

On Hamiltonian Perturbations of Hyperbolic Systems of Conservation Laws, II: Universality of Critical Behaviour

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Abstract: Hamiltonian perturbations of the simplest hyperbolic equation $u_t + a(u)u_x = 0$ are studied. We argue that the behaviour of solutions to the perturbed equation near the point of gradient catastrophe of the unperturbed one should be essentially independent on the choice of generic perturbation neither on the choice of generic solution. Moreover, this behaviour is described by a special solution to an integrable fourth order ODE.

1. Introduction

In the present work we continue the study of Hamiltonian perturbations of hyperbolic PDEs initiated by the paper [10]. We consider here the simplest case of a single equation in one spatial dimension,

$$u_t + a(u)u_x + \epsilon \left[b_1(u)u_{xx} + b_2(u)u_x^2 \right] + \epsilon^2 \left[b_3(u)u_{xxx} + b_4(u)u_x u_{xx} + b_5(u)u_x^3 \right] + \dots = 0. \quad (1.1)$$

Here ϵ is a small parameter; the coefficient of ϵ^k is a graded homogeneous polynomial in the derivatives u_x, u_{xx}, \dots of the total degree $(k + 1)$,

$$\deg u^{(n)} = n, \quad n > 0.$$

The unperturbed equation

$$u_t + a(u)u_x = 0 \quad (1.2)$$

can be considered as the simplest example of a nonlinear hyperbolic system; the smooth functions $b_1(u), b_2(u)$, etc. determine the structure of the perturbation.

Such expansions arise, e.g., in the study of the long wave (also called dispersionless) approximations of evolutionary PDEs; see Sect. 5 below for other mechanisms that yield perturbed equations of the form (1.1).

The unperturbed equation (1.2) admits a Hamiltonian description of the form

$$u_t + \{u(x), H_0\} \equiv u_t + \partial_x \frac{\delta H_0}{\delta u(x)} = 0, \quad (1.3)$$

$$H_0 = \int f(u) dx, \quad f''(u) = a(u),$$

$$\{u(x), u(y)\} = \delta'(x - y). \quad (1.4)$$

The perturbed equations of the form (1.1) are considered up to equivalencies defined by *Miura-type transformations* [9] of the form

$$u \mapsto u + \sum_{k \geq 1} \epsilon^k F_k \left(u; u_x, \dots, u^{(k)} \right), \quad (1.5)$$

where $F_k(u; u_x, \dots, u^{(k)})$ is a graded homogeneous polynomial in the derivatives u_x, u_{xx}, \dots of the degree

$$\deg F_k = k.$$

Using results of [15] (see also [6, 9]) one can show that any Hamiltonian perturbation of Eq. (1.2) can be reduced to the form

$$u_t + \partial_x \frac{\delta H}{\delta u(x)} = 0, \quad H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \dots,$$

$$H_k = \int h_k \left(u; u_x, \dots, u^{(k)} \right) dx, \quad \deg h_k \left(u; u_x, \dots, u^{(k)} \right) = k. \quad (1.6)$$

Recall that for $H = \int h(u; u_x, u_{xx}, \dots) dx$,

$$\frac{\delta H}{\delta u(x)} = \mathcal{E} h,$$

where

$$\mathcal{E} = \frac{\partial}{\partial u} - \partial_x \frac{\partial}{\partial u_x} + \partial_x^2 \frac{\partial}{\partial u_{xx}} - \dots$$

is the Euler – Lagrange operator. The following well known property of the Euler – Lagrange operator will be often used in this paper: $\mathcal{E} h = 0$ iff there exists $h_1 = h_1(u; u_x, \dots)$ such that $h = \text{const} + \partial_x h_1$. Note that we do not specify here the class of functions $u(x)$. The Hamiltonians $H = H[u]$ can be ill defined (e.g., a divergent integral) but the evolutionary PDE (1.6) makes sense. The crucial point for the subsequent considerations is the following statement (see, e.g., [7]): for two commuting Hamiltonians

$$\{H, F\} = 0 \quad \Leftrightarrow \quad \mathcal{E} \left(\frac{\delta H}{\delta u(x)} \partial_x \frac{\delta F}{\delta u(x)} \right) = 0$$

the evolutionary PDEs

$$u_t + \partial_x \frac{\delta H}{\delta u(x)} = 0 \quad \text{and} \quad u_s + \partial_x \frac{\delta F}{\delta u(x)} = 0$$

commute,

$$(u_t)_s = (u_s)_t.$$

For sufficiently small ϵ one expects to see no major differences in the behaviour of solutions to the perturbed and unperturbed Eqs. (1.1) and (1.2) within the regions where the x -derivatives are bounded. However the differences become quite serious near the critical point (also called the point of *gradient catastrophe*) where the derivatives of solution to the unperturbed equation tend to infinity.

Although the case of small viscosity perturbations has been well studied and understood (see [3] and references therein), the critical behaviour of solutions to *general conservative perturbations* (1.6) to our best knowledge has not been investigated (see the papers [32, 12, 17–19, 23–25, 28] for the study of various particular cases).

The main goal of this paper is to formulate the Universality Conjecture about the behaviour of a generic solution to the general perturbed Hamiltonian equation near the point of gradient catastrophe of the unperturbed solution. We argue that, up to shifts, Galilean transformations and rescalings this behaviour essentially *does not* depend on the choice of solution neither on the choice of the equation (provided certain genericity assumptions hold valid). Moreover, this behaviour near the point (x_0, t_0, u_0) is given by

$$u \simeq u_0 + a \epsilon^{2/7} U \left(b \epsilon^{-6/7} (x - a_0(t - t_0) - x_0); c \epsilon^{-4/7} (t - t_0) \right) + O \left(\epsilon^{4/7} \right), \tag{1.7}$$

where $U = U(X; T)$ is the unique real *smooth for all* $X \in \mathbb{R}$ solution to the fourth order ODE,

$$X = T U - \left[\frac{1}{6} U^3 + \frac{1}{24} \left(U'^2 + 2U U'' \right) + \frac{1}{240} U^{IV} \right], \quad U' = \frac{dU}{dX}, \quad \text{etc.}, \tag{1.8}$$

depending on the parameter T . Here a, b, c are some constants that depend on the choice of the equation and the solution, $a_0 = a(v_0)$.

Equation (1.8) appeared in [4] (for the particular value of the parameter $T = 0$) in the study of the double scaling limit for the matrix model with the multicritical index $m = 3$. It was observed that generic solutions to (1.8) blow up at some point of real line; the conjecture about existence of a unique smooth solution has been formulated. To our best knowledge, this conjecture remains open, although there are some supporting evidences [20].

The present paper is organized as follows. In Sect. 2 we classify all Hamiltonian perturbations up to the order ϵ^4 . They are parametrized by two arbitrary functions $c(u), p(u)$. For the simplest example the perturbations of the Riemann wave equation $u_t + u u_x = 0$ read

$$\begin{aligned} u_t + u u_x + \frac{\epsilon^2}{24} \left[2c u_{xxx} + 4c' u_x u_{xx} + c'' u_x^3 \right] \\ + \epsilon^4 \left[2p u_{xxxx} + 2p' (5u_{xx} u_{xxx} + 3u_x u_{xxxx}) \right. \\ \left. + p'' \left(7u_x u_{xx}^2 + 6u_x^2 u_{xxx} \right) + 2p''' u_x^3 u_{xx} \right] = 0. \end{aligned} \tag{1.9}$$

For $c(u) = \text{const}$, $p(u) = 0$ this is nothing but the Korteweg - de Vries (KdV) equation; for other choices of the functions $c(u)$, $p(u)$ it seems not to be an integrable PDE. Remarkably, for arbitrary choice of the functional parameters the perturbed equation possesses an infinite family of *approximate symmetries* (see [2, 9, 22, 30] for discussion of approximate symmetries). In principle our approach can be applied to classifying the Hamiltonian perturbations of higher orders. However, higher order terms do not affect the type of critical behaviour.

