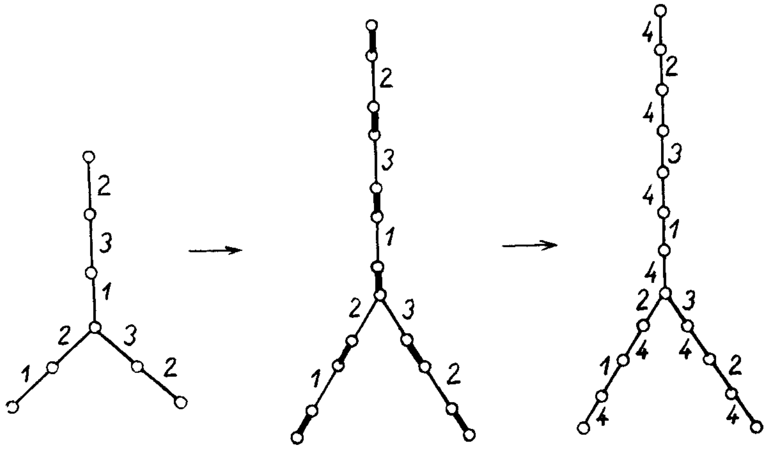

Fig. 3.6.

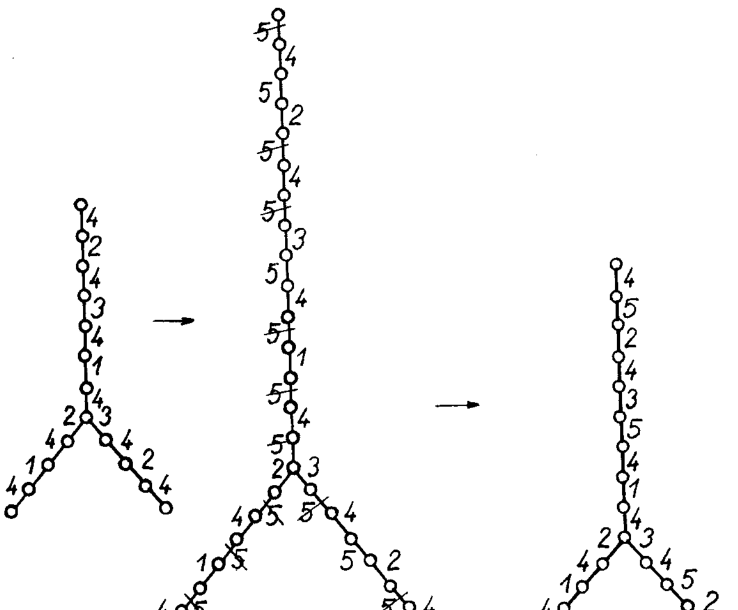

Fig. 3.7.

A lower bound $\dim B_n^{(2^s)} \geqq n + s$ is trivial, since $|V(B_n^{(2^s)})| = 2^{n+s}$. Fig. 3.6 illustrates the construction for the case $n = 2$, $s = 1$, the new edges being again drawn by thick lines.