In Sect. 3 we establish an important property of *quasitriviality* of all perturbations (cf. [9, 10, 27]). The quasitriviality is given by a substitution

$$u \mapsto u + \epsilon^2 K_2(u; u_x, u_{xx}, u_{xxx}) + \epsilon^4 K_4(u; u_x, \dots, u^{(6)}) \quad (1.10)$$

that transforms, modulo $O(\epsilon^6)$ the unperturbed equation (1.2) to (1.6). Here the functions K_2 and K_4 depend *rationally* on the x -derivatives. We also formulate the first part of our Main Conjecture that says that, for sufficiently small ϵ the solution to the perturbed system exists at least on the same domain of the (x, t) -plane where the unperturbed solution is defined. In Sect. 4 we briefly discuss existence of a bihamiltonian structure compatible with the perturbation (see also Appendix below). Some examples of perturbed Hamiltonian equations are described in Sect. 5. In Sect. 6 we recollect some properties of the ODE (1.8) and we formulate the second part of the Main Conjecture describing the special function $U(X; T)$ in (1.7) as a particular solution to (1.8). Finally, in Sect. 7 we give the precise formulation of the Universality Conjecture (Main Conjecture, Part 3) and give some evidences supporting it¹. Because of lack of space we do not consider the numerical evidences supporting the idea of Universality; they will be given in a subsequent publication (see also [16]). In the last section we outline the programme of further researches towards understanding of universality phenomena of critical behaviour in general Hamiltonian perturbations of hyperbolic systems.

2. Hamiltonian Perturbations of the Riemann Wave Equation

Let us start with the simplest case of Hamiltonian perturbations of the equation

$$\begin{aligned} v_t + v v_x &= 0 \quad \Leftrightarrow \quad v_t + \{v(x), H_0\} = 0, \\ \{v(x), v(y)\} &= \delta'(x - y), \\ H_0 &= \int \frac{v^3}{6} dx. \end{aligned} \quad (2.1)$$

Lemma 2.1. *Up to the order $O(\epsilon^4)$, all Hamiltonian perturbations of (2.1) can be reduced to the form*

$$\begin{aligned} u_t + \partial_x \frac{\delta H}{\delta u(x)} &= 0, \\ H &= \int \left[\frac{u^3}{6} - \epsilon^2 \frac{c(u)}{24} u_x^2 + \epsilon^4 \left(p(u) u_{xx}^2 + s(u) u_x^4 \right) \right] dx, \end{aligned} \quad (2.2)$$

where $c(u)$, $p(u)$, $s(u)$ are arbitrary functions. Moreover, the function $s(u)$ can be eliminated by a Miura-type transform.

¹ Perhaps, only this Part 3 deserves the name of the Main Conjecture. However, the precise formulation of it depends on the first two parts.

Proof. The Hamiltonian must have the form

$$H = H_0 + \epsilon H_1 + \dots + \epsilon^4 H_4,$$

where the density of H_k is a graded homogeneous polynomial of the degree k . So, the density of H_1 is a total derivative:

$$H_1 = \int \alpha(u) u_x dx, \quad \alpha(u) u_x = \partial_x A(u), \quad A'(u) = \alpha(u).$$

The density of the Hamiltonian H_2 modulo total derivatives must have the form

$$-\frac{c(u)}{24} u_x^2$$

for some function $c(u)$. Similarly, H_3 must have the form

$$H_3 = \int c_1(u) u_x^3 dx.$$

Here $c_1(u)$ is another arbitrary function.

Let us show that H_3 can be eliminated by a Miura-type transform. Let us look for it in the form

$$u \mapsto u + \epsilon \{u(x), F\} + \frac{\epsilon^2}{2} \{\{u(x), F\}, F\} + \dots, \quad (2.3)$$

choosing

$$F = \epsilon^2 \int \alpha(u) u_x^2 dx.$$

Such a transformation preserves the Poisson bracket. The change of the Hamiltonian H will be given by

$$\delta H = \epsilon \{F, H\} + O(\epsilon^4).$$

At the order ϵ^3 one has

$$\delta H = \epsilon^3 \int \left[\frac{1}{2} \alpha'(u) u_x^2 - \partial_x (\alpha u_x) \right] u u_x dx = \frac{\epsilon^3}{2} \int \alpha(u) u_x^3 dx.$$

So, choosing $\alpha(u) = -2c_1(u)$ we kill the terms cubic in ϵ .

The rest of the proof is obvious: in order 4 all the Hamiltonians have the form

$$H_4 = \int \left[p(u) u_{xx}^2 + s(u) u_x^4 \right] dx$$

for some functions $p(u), s(u)$. The last term can be killed by the canonical transformation of the form (2.3) generated by the Hamiltonian

$$F = -\frac{\epsilon^3}{2} \int s(u) u_x^3 dx.$$

The lemma is proved. \square

Choosing $s(u) = 0$ one obtains the family (1.9) of Hamiltonian perturbations of the Riemann wave equation depending on two arbitrary functions $c = c(u)$, $p = p(u)$.

We will now compare the symmetries of (2.1) and those of the perturbed system (2.2). It is easy to see that the Hamiltonian equation

$$\begin{aligned} v_s + a(v)v_x = 0 &\Leftrightarrow v_s + \{v(x), H_f^0\} = 0, \\ H_f^0 = \int f(v) dx, \quad f''(v) = a(v) \end{aligned} \quad (2.4)$$

is a symmetry of (2.1) for any $a(v)$,

$$(v_t)_s = (v_s)_t.$$

Moreover, the Hamiltonians H_f^0 commute pairwise,

$$\{H_f^0, H_g^0\} = 0 \quad \forall f = f(u), \quad \forall g = g(u).$$

This family of commuting Hamiltonians is *complete* in the following sense.

Lemma 2.2. *The family of commuting Hamiltonians H_f^0 is maximal, i.e., if $H = \int h(u; u_x, u_{xx}, \dots) dx$ commutes with all functionals of the form H_f^0 then*

$$h(u; u_x, u_{xx}, \dots) = g(u) + \partial_x(\dots)$$

for some function $g(u)$.

We will now construct a perturbation of the Hamiltonians H_f^0 preserving the commutativity modulo $O(\epsilon^6)$. Like in Lemma 2.1 one can easily check that all the perturbations up to the order 4 must have the form

$$H_f = \int \left\{ f(u) - \epsilon^2 \frac{c_f(u)}{24} u_x^2 + \epsilon^4 \left[p_f(u) u_{xx}^2 + s_f(u) u_x^4 \right] \right\} dx$$

for some functions $c_f(u)$, $p_f(u)$, $s_f(u)$. To ensure commutativity one has to choose these functions as follows.

Lemma 2.3. *For any $f = f(u)$ the Hamiltonian flow*

$$\begin{aligned} u_s + \partial_x \frac{\delta H_f}{\delta u(x)} = 0, \quad H_f = \int h_f dx, \\ h_f = f - \frac{\epsilon^2}{24} c f''' u_x^2 + \epsilon^4 \left[\left(p f''' + \frac{c^2 f^{(4)}}{480} \right) u_{xx}^2 \right. \\ \left. - \left(\frac{c c'' f^{(4)}}{1152} + \frac{c c' f^{(5)}}{1152} + \frac{c^2 f^{(6)}}{3456} + \frac{p' f^{(4)}}{6} + \frac{p f^{(5)}}{6} - s f''' \right) u_x^4 \right] \end{aligned} \quad (2.5)$$

is a symmetry, modulo $O(\epsilon^6)$, of (2.2). Moreover, the Hamiltonians H_f commute pairwise:

$$\{H_f, H_g\} = O(\epsilon^6)$$

for arbitrary two functions $f(u)$ and $g(u)$.

Proof. One has to check the identity

$$\mathcal{E} \left(\frac{\delta H_f}{\delta u(x)} \partial_x \frac{\delta H_g}{\delta u(x)} \right) = 0,$$

where \mathcal{E} is the Euler – Lagrange operator. We leave this calculation as an exercise for the reader.

Observe that for $f = \frac{u^3}{6}$ the Hamiltonian H_f coincides with (2.2). Also for $f = u$ (the Casimir of the Poisson bracket) and $f = \frac{u^2}{2}$ (the momentum) the perturbation is trivial,

$$H_f = H_f^0.$$

We do not know under what conditions on the functional parameters $c(u), p(u)$ higher order perturbations can be added to the Hamiltonians (2.5) preserving the commutativity. The examples of Sect. 5 show that this can be done at least for some particular choices of the functions. However, the remark at the end of Sect. 4 suggests that the answer is not always affirmative. \square

3. Solutions to the Perturbed Equations. Quasitriviality

We address now the problem of existence of solutions to the perturbed equation for $t < t_C$. We will construct a *formal asymptotic solution* to (2.2) (and also to all commuting flows (2.5)) valid on the entire interval $t < t_C$. The basic idea is to find a substitution

$$v \mapsto u = v + O(\epsilon)$$

that transforms *all* solutions to *all unperturbed equations* of the form (2.4) to solutions to the corresponding *perturbed equations* (2.5).

Quasitriviality Theorem. *There exists a transformation*

$$v \mapsto u = v + \sum_{k=1}^4 \epsilon^k F_k(u; u_x, \dots, u^{(nk)}), \tag{3.1}$$

where F_k are rational functions in the derivatives homogeneous of the degree k , independent of $f = f(u)$, that transforms all monotone solutions of (2.4) to solutions, modulo $O(\epsilon^6)$, of (2.5) and vice versa.

The general quasitriviality theorem for evolutionary PDEs admitting a *bihamiltonian* description was obtained in [10]². As we do not assume *a priori* existence of a bihamiltonian structure (see, however, the next section), we will give here a direct proof of quasitriviality for the family of commuting Hamiltonians (2.5).

For convenience we chose

$$s(u) = \frac{c(u) c'''(u)}{3456}.$$

² In a very recent paper [27] the quasitriviality result was proved, in all orders in ϵ , for an *arbitrary* perturbation of the Riemann wave equation $v_t + v v_x = 0$. It has also been shown that the same transformation trivializes also all symmetries of the perturbed equation.

Theorem 3.1. *Introduce the following Hamiltonian*

$$K = \int \left[\frac{1}{24} \epsilon c(u) u_x \log u_x + \epsilon^3 \left(\frac{c^2(u) u_{xx}^3}{5760 u_x^3} - \frac{p(u) u_{xx}^2}{4 u_x} \right) \right] dx.$$

Then the canonical transformation

$$u \mapsto v = u + \epsilon \{u(x), K\} + \frac{\epsilon^2}{2} \{\{u(x), K\}, K\} + \dots$$

satisfies

$$H_f = \int f(v) dx + O(\epsilon^6) \quad \forall f(u).$$

The inverse transformation is the needed quasitriviality. It is generated by the Hamiltonian

$$-K = \int \left[-\frac{1}{24} \epsilon c(v) v_x \log v_x - \epsilon^3 \left(\frac{c^2(v) v_{xx}^3}{5760 v_x^3} - \frac{p(v) v_{xx}^2}{4 v_x} \right) \right] dx,$$

that is

$$\begin{aligned} v \mapsto u &= v - \epsilon \{v(x), K\} + \frac{\epsilon^2}{2} \{\{v(x), K\}, K\} + \dots \\ &= v + \frac{\epsilon^2}{24} \partial_x \left(c \frac{v_{xx}}{v_x} + c' v_x \right) + \epsilon^4 \partial_x \left[c^2 \left(\frac{v_{xx}^3}{360 v_x^4} - \frac{7 v_{xx} v_{xxx}}{1920 v_x^3} + \frac{v_{xxxx}}{1152 v_x^2} \right)_x \right. \\ &\quad + c c' \left(\frac{47 v_{xx}^3}{5760 v_x^3} - \frac{37 v_{xx} v_{xxx}}{2880 v_x^2} + \frac{5 v_{xxxx}}{1152 v_x} \right) + c'^2 \left(\frac{v_{xxx}}{384} - \frac{v_{xx}^2}{5760 v_x} \right) \\ &\quad + c c'' \left(\frac{v_{xxx}}{144} - \frac{v_{xx}^2}{360 v_x} \right) \\ &\quad + \frac{1}{1152} \left(7 c' c'' v_x v_{xx} + c''^2 v_x^3 + 6 c c''' v_x v_{xx} + c' c''' v_x^3 + c c^{(4)} v_x^3 \right) \\ &\quad \left. + p \left(\frac{v_{xx}^3}{2 v_x^3} - \frac{v_{xx} v_{xxx}}{v_x^2} + \frac{v_{xxxx}}{2 v_x} \right) + p' v_{xxx} + p'' \frac{v_x v_{xx}}{2} \right]. \end{aligned} \quad (3.2)$$

In this formula $c = c(v)$, $p = p(v)$.

Main Conjecture, Part 1. *Let $v = v(x, t)$ be a smooth solution to the unperturbed equation $v_t + a(v) v_x = 0$ defined for all $x \in \mathbb{R}$ and $0 \leq t < t_0$ monotone in x for any t . Then there exists a solution $u = u(x, t; \epsilon)$ to the perturbed equation*

$$u_t + \partial_x \frac{\delta H_f}{\delta u(x)} = 0, \quad f''(u) = a(u)$$

defined on the same domain in the (x, t) -plane with the asymptotic at $\epsilon \rightarrow 0$ of the form (3.2).

4. Are All Hamiltonian Perturbations Also Bihamiltonian?

All unperturbed equations

$$v_s + a(v) v_x = 0$$

are bihamiltonian w.r.t. the Poisson pencil (see the definition in [9])

$$\begin{aligned} \{v(x), v(y)\}_1 &= \delta'(x - y), \\ \{v(x), v(y)\}_2 &= q(v(x))\delta'(x - y) + \frac{1}{2}q'(v)v_x\delta(x - y) \end{aligned} \quad (4.1)$$

for an arbitrary function $q(u)$,

$$\begin{aligned} v_s + \{v(x), H_1\}_1 = v_s + \{v(x), H_2\}_2 = 0, \quad H_1 &= \int f_1(v) dx, \quad H_2 = \int f_2(v) dx \\ f_1''(v) = a(v) = q(v)f_2''(v) + \frac{1}{2}q'(v)f_2'(v). \end{aligned}$$

To show that (4.1) is a Poisson pencil it suffices to observe that the linear combination

$$\begin{aligned} \{v(x), v(y)\}_2 - \lambda \{v(x), v(y)\}_1 &= (q(v(x)) - \lambda) \delta'(x - y) \\ &\quad + \frac{1}{2}q'(v)v_x\delta(x - y) \end{aligned} \quad (4.2)$$

is the Poisson bracket associated [11] with the flat metric

$$ds^2 = \frac{dv^2}{q(v) - \lambda}.$$

Theorem 4.1. *For $c(u) \neq 0$ the commuting Hamiltonians (2.5) admit a unique bihamiltonian structure obtained by a deformation of (4.1) with $q(u)$ satisfying*

$$p(u) = \frac{c^2}{960} \left[5 \frac{c'}{c} - \frac{q''}{q'} \right], \quad s(u) = 0. \quad (4.3)$$

The proof of this result along with the explicit formula for the deformed bihamiltonian structure is sketched in the Appendix below.