3.4. Corollary. *For $n \geqq 2$ and $k > 1$, $\dim B_n^{(k)} = n +]\log_2 k[$.*

Proof. The case $k = 2^s$ for some $s > 0$ is solved by 3.3. For k different from the powers of 2 we first prove the lower bound $\dim B_n^{(k)} \geqq n +]\log_2 k[$, which immediately follows by comparing the cardinalities of the vertex sets ($B_n^{(k)} \subseteq Q_m$ implies the inequality $k \cdot 2^n \leqq 2^m$). For the proof of $\dim B_n^{(k)} \leqq n +]\log_2 k[$ let $s =]\log_2 k[$ and let φ be a C_{n+s}-valuation of $B_n^{(2^s)}$ constructed according to 3.3. Investigating the last step of this proof, i.e., the construction of the C_{n+s}-valued $B_n^{(2^s)}$ from the C_{n+s-1}-valued $B_n^{(2^{s-1})}$, we can see that each path of length 2^{s-1} of $B_n^{(2^{s-1})}$ between two vertices of degree 3 or between a vertex of degree 3 and a leaf was extended by adding 2^{s-1} new edges to the path of length 2^{s+1} (the main branch of $B_n^{(2^{s-1})}$ being extended by adding 2^s new edges to the path of length $2^{s+1} - 1$). The desired C_{n+s}-valued tree $B_n^{(k)}$ can be then obtained by removing arbitrarily chosen $2^s - k$ new edges from every such path (and by removing arbitrarily chosen $2(2^s - k)$ new edges from the new main branch). As an example, Fig. 3.7 shows the construction of the C_5-valued $B_2^{(5)}$ from C_4-valued $B_2^{(4)}$ via the C_5-valued $B_2^{(8)}$.

4. FURTHER SPANNING TREES OF HYPERCUBES

4.1. Definition. *For $n \geqq 3$, any graph homeomorphic to an n-star will be called an n-quasistar.*

The paths joining the centre of an n-quasistar with its leaves will be called rays of a quasistar and will be denoted by $R_1, \ldots, R_n$. A ray is even (odd), if its length – i.e. the number of edges in it – is even (odd).

Fig. 4.1 shows two different 3-quasistars.

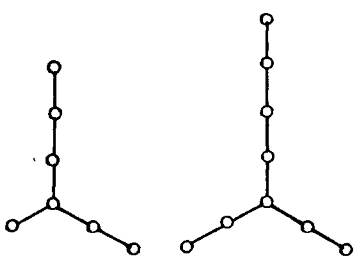

Fig. 4.1.

4.2. Remark. A bipartite graph is called balanced, if it may be regularly coloured by colours c_1, c_2 in such a way that the number of vertices coloured by c_1 equals that of vertices coloured by c_2. Obviously, Q_n is balanced ($n \geqq 1$) and if T is a spanning tree of Q_n, then $|V(T)| = 2^n$, maxdeg $(T) \leqq n$ and T is balanced as well. Further, an n-quasistar is balanced if and only if it has just one odd ray.

4.3. Proposition. *Let S be a balanced 3-quasistar with $|V(S)| = 2^n$ for some $n \geqq 3$. Then S is a spanning tree of Q_n and there is such an embedding of S into Q_n that the images of the end-vertices of the two even rays of S have distance 2 in Q_n.*

Proof. The proof proceeds by induction on n. A 3-quasistar S is uniquely determined by the triple (r_1, r_2, r_3) of positive integers $r_1 \leqq r_2 \leqq r_3$, denoting the lengths of its rays. There are exactly two balanced 3-quasistars having 8 vertices; these are $(1, 2, 4)$ and $(2, 2, 3)$ and both of them are embeddable in Q_3 (and therefore also its spanning trees). Their embeddings satisfying the condition on the end-vertices of the even rays are shown in Fig. 4.2.

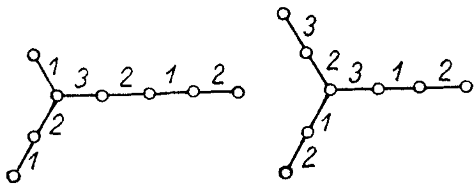

Fig. 4.2.

Assume now $n > 3$, let $S = (r_1, r_2, r_3)$ be a balanced 3-quasistar with rays R_1, R_2, R_3, let $|V(S)| = 2^n$ and hence $r_1 + r_2 + r_3 = 2^n - 1$. Recall that $r_1 \leqq r_2 \leqq r_3$.

1. Suppose $r_3 = 2^{n-1}$, let e.g. r_1 be even (in the case of r_1 odd and r_2 even we proceed quite similarly). From 2.2 we conclude that it is possible to construct a C_{n-1}-valuation of the path of length $2^{n-1} - 1$ formed by R_1 and R_2 so that $O(R_1) = \{1, 2\}$. We shall now extend this valuation to a C_n-valuation of the whole S as follows: the edge of R_3 incident with the centre of S obtains the value n and the remaining part of R_3 which is a path of length $2^{n-1} - 1$ will be C_{n-1}-valued (using e.g. the basic

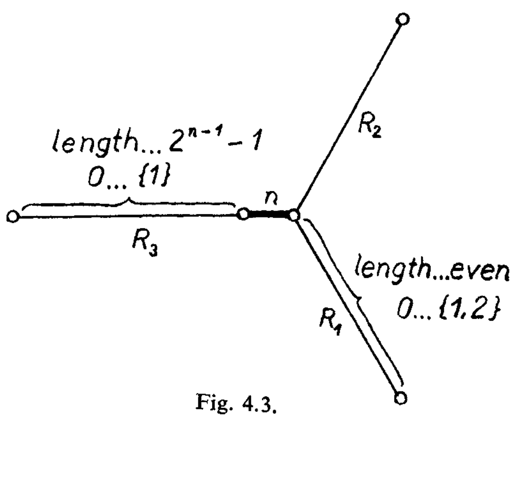

Fig. 4.3.

C_{n-1}-valuation) so that $O(R_3) = \{1\}$. The valuation obtained is obviously a C_n-valuation of S; moreover, if we denote by $R_1 + R_3$ the path formed by R_1 and R_3, then $O(R_1 + R_3) = \{2, n\}$, q.e.d. (cf. Fig. 4.3).

2. Suppose now $r_3 > 2^{n-1}$, let e.g. r_1 be even. It follows by induction that there is a C_{n-1}-valuation of a 3-quasistar $(r_1, r_2, r_3 - 2^{n-1})$ with rays R_1, R_2 and R'_3 (where R'_3 arises from R_3 by removing the path of length 2^{n-1}). Moreover, if r_3 is odd, then $O(R_1 + R_2) = \{1, 2\}$; if r_3 is even, then $O(R_1 + R'_3) = \{1, 2\}$. In both cases we extend this valuation by assigning values to edges of $R_3 - R'_3$ as follows: the edge nearest to the centre of S obtains n, the remaining path p of length $2^{n-1} - 1$ will be C_{n-1}-valued (using again e.g. the basic C_{n-1}-valuation) in such a way that $O(p) = \{1\}$. Obviously we obtain a C_n-valuation of S; if r_3 is odd, $O(R_1 + R_2) = \{1, 2\}$; if r_3 is even, $O(R_1 + R_3) = \{2, n\}$), q.e.d. (cf. Fig. 4.4.).

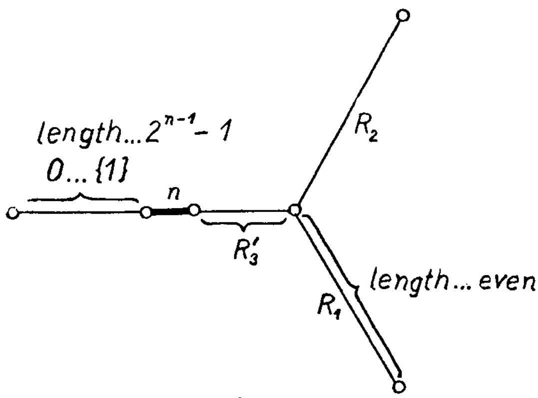

Fig. 4.4.