The assumption $c \neq 0$ is essential: one can check that for $c(u) \equiv 0$ the Hamiltonians (2.5) commute, modulo $O(\epsilon^6)$, only w.r.t. the standard Poisson bracket (1.4). On the other hand it turns out that for this particular choice of the functional parameters the deformation of commuting Hamiltonians *cannot* be extended to the order $O(\epsilon^8)$.

5. Examples

Example 1. For $c(u) = c_0 = \text{const}$, $p(u) = s(u) = 0$ one obtains from (2.2) the KdV equation

$$u_t + u u_x + c_0 \frac{\epsilon^2}{12} u_{xxx} = 0.$$

Choosing in (2.5)

$$f(u) = \frac{u^{k+2}}{(k+2)!}$$

one obtains the Hamiltonians of the KdV hierarchy

$$\begin{aligned} \frac{\partial u}{\partial t_k} + \partial_x \frac{\delta H_k}{\delta u(x)} &= 0, \quad H_k = \int h_k dx, \quad k \geq 0 \\ h_k &= \frac{u^{k+2}}{(k+2)!} - c_0 \frac{\epsilon^2}{24} \frac{u^{k-1}}{(k-1)!} u_x^2 \\ &\quad + c_0^2 \frac{\epsilon^4}{96} \left[\frac{u^{k-2}}{5(k-2)!} u_{xx}^2 - \frac{u^{k-4}}{36(k-4)!} u_x^4 \right] + O(\epsilon^6). \end{aligned}$$

The quasitriviality transformation (3.2) takes the form [2, 9]

$$\begin{aligned} v \mapsto u &= v + \partial_x^2 \left[\frac{\epsilon^2}{24} c_0 \log v_x + c_0^2 \epsilon^4 \left(\frac{v_{xx}^3}{360 v_x^4} - \frac{7 v_{xx} v_{xxx}}{1920 v_x^3} + \frac{v_{xxxx}}{1152 v_x^2} \right) \right] \\ &\quad + O(\epsilon^6). \end{aligned} \quad (5.1)$$

Example 2. The Volterra lattice

$$\dot{q}_n = q_n(q_{n+1} - q_{n-1}) \quad (5.2)$$

(also called difference KdV) has the following bihamiltonian structure [13]

$$\{q_n, q_m\}_1 = 2q_n q_m (\delta_{n+1, m} - \delta_{n, m+1}), \quad (5.3)$$

$$\dot{q}_n = \{q_n, H_1\}_1, \quad H_1 = \frac{1}{2} \sum \log q_n,$$

$$\begin{aligned} \{q_n, q_m\}_2 &= q_n q_m \left\{ \left[\frac{q_n + q_m}{2} - 2 \right] (\delta_{n, m+1} - \delta_{n, m-1}) \right. \\ &\quad \left. + \frac{1}{2} \delta_{n, m+2} - \frac{1}{2} \delta_{n, m-2} \right\}, \end{aligned} \quad (5.4)$$

$$\dot{q}_n = \{q_n, H_2\}_2, \quad H_2 = \sum q_n.$$

After substitution

$$q_n = e^{v(n\epsilon)}$$

and division by 4ϵ one arrives at the following bihamiltonian structure:

$$\begin{aligned} \{v(x), v(y)\}_1 &= \frac{1}{4\epsilon} [\delta(x-y+\epsilon) - \delta(x-y-\epsilon)] \\ &= \delta'(x-y) + \frac{\epsilon^2}{3} \delta'''(x-y) + \dots, \end{aligned} \quad (5.5)$$

$$\begin{aligned}
\{v(x), v(y)\}_2 &= \left(1 - e^{v(x)}\right) \delta'(x - y) - \frac{1}{2} e^v v_x \delta(x - y) \\
&+ \epsilon^2 \left[\frac{1}{12} (2 - 5 e^v) \delta'''(x - y) - \frac{5}{8} e^v v_x \delta''(x - y) \right. \\
&\left. - \frac{3}{8} e^v (v_{xx} + v_x^2) \delta'(x - y) - \frac{1}{12} e^v (v_{xxx} + 3 v_x v_{xx} + v_x^3) \delta(x - y) \right] + O(\epsilon^4). \quad (5.6)
\end{aligned}$$

To compare this bihamiltonian structure with the one obtained in Theorem 4.1 the Poisson bracket (5.5) must be reduced to the standard form

$$\{u(x), u(y)\}_1 = \delta'(x - y) \quad (5.7)$$

by means of the transformation

$$u = \sqrt{\frac{\epsilon \partial_x}{\sinh \epsilon \partial_x}} v = v - \frac{\epsilon^2}{12} v_{xx} + \frac{\epsilon^4}{160} v_{xxxx} + O(\epsilon^6).$$

After the transformation the second bracket takes the form

$$\begin{aligned}
\{u(x), u(y)\}_2 &= \left(1 - e^{u(x)}\right) \delta'(x - y) - \frac{1}{2} e^u u_x \delta(x - y) \\
&- \epsilon^2 e^{u(x)} \left[\frac{1}{4} \delta'''(x - y) + \frac{3}{8} u_x \delta''(x - y) + \frac{1}{24} (7 u_{xx} + 5 u_x^2) \delta'(x - y) \right. \\
&\left. + \frac{1}{24} (2 u_{xxx} + 4 u_x u_{xx} + u_x^3) \delta(x - y) \right] + O(\epsilon^4). \quad (5.8)
\end{aligned}$$

We leave as an exercise for the reader to compute the terms of order ϵ^4 and to verify that the Poisson bracket (5.8) is associated with the functional parameters ϵ chosen as follows

$$c(u) = 2, \quad p(u) = -\frac{1}{240}, \quad q(u) = 1 - e^u, \quad s(u) = \frac{1}{4320}.$$

Example 3. The Camassa – Holm equation [5] (see also [14])

$$v_t - \epsilon^2 v_{xxt} = \frac{3}{2} v v_x - \epsilon^2 \left[v_x v_{xx} + \frac{1}{2} v v_{xxx} \right] \quad (5.9)$$

admits a bihamiltonian description (cf. [21]) after doing the following Miura-type transformation

$$u = v - \epsilon^2 v_{xx}. \quad (5.10)$$

The bihamiltonian structure reads

$$\{u(x), u(y)\}_1 = \delta'(x - y) - \epsilon^2 \delta'''(x - y), \quad (5.11)$$

$$\{u(x), u(y)\}_2 = u(x) \delta'(x - y) + \frac{1}{2} u_x \delta(x - y). \quad (5.12)$$

The Casimir H_{-1} of the first Poisson bracket analytic in ϵ has the form

$$H_{-1} = \int h_{-1} dx, \quad h_{-1} = u(x).$$

Applying the bihamiltonian recursion procedure one obtains a sequence of commuting Hamiltonians $H_k = \int h_k dx$ of the hierarchy,

$$h_0 = \frac{1}{2} u v, \quad h_1 = \frac{1}{8} [v^3 + u v^2], \dots$$

The corresponding Hamiltonian flows

$$u_{t_k} = \{u(x), H_k\}_1 \equiv (1 - \epsilon^2 \partial_x^2) \partial_x \frac{\delta H_k}{\delta u(x)}$$

read

$$u_{t_0} = u_x, \quad u_{t_1} = \frac{3}{2} v v_x - \epsilon^2 \left[v_x v_{xx} + \frac{1}{2} v v_{xxx} \right], \dots$$

The last equation reduces to (5.9) after the substitution (5.10).

To compare the commuting Hamiltonians with those given in (2.5) one must first reduce the first Poisson bracket to the standard form $\{\tilde{u}(x), \tilde{u}(y)\}_1 = \delta'(x - y)$ by the transformation

$$\tilde{u} = \left(1 - \epsilon^2 \partial_x^2\right)^{-1/2} u = u + \frac{1}{2} \epsilon^2 u_{xx} + \frac{3}{8} \epsilon^4 u_{xxxx} + \dots$$

After the transformation the Camassa – Holm equation will read

$$\tilde{u}_t = \frac{3}{2} \tilde{u} \tilde{u}_x + \epsilon^2 (2 \tilde{u}_x \tilde{u}_{xx} + \tilde{u} \tilde{u}_{xxx}) + \epsilon^4 (5 \tilde{u}_{xx} \tilde{u}_{xxx} + 3 \tilde{u}_x \tilde{u}_{xxxx} + \tilde{u} \tilde{u}_{xxxxx}) + \dots$$

It is easy to see that the commuting Hamiltonians of Camassa – Holm hierarchy are obtained from (2.5) by the specialization

$$c(u) = 8u, \quad p(u) = \frac{u}{3}, \quad q(u) = u, \quad s(u) = 0.$$

6. Introducing a Special Function

Let us recall some properties of the differential equation

$$X = T U - \left[\frac{1}{6} U^3 + \frac{1}{24} (U'^2 + 2U U'') + \frac{1}{240} U^{IV} \right] \tag{6.1}$$

often considered as a 4th order analogue of the classical Painlevé-I equation. First, it can be interpreted as a monodromy preserving deformation of the following linear differential operator with polynomial coefficients

$$\frac{\partial \psi}{\partial z} = \mathbf{W} \psi, \tag{6.2}$$

where the matrix \mathbf{W} reads

$$\mathbf{W} = -\frac{1}{120} \begin{pmatrix} 12UU' + 8zU' + U''' & 2(16z^2 + 8zU + 6U^2 + U'' - 60T) \\ 2w_{21} & -12UU' - 8zU' - U''' \end{pmatrix},$$

where

$$w_{21} = 32 z^3 - 16z^2 U - 2z(2U^2 + U'' + 60 T) + 8U^3 + 2U''U - U'^2 + 120X.$$

Indeed, it coincides with the compatibility conditions

$$\mathbf{W}_X - \mathbf{U}_z + [\mathbf{W}, \mathbf{U}] = 0$$

of the linear system (6.2) with

$$\frac{\partial \psi}{\partial X} = \mathbf{U} \psi, \quad \mathbf{U} = \begin{pmatrix} 0 & -1 \\ 2U - 2z & 0 \end{pmatrix}. \quad (6.3)$$

Moreover, the dependence of (6.2) on T is isomonodromic *iff* the function $U(X)$ depends also on the parameter T according to the KdV equation

$$U_T + U U' + \frac{1}{12} U''' = 0. \quad (6.4)$$

This is the spelling of the compatibility condition of the linear system (6.2), (6.3) with

$$\frac{\partial \psi}{\partial T} = \mathbf{V} \psi, \quad \mathbf{V} = \frac{1}{6} \begin{pmatrix} U' & 2U + 4z \\ 8z^2 - 4zU - 4U^2 - U'' & -U' \end{pmatrix}. \quad (6.5)$$

The *Painlevé property* readily follows from the isomonodromicity: singularities in the complex (X, T) -plane of general solution to (6.1), (6.4) are poles [20].

Main Conjecture, Part 2 (cf. [4]). *The ODE (6.1) has unique solution $U = U(X; T)$ smooth for all real $X \in \mathbb{R}$ for all real values of the parameter T .*

Note that due to the uniqueness the solution in question satisfies the KdV equation (6.4).

For $T \ll 0$ the solution of interest is very close to the unique root of the cubic equation

$$X \simeq T U - \frac{U^3}{6},$$

that is,

$$\begin{aligned} U &\simeq (-T)^{1/2} \left[w + (-T)^{-7/2} \frac{3w^2 - 2}{3(w^2 + 2)^4} \right. \\ &\quad \left. - (-T)^{-7} w \frac{189w^4 - 972w^2 + 436}{9(w^2 + 2)^9} + O\left((-T)^{-21/2}\right) \right] \\ X &= -(-T)^{3/2} \left(w + \frac{1}{6} w^3 \right). \end{aligned} \quad (6.6)$$

Same is true for any T for $|X| \gg 0$. For $T \gg 0$ the solution develops oscillations typical for dispersive waves [32] within a region around the origin; one can use the Whitham method to approximate $U(X; T)$ by modulated elliptic functions within the oscillatory zone [18, 29]. Thus the solution in question interpolates between the two types of asymptotic behaviour (cf. [23] where the role of the special solution $U(X; T)$ in the KdV theory was discussed).

The solutions to the fourth order ODE (6.1) can be parametrized [20] by the monodromy data (i.e., the collection of Stokes multipliers) of the linear differential operator (6.3) with coefficients polynomial in z . The solution corresponding to given Stokes multipliers can be reconstructed by solving a certain Riemann – Hilbert problem. The particular values of the Stokes multipliers associated with the smooth solution in question have been conjectured in [20].

7. Local Galilean Symmetry and Critical Behaviour

We will now proceed to discussing the universality problem. Consider the perturbed PDE

$$u_t + \{u(x), H_f\} = u_t + a(u)u_x + O(\epsilon^2) = 0, \quad f''(u) = a(u). \quad (7.1)$$

Let us apply the transformation (3.2) to the unperturbed solution $v = v(x, t)$ of

$$v_t + a(v)v_x = 0 \quad (7.2)$$

obtained by the method of characteristics:

$$x = a(v)t + b(v) \quad (7.3)$$

for some smooth function $b(v)$. Let the solution arrive at the point of gradient catastrophe for some $x = x_0, t = t_0, v = v_0$. At this point one has

$$\begin{aligned} x_0 &= a(v_0)t_0 + b(v_0), \\ 0 &= a'(v_0)t_0 + b'(v_0), \\ 0 &= a''(v_0)t_0 + b''(v_0) \end{aligned} \quad (7.4)$$

(inflection point). Let us assume the following *genericity assumption*:

$$\kappa := -(a'''(v_0)t_0 + b'''(v_0)) \neq 0. \quad (7.5)$$

Let us first recall the universality property for the critical behaviour of the unperturbed solutions: up to shifts, Galilean transformations and rescalings a generic solution to (7.2) near (x_0, t_0) behaves like the cubic root function. We will present this well known statement in the following form. Introduce the new variables

$$\begin{aligned} \bar{x} &= x - a_0(t - t_0) - x_0, \\ \bar{t} &= t - t_0, \\ \bar{v} &= v - v_0. \end{aligned}$$