3. Suppose $r_2 > 2^{n-2}$ (and therefore also $r_3 > 2^{n-2}$). We remove the paths of length 2^{n-2} from R_2 and R_3 and obtain in this way a 3-quasistar $S' = (r_1, r_2 - 2^{n-2}, r_3 - 2^{n-2})$ with rays R_1, R'_2, R'_3. It follows by induction that there is a C_{n-1}-valuation of S' satisfying the additonal condition concerning the end-vertices of the even rays. Again we will extend this C_{n-1}-valuation to the C_n-valuation of the whole S; we proceed as follows:

3a. If r_2 and r_3 are even, then $|O(R'_2 + R'_3)| = 2$. Consider a canonical decomposition of Q_n into Q'_{n-1} and Q''_{n-1}; by induction there is an embedding of S' in Q'_{n-1} such that the images u' and v' of the end-vertices of R'_2 and R'_3 have distance 2 in Q'_{n-1}. We assign value n to the first edges (nearest to the centre of S) removed from R_2 and R_3. This means (in terms of the embedding) a transition from Q'_{n-1} into Q''_{n-1}. The vertices u, v obtained in this way have again distance 2; choose vertices u'' and v'' in Q''_{n-1} such that $\varrho(u, u'') = \varrho(v, v'') = 1$, $\varrho(u'', v'') = \varrho(u, v) = 2$. According to 2.7 there are two vertex-disjoint paths p_1, p_2 in Q''_{n-1} such that p_1 joins u with u'', p_2 joins v with v'' and both p_1 and p_2 have length $2^{n-2} - 1$. Hence we can use p_1 and p_2 for embedding the parts of R_2 and R_3 removed from them at the beginning, q.e.d. (cf. Fig. 4.5).

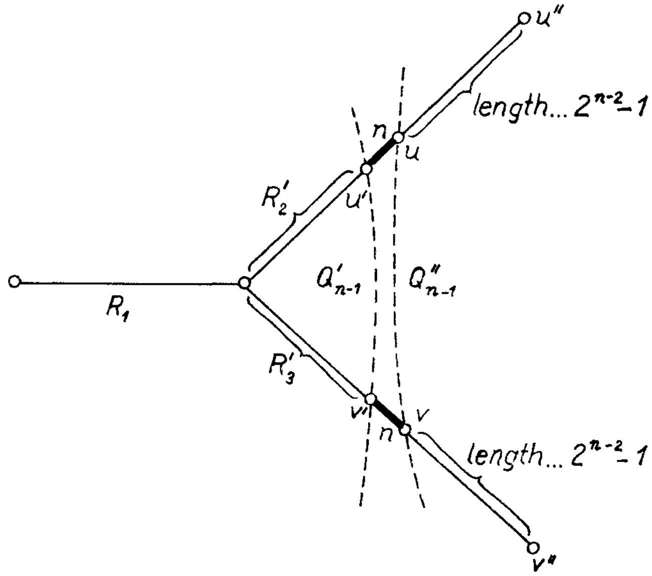

Fig. 4.5.

3b. Let r_1 be even (and therefore $r_2 \not\equiv r_3 \pmod 2$). Without loss of generality let r_2 be even. Again, there is an embedding of S' in Q_{n-1}' such that $O(R_1 + R_2') = \{1, 2\}$; the first edges removed from R_2 and R_3 will be assigned n. The vertices u and v obtained in this way in Q_{n-1}'' of the canonical decomposition of Q_n have an

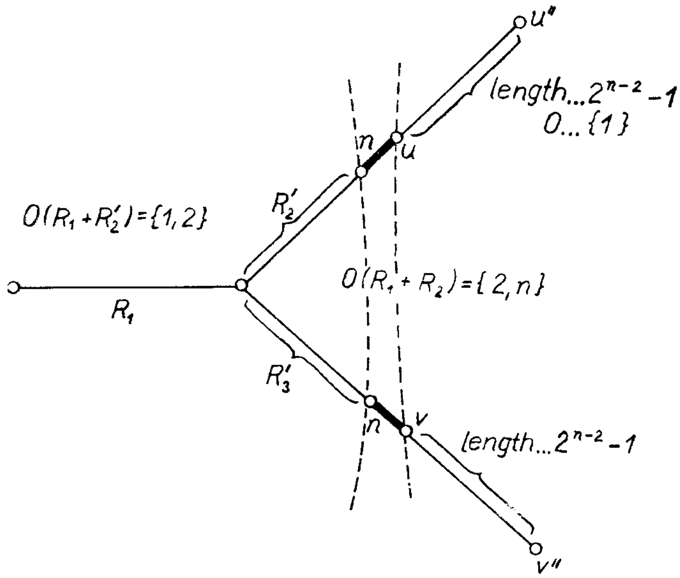

Fig. 4.6.