Let us do the following scaling transformation

$$\begin{aligned} \bar{x} &\mapsto \lambda \bar{x}, \\ \bar{t} &\mapsto \lambda^{\frac{2}{3}} \bar{t}, \\ \bar{v} &\mapsto \lambda^{\frac{1}{3}} \bar{v}. \end{aligned} \quad (7.6)$$

Lemma 7.1. *After the rescaling (7.6) any generic solution to (7.2) at the limit $\lambda \rightarrow 0$ for $t < t_0$ goes to the solution of the cubic equation*

$$\bar{x} = a'_0 \bar{v} \bar{t} - \kappa \frac{\bar{v}^3}{6}. \quad (7.7)$$

In these formulae $a_0 = a(v_0)$, $a'_0 = a'(v_0)$. Note that the inequality

$$\kappa a'_0 > 0 \quad (7.8)$$

must hold true in order to have the solution well defined for $t < t_0$ near the point of generic gradient catastrophe (7.4).

To prove the lemma it suffices to observe that, after the rescaling (7.6) and division by λ Eq. (7.3) yields

$$\bar{x} = a'_0 \bar{v} \bar{t} - \kappa \frac{\bar{v}^3}{6} + O\left(\lambda^{1/3}\right).$$

The parameter κ can be eliminated from (7.7) by a rescaling. The resulting cubic function can be interpreted as the universal unfolding of the A_2 singularity [1]. Our basic observation we are going to explain now is that, after a Hamiltonian perturbation the A_2 singularity transforms to the special solution of (1.8) described above.

Let us look for a solution to the perturbed PDE (7.1) in the form of a formal power series

$$u = u(x, t; \epsilon) = v(x, t) + \sum_{k \geq 1} \epsilon^k v_k(x, t) \quad (7.9)$$

with $v(x, t)$ given by (7.3) satisfying (7.1) modulo $O(\epsilon^5)$. We will say that such a solution is *monotone* at the point $x = x_0$, $t = t_0$ if

$$u_x(x_0, t_0; 0) \equiv v_x(x_0, t_0) \neq 0.$$

According to the results of Sect. 3 all monotone solutions of the form (7.9) can be obtained by applying the transformation (3.2) to the nonperturbed solution (7.2) (more precisely, one has to allow ϵ -dependence of the function $b(u)$).

Lemma 7.2. *Let us perform the rescaling (7.6) along with*

$$\epsilon \mapsto \lambda^{7/6} \epsilon \quad (7.10)$$

in the quasitriviality transformation (3.2). Then the resulting solution to the perturbed PDE will be equal to

$$u = v_0 + \lambda^{1/3} \left\{ \bar{v} + \partial_x^2 \left[\frac{\epsilon^2}{24} c_0 \log \bar{v}_x + c_0^2 \epsilon^4 \right. \right. \\ \left. \left. \times \left(\frac{\bar{v}_{xx}^3}{360 \bar{v}_x^4} - \frac{7 \bar{v}_{xx} \bar{v}_{xxx}}{1920 \bar{v}_x^3} + \frac{\bar{v}_{xxxx}}{1152 \bar{v}_x^2} \right) \right] \right\} + O\left(\lambda^{2/3}\right) \quad (7.11)$$

(cf. (5.1)) where

$$c_0 = c(v_0), \quad (7.12)$$

$\bar{v} = \bar{v}(x, t)$ is the solution to the cubic equation (7.7).

Proof is straightforward.

It remains to identify (7.11) with the formal asymptotic solution (6.6) to the ODE (6.1). This can be done by a direct substitution. An alternative way is to observe that, near the point of gradient catastrophe the perturbed PDE acquires an *additional Galilean symmetry*. Indeed, according to the previous lemma, locally one can replace the functions $c(u)$, $p(u)$ by constants $c_0 = c(v_0)$, $p_0 = p(v_0)$ (the constant p_0 , however, does not enter in the leading term of the asymptotic expansion in powers of $\lambda^{1/3}$). Let us show that in this situation any solution to the perturbed PDE of the form (7.9) satisfies also a fourth order ODE.

Lemma 7.3. *Let $c(u) = c_0$, $p(u) = p_0$. Then for any solution $u(x, t; \epsilon)$ of the form (7.9) monotone at the point (x_0, t_0) there exists a formal series*

$$g(u; \epsilon) = g_0(u) + \sum_{k \geq 1} \epsilon^k g_k(u)$$

such that for arbitrary x, t sufficiently close to x_0, t_0 the function $u(x, t; \epsilon)$ satisfies, modulo $O(\epsilon^5)$, the following fourth order ODE:

$$x = t \frac{\delta H_{f'}}{\delta u(x)} + \frac{\delta H_{g'}}{\delta u(x)}. \tag{7.13}$$

Here

$$g_0''(u) = b(u).$$

Proof. It is easy to see that the flow

$$u_\tau = 1 - t \partial_x \frac{\delta H_{f'}}{\delta u(x)} \tag{7.14}$$

is a symmetry of (7.1). Combining this symmetry with one of the commuting flows

$$u_s + \partial_x \frac{\delta H_{g'}}{\delta u(x)} = 0$$

one obtains another symmetry. The set of stationary points of this combination

$$\partial_x \left(t \frac{\delta H_{f'}}{\delta u(x)} + \frac{\delta H_{g'}}{\delta u(x)} - x \right) = 0$$

is therefore invariant for the t -flow. Considering the limit $\epsilon \rightarrow 0$ it is easy to see that the integration constant vanishes on the solution (3.2), (7.2). The lemma is proved.

The ODE for the function $u(x)$ is closely related to the so-called *string equation* known in matrix models and topological field theory (see, e.g., [9]). Explicitly

$$\begin{aligned} x = & t a(u) + b(u) + c_0 \frac{\epsilon^2}{24} \left\{ t \left[2 a'' u_{xx} + a''' u_x^2 \right] + \left[2 b'' u_{xx} + b''' u_x^2 \right] \right\} \\ & + \epsilon^4 \left\{ \left[2 p_0 (t a'' + b'') + \frac{1}{240} c_0^2 (t a''' + b''') \right] u_{xxx} \right. \\ & + \left[4 p_0 (t a''' + b''') + \frac{1}{120} c_0^2 (t a^{IV} + b^{IV}) \right] u_{xxx} u_x \\ & + \left[4 p_0 (t a^{IV} + b^{IV}) + \frac{11}{1440} c_0^2 (t a^V + b^V) \right] u_{xx} u_x^2 \\ & \left. + \left[\frac{1}{2} p_0 (t a^V + b^V) + \frac{1}{1152} c_0^2 (t a^{VI} + b^{VI}) \right] u_x^4 \right\} + O(\epsilon^5). \tag{7.15} \end{aligned}$$

Let us call the solution *generic* if, along with the condition $\kappa := -(a'''(v_0)t_0 + b'''(v_0)) \neq 0$ it also satisfies

$$c_0 := c(v_0) \neq 0. \tag{7.16}$$

Main Conjecture, Part 3. *The generic solution described in the Main Conjecture, Part 1 can be extended up to $t = t_0 + \delta$ for sufficiently small positive $\delta = \delta(\epsilon)$; near the point (x_0, t_0) it behaves in the following way:*

$$u \simeq v_0 + \left(\frac{\epsilon^2 c_0}{\kappa^2}\right)^{1/7} U\left(\frac{x - a_0(t - t_0) - x_0}{(\kappa c_0^3 \epsilon^6)^{1/7}}; \frac{a'_0(t - t_0)}{(\kappa^3 c_0^2 \epsilon^4)^{1/7}}\right) + O(\epsilon^{4/7}). \tag{7.17}$$

Here $U = U(X; T)$ is the solution to the ODE (1.8) specified in the Main Conjecture, Part 2.