odd distance; in order to extend the existing valuation to the whole S we use again 2.7 in such a way that $|O(R_1 + R_2)| = 2$ (cf. Fig. 4.6).

4. Suppose that neither the case 1 nor 2 nor 3 holds. Then necessarily $r_1 = r_2 = 2^{n-2}$ and $r_3 = 2^{n-1} - 1$. To see it recall that $r_1 \leqq r_2 \leqq r_3$ and $r_1 + r_2 + r_3 = 2^n - 1$. Hence $r_1 + r_2 < 2^{n-1}$ implies $r_3 \geqq 2^{n-1}$ and the case 1 or 2 would follow; therefore $r_1 + r_2 \geqq 2^{n-1}$ and since $r_1 \leqq r_2$ and $r_2 \leqq 2^{n-2}$ (otherwise the case 3 would take place), the desired equalities follow.

In order to construct a C_n-valuation of S we proceed in this case as follows (cf. Fig. 4.7): assign to edges of R_1 (which is of length 2^{n-2}) from the end to the centre of S the values of the basic C_{n-2}-valuation; to the edge incident with the centre give the value $n - 1$; to edges of R_2 (which is of length 2^{n-2} as well) we assign (from the centre to the end) again the values of the basic C_{n-2}-valuation, while the edge incident with the leaf obtains n. The edges of R_3 (of length $2^{n-1} - 1$) are treated in the following way: the edge incident with the centre obtains n, the others (in the direction to the leaf) the values of the basic C_{n-1}-valuation, the last value $(=1)$ not being used, since the length of the whole R_3 is only $2^{n-1} - 1$. Thereafter we interchange the values $n - 1$ and $n - 2$. It may be easily checked that the valuation of S obtained is its C_n-valuation fulfilling $O(R_1) = \{n - 2, n - 1\}$, $O(R_2) = \{n - 2, n\}$, therefore $|O(R_1 + R_2)| = 2$, which completes the proof of the whole proposition.

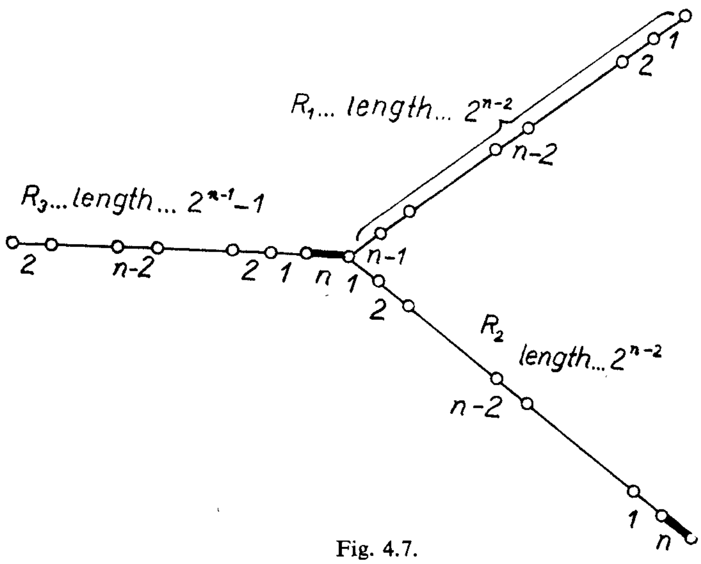

Fig. 4.7.

The following statement describes another class of spanning trees of hypercubes.

4.4. Proposition. *Let T be a tree fulfilling the following conditions: T is balanced, $|V(T)| = 2^n$ for some $n \geqq 3$,* maxdeg $(T) = 3$ *and T has exactly 2 vertices of degree 3. Then T is a spanning tree of Q_n.*

Proof. A tree T fulfilling the assumptions is uniquely determined by the 5-tuple of positive integers (r_1, r_2, a, r_3, r_4), where $r_1, \ldots, r_4$ are the lengths of the four rays $R_1, \ldots, R_4$ of T and a is the length of the axial path A of T (cf. Fig. 4.8).

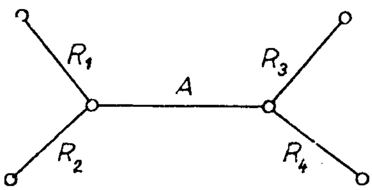

Fig. 4.8.

We have $a + r_1 + r_2 + r_3 + r_4 = 2^n - 1$, assume without loss of generality $r_1 + r_2 \leqq r_3 + r_4$.

1. Let $r_3 + r_4 > 2^{n-1}$; suppose e.g. $r_3 \leqq r_4$. If $r_3 > 2$, then it is possible to remove an even positive number of edges both from R_3 and R_4 in such a manner that altogether 2^{n-1} edges are deleted; the tree $(r_1, r_2, a, r'_3, r'_4)$ obtained in this way has $2^{n-1} - 1$ edges and obviously is balanced. Therefore (by induction), it is a spanning tree of Q_{n-1}; let us assign n to the first edges of the removed parts of R_3 and R_4 (in terms of the embedding this means a transition from Q'_{n-1} to Q''_{n-1} in the canonical

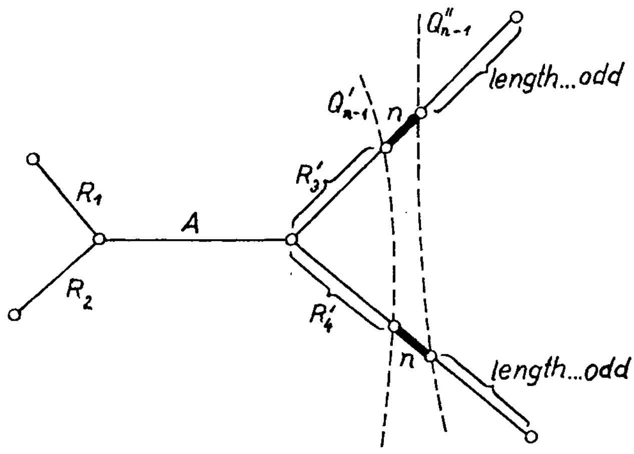

Fig. 4.9.

decomposition of Q_n into Q'_{n-1} and Q''_{n-1}). Since the remaining parts of R_3 and R_4 have odd lengths it is possible to extend the construction of the C_n-valuation to the whole T according to 2.4 or 2.5 (cf. Fig. 4.9).

We proceed similarly also in the case $r_3 = 2$ – then we remove the whole R_3 and obtain a balanced 3-quasistar $(r_1, r_2, a + r_4 - 2^{n-1} + 2)$ (cf. Fig. 4.10).

Let now $r_3 = 1$, then $r_4 \geqq 2^{n-1}$. We delete 2^{n-1} edges from R_4 and obtain either a quasistar or a tree $(r_1, r_2, a, 1, r_4 - 2^{n-1})$; both of them are spanning trees of Q_{n-1}.

Let us extend the corresponding C_{n-1}-valuation as follows: the first edge obtains n (as usual it means a transition to Q_{n-1} in a canonical decomposition) and for the remaining path of length $2^{n-1} - 1$ we can use e.g. the basic C_{n-1}-valuation, q.e.d.

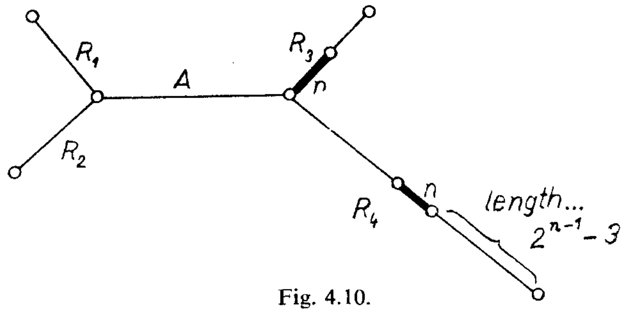

Fig. 4.10.

2. Let $r_3 + r_4 < 2^{n-1}$ (and therefore $r_1 + r_2 < 2^{n-1}$ as well). Then there is an edge of the axial path such that by removing it we obtain from T two graphs having $2^{n-1} - 1$ edges which are either balanced 3-quasistars or paths. Hence, by induction, they are spanning trees of Q_{n-1} and it suffices to take their corresponding C_{n-1}-valuations and to assign n to the edge that has been previously removed.

3. Let $r_3 + r_4 = 2^{n-1}$, both r_3 and r_4 being even. We delete from T the whole rays R_3 and R_4 and obtain in this way a balanced 3-quasistar S which is a spanning tree of Q_{n-1}. Let u be a vertex of T incident with R_3, R_4 and A (then u is obviously a leaf of S – cf. Fig. 4.11). We extend an existing C_{n-1}-valuation of S to the whole T as follows: change the value i of the (only) edge of S incident with u to n and assign n also to the last edge of R_4. Let R_4' denote the rest of R_4 after removing the last edge; it is possible (using 2.2) to construct a C_{n-1}-valuation of $R_3 + R_4'$ (whose length is $2^{n-1} - 1$) such that $O(R_4') = \{i\}$. Then it may be easily checked that in this way a C_n-valuation of T arises, q.e.d.

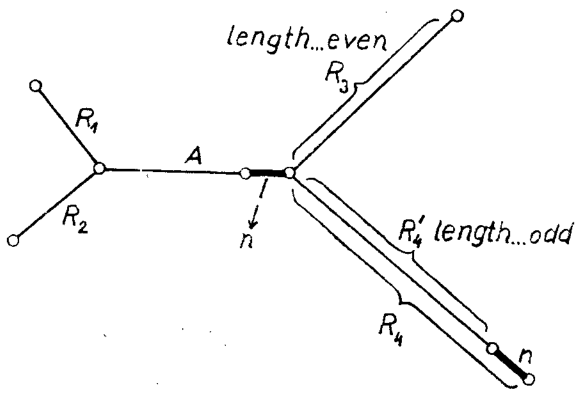

Fig. 4.11.

4. Let $r_3 + r_4 = 2^{n-1}$, both r_3 and r_4 being odd. Since T is balanced, r_1, r_2 and a have to be odd as well. We proceed as follows (cf. Fig. 4.12): let u be the vertex of A whose distance from the common vertex of R_3 and R_4 equals 1 (if $a = 1$, then u is incident with both R_1 and R_2); let us remove from T the whole R_3 and R_4 and also

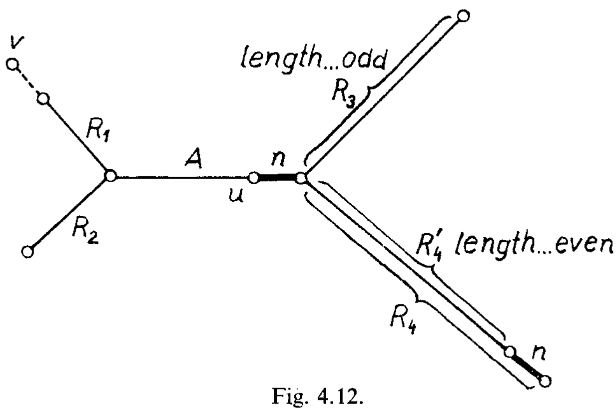

Fig. 4.12.

the last edge of A (incident with u). Denote the graph obtained by S. Obviously S has $2^{n-1} - 2$ edges; if $a > 1$, then S is a non-balanced 3-quasistar $(r_1, r_2, a - 1)$, if $a = 1$, then S is a path. Add for a moment a new edge to R_1 in S, let S' be the graph obtained and let v be the new end-vertex of the extended R_1 in S'. Denote by p the path in S' joining u and v. We shall construct a C_{n-1}-valuation of S' such that $|O(p)| = 2$; use for it 3.3 in case that $a > 1$ and therefore $S' = (r_1 + 1, r_2, a - 1)$ is a balanced 3-quasistar, and 2.2 if $a = 1$ and S' is a path. This C_{n-1}-valuation partialized to S will be the starting point of the construction of the desired C_n-valuation of T: let us assign n to the last edge of A (having been previously removed) and also to the last edge of R_4; further, we construct a C_{n-1}-valuation of the remaining part R_4' of R_4 and of the whole R_3 in such a way that $O(R_4') = O(p)$. It may be easily checked that we have obtained a C_n-valuation of T, q.e.d. This completes the proof of the whole proposition.

5. CONCLUDING REMARKS, OPEN PROBLEMS AND CONJECTURES

The propositions proved in the previous sections might be useful when trying to solve the following

5.1. Open problem. *Characterize the spanning trees of Q_n!*

Let us note here that the conditions mentioned in 4.2, necessary for T to be a spanning tree of Q_n (namely, that $|V(T)| = 2^n$, T is balanced and $\operatorname{maxdeg}(T) \leqq n$) are not sufficient. In order to see this start from the so called 4-tomic tree on 2 levels of edges, denoted by $T_2^{(4)}$ (cf. Fig. 5.1). It is proved in [4] that for $k \geqq 2$, $\dim T_2^{(k)} =$

$=](3k + 1)/2[$, hence $\dim T_2^{(4)} = 7$. We can easily construct (by adding new vertices and edges to $T_2^{(4)}$) a balanced tree T' with 64 vertices and $\operatorname{maxdeg}(T') = 5$ such that $\dim T' \geqq 7$; hence, T' cannot be a spanning tree of Q_6.

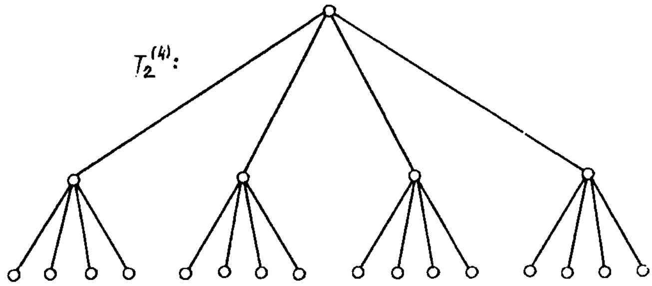

Fig. 5.1.

On the other hand, it seems not quite hopeless to try to strengthen further 4.4 (cf. also 4.3), possibly to the following

5.2. Conjecture. *Let T be a balanced tree,* $|V(T)| = 2^n$, $\operatorname{maxdeg}(T) \leqq 3$. *Then T is a spanning tree of Q_n.*

We recall in this connection that [8] contains two examples of spanning trees of Q_n with maximal degree 3 having a large number of vertices of degree 3.

Another way of generalizing 4.3 is the following

5.3. Conjecture. *Let T be a balanced l-quasistar,* $|V(T)| = 2^n$, $l \leqq n$. *Then T is a spanning tree of Q_n.*

[9] contains the proof of the latter conjecture for $l = 4$ and 5.

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