To arrive at the asymptotic formula (7.17) we do in (7.15) the rescaling of the form (7.6) along with (7.10). After substitution to Eq. (7.15) and division by λ , one obtains

$$\bar{x} = a'_0 \bar{u} \bar{t} - \kappa \left[\frac{\bar{u}^3}{6} + \frac{\epsilon^2}{24} c_0 (\bar{u}_x^2 + 2\bar{u} \bar{u}_{xx}) + \frac{\epsilon^4}{240} c_0^2 \bar{u}_{xxx} \right] + O(\lambda^{1/3}).$$

In derivation of this formula we use that the monomial of the form

$$\epsilon^k u_x^{i_1} u_{xx}^{i_2} u_{xxx}^{i_3} \dots$$

after the rescaling will be multiplied by λ^D with

$$D = \frac{1}{6}k + \frac{1}{3}(i_1 + i_2 + \dots)$$

due to the degree condition

$$i_1 + 2i_2 + 3i_3 + \dots = k.$$

Adding the terms of higher order $k > 4$ will not change the leading term. Choosing

$$\lambda = \epsilon^{6/7} c_0^{3/7}$$

we arrive at the needed asymptotic formula.

Clearly the above arguments require existence and uniqueness of the solution to (1.8) smooth on the real line described in the Main Conjecture, Part 2. \square

8. Concluding Remarks

We have presented arguments supporting the conjectural universality of critical behaviour of solutions to generic Hamiltonian perturbations of a hyperbolic equation of the form (1.2). In subsequent publications we will study the Main Conjecture in more details. The possibilities of using the idea of Universality in numerical algorithms to dealing with oscillatory behaviour of solutions to Hamiltonian PDEs will be explored. We will also proceed to the study of singularities of generic solutions to integrable Hamiltonian hyperbolic systems of conservation laws

$$u_t^i + \partial_x \left(\eta^{ij} \frac{\partial h(u)}{\partial u^j} \right) = 0, \quad \eta^{ji} = \eta^{ij}, \quad \det(\eta^{ij}) \neq 0. \tag{8.1}$$

Recall that, according to the results of [31] the system (8.1) is integrable if it diagonalizes in a system of curvilinear coordinates $v^k = v^k(u)$, $k = 1, \dots, n$ for the Euclidean/pseudo-Euclidean metric

$$ds^2 = \eta_{ij} du^i du^j = \sum_{k=1}^n g_k(v) (dv^k)^2, \quad (\eta_{ij}) := (\eta^{ij})^{-1},$$

$$v_t^k + \lambda^k(v) v_x^k = 0, \quad k = 1, \dots, n$$

(in this formula there is no summation over repeated indices!). All Hamiltonian perturbations of the hyperbolic system (8.1) can be written in the form

$$u_t^i + \partial_x \left(\eta^{ij} \frac{\delta H}{\delta u^j(x)} \right) = 0, \quad H = \int \left[h(u) + \sum_{k \geq 1} \epsilon^k h_k(u; u_x, \dots, u^{(k)}) \right] dx,$$

$$\deg h_k = k.$$

We plan to study symmetries of the perturbed Hamiltonian hyperbolic systems. In particular, we will classify the perturbations preserving integrability and study the correspondence between the types of critical behaviour of the perturbed and unperturbed systems. The next step would be to extend our approach to Hamiltonian perturbations of spatially multidimensional hyperbolic systems (cf. [8]).

Appendix: Bihamiltonian Structures Associated with the Perturbations of the Riemann Wave Hierarchy

Theorem A.1. *For arbitrary two functions $c = c(u) \neq 0$, $q = q(u)$ the family of Hamiltonians (2.5) with*

$$p(u) = \frac{c^2}{960} \left[5 \frac{c'}{c} - \frac{q''}{q'} \right], \quad s(u) = 0 \tag{A.1}$$

is commutative

$$\{H_f, H_g\}_{1,2} = 0 \pmod{O(\epsilon^6)} \quad \forall f = f(u), \quad \forall g = g(u) \tag{A.2}$$

with respect to the Poisson pencil of the form

$$\{u(x), u(y)\}_1 = \delta'(x - y),$$

$$\{u(x), u(y)\}_2 = \{u(x), u(y)\}_1^{[0]} + \epsilon^2 \{u(x), u(y)\}_1^{[2]} + \epsilon^4 \{u(x), u(y)\}_1^{[4]} + O(\epsilon^6).$$

Here the terms of order 0:

$$\{u(x), u(y)\}_2^{[0]} = q(u) \delta'(x - y) + \frac{1}{2} q'(u) u_x \delta(x - y).$$

All terms of higher orders are uniquely determined from the bicommutativity (A.2) provided validity of the constraint (A.1). Namely, the terms of order 2:

$$\begin{aligned}
\{u(x), u(y)\}_2^{[2]} &= \frac{cq'}{8} \delta'''(x-y) + \frac{3}{16} (cq')' u_x \delta''(x-y) \\
&+ \left[\left(\frac{c''q'}{16} + \frac{c'q''}{6} + \frac{5cq'''}{48} \right) u_x^2 + \frac{c'q'u_{xx}}{16} + \frac{7cq''u_{xx}}{48} \right] \delta'(x-y) \\
&+ \left[\left(\frac{c''q''}{48} + \frac{c'q'''}{24} + \frac{cq^{(4)}}{48} \right) u_x^3 + \frac{1}{12} (c'q'' + cq''') u_x u_{xx} + \frac{cq''}{24} u_{xxx} \right] \delta(x-y).
\end{aligned}$$

The terms of order 4:

$$\begin{aligned}
\{u(x), u(y)\}_2^{[4]} &= \frac{1}{192} (3cc'q' + c^2q'') \delta^V(x-y) + \frac{5}{384} (3cc'q' + c^2q'')' u_x \delta^{IV}(x-y) \\
&+ \left[\left(\frac{3c'c''q'}{32} + \frac{cc''q''}{32} + \frac{3c'^2q''}{32} + \frac{5cc''q''}{48} - \frac{cc'q''^2 u_x^2}{240q'} \right. \right. \\
&\quad \left. \left. + \frac{c^2q''^3}{480q'^2} + \frac{19cc'q'''}{192} - \frac{3c^2q''q'''}{640q'} + \frac{c^2q^{(4)}}{64} \right) u_x^2 \right. \\
&\quad \left. + \left(\frac{3c'^2q'}{64} + \frac{3cc''q'}{64} + \frac{17cc'q''}{192} - \frac{c^2q''^2}{480q'} + \frac{19c^2q'''}{960} \right) u_{xx} \right] \delta'''(x-y) \\
&+ \left[\left(\frac{3c''^2q'}{128} + \frac{c'c''q''}{32} + \frac{cc^{(4)}q'}{128} + \frac{19c'c''q''}{128} + \frac{23cc''q''}{384} + \frac{5cc'q^{(4)}}{64} + \frac{7cc''q''}{64} \right. \right. \\
&\quad \left. \left. + \frac{c^2q^{(5)}}{96} + \frac{3c'^2q'''}{32} - \frac{c'^2q''^2}{160q'} - \frac{cc''q''^2}{160q'} + \frac{cc'q''^3}{80q'^2} - \frac{c^2q''^4}{160q'^3} - \frac{17cc'q''q''}{640q'} \right. \right. \\
&\quad \left. \left. + \frac{21c^2q''^2q'''}{1280q'^2} - \frac{9c^2q''^2}{1280q'} - \frac{9c^2q''q^{(4)}}{1280q'} \right) u_x^3 + \left(\frac{9c'c'q'}{64} + \frac{3cc''q''}{64} \right. \right. \\
&\quad \left. \left. + \frac{11c'^2q''}{64} + \frac{13cc''q''}{64} - \frac{3cc'q''^2}{160q'} + \frac{3c^2q''^3}{320q'^2} + \frac{69cc'q'''}{320} \right. \right. \\
&\quad \left. \left. - \frac{13c^2q''q'''}{640q'} + \frac{3c^2q^{(4)}}{80} \right) u_x u_{xx} \right. \\
&\quad \left. + \left(\frac{c'^2q'}{32} + \frac{cc''q'}{32} + \frac{13cc'q''}{192} - \frac{c^2q''^2}{320q'} + \frac{c^2q'''}{60} \right) u_{xxx} \right] \delta''(x-y) \\
&+ \left[\left(\frac{c''^2q''}{48} + \frac{c'c''q''}{32} + \frac{cc^{(4)}q''}{96} - \frac{c'c''q''^2}{160q'} - \frac{cc''q''^2}{480q'} + \frac{c'^2q''^3}{160q'^2} + \frac{cc''q''^3}{160q'^2} \right. \right. \\
&\quad \left. \left. - \frac{cc'q''^4}{80q'^3} + \frac{c^2q''^5}{160q'^4} + \frac{35c'c''q'''}{384} + \frac{5cc''q'''}{128} - \frac{9c'^2q''q'''}{640q'} - \frac{9cc''q''q''}{640q'} \right. \right. \\
&\quad \left. \left. + \frac{11cc'q''^2q'''}{320q'^2} - \frac{13c^2q''^3q'''}{640q'^3} - \frac{cc'q''^2}{64q'} + \frac{19c^2q''q''^2}{1280q'^2} + \frac{17c'^2q^{(4)}}{384} \right. \right. \\
&\quad \left. \left. + \frac{5cc''q^{(4)}}{96} - \frac{cc'q''q^{(4)}}{64q'} + \frac{17c^2q''^2q^{(4)}}{1920q'^2} - \frac{11c^2q''q^{(4)}}{1280q'} + \frac{35cc'q^{(5)}}{1152} \right) \right] \delta(x-y)
\end{aligned}$$

$$\begin{aligned}
& -\frac{11c^2q''q^{(5)}}{3840q'} + \frac{c^2q^{(6)}}{288} \Big) u_x^4 + \left(\frac{3c'^2q'}{128} + \frac{c'c'''q'}{32} + \frac{cc^{(4)}q'}{128} + \frac{91c'c''q''}{384} \right. \\
& + \frac{37cc'''q''}{384} - \frac{c'^2q''^2}{60q'} - \frac{cc''q''^2}{60q'} + \frac{cc'q''^3}{30q'^2} - \frac{c^2q''^4}{60q'^3} + \frac{59c'^2q''''}{320} + \frac{53cc''q''''}{240} \\
& - \frac{47cc'q''q''''}{640q'} + \frac{173c^2q''^2q''''}{3840q'^2} - \frac{77c^2q''''^2}{3840q'} + \frac{169cc'q^{(4)}}{960} - \frac{77c^2q''q^{(4)}}{3840q'} \\
& + \frac{73c^2q^{(5)}}{2880} \Big) u_x^2 u_{xx} + \left(\frac{3c'c''q'}{128} + \frac{cc'''q'}{128} + \frac{5c'^2q''}{96} + \frac{cc''q''}{16} - \frac{cc'q''^2}{80q'} \right. \\
& + \frac{c^2q''^3}{160q'^2} + \frac{157cc'q''''}{1920} - \frac{5c^2q''q''''}{384q'} + \frac{31c^2q^{(4)}}{1920} \Big) u_{xx}^2 \\
& + \left(\frac{3c'c''q'}{64} + \frac{cc'''q'}{64} + \frac{c'^2q''}{12} + \frac{3cc''q''}{32} - \frac{cc'q''^2}{60q'} \right. \\
& + \frac{c^2q''^3}{120q'^2} + \frac{19cc'q''''}{160} - \frac{11c^2q''q''''}{640q'} + \frac{11c^2q^{(4)}}{480} \Big) u_x u_{xxx} \\
& + \left(\frac{c'^2q'}{128} + \frac{cc''q'}{128} + \frac{11cc'q''}{384} - \frac{c^2q''^2}{320q'} + \frac{17c^2q''''}{1920} \right) u_{xxxx} \Big] \delta'(x-y) \\
& + \left[\left(\frac{c''^2q''''}{192} + \frac{c'c'''q''''}{128} + \frac{cc^{(4)}q''''}{384} - \frac{c'c''q''q''''}{640q'} - \frac{cc'''q''q''''}{1920q'} + \frac{c'^2q''^2q''''}{640q'^2} \right. \right. \\
& + \frac{cc''q''^2q''''}{640q'^2} - \frac{cc'q''^3q''''}{320q'^3} + \frac{c^2q''^4q''''}{640q'^4} - \frac{c'^2q''^2}{640q'} - \frac{cc''q''^2}{640q'} + \frac{3cc'q''q''^2}{640q'^2} \\
& - \frac{c^2q''^2q''^2}{320q'^3} + \frac{c^2q''^3}{1280q'^2} + \frac{7c'c''q^{(4)}}{384} + \frac{cc'''q^{(4)}}{128} - \frac{c'^2q''q^{(4)}}{640q'} - \frac{cc''q''q^{(4)}}{640q'} \\
& + \frac{cc'q''^2q^{(4)}}{320q'^2} - \frac{c^2q''^3q^{(4)}}{640q'^3} - \frac{3cc'q''^3q^{(4)}}{640q'} + \frac{13c^2q''q''''q^{(4)}}{3840q'^2} - \frac{c^2q^{(4)2}}{1280q'} \\
& + \frac{17c'^2q^{(5)}}{2304} + \frac{5cc''q^{(5)}}{576} - \frac{cc'q''q^{(5)}}{640q'} + \frac{c^2q''^2q^{(5)}}{1280q'^2} - \frac{c^2q''''q^{(5)}}{960q'} + \frac{5cc'q^{(6)}}{1152} \\
& - \frac{c^2q''q^{(6)}}{3840q'} + \frac{c^2q^{(7)}}{2304} \Big) u_x^5 + \left(\frac{c''^2q''}{64} + \frac{c'c'''q''}{48} + \frac{cc^{(4)}q''}{192} - \frac{c'c''q''^2}{160q'} \right. \\
& - \frac{cc'''q''^2}{480q'} + \frac{c'^2q''^3}{160q'^2} + \frac{cc''q''^3}{160q'^2} - \frac{cc'q''^4}{80q'^3} + \frac{c^2q''^5}{160q'^4} + \frac{97c'c''q''''}{960} \\
& + \frac{13cc'''q''''}{320} - \frac{c'^2q''q''''}{60q'} - \frac{cc''q''q''''}{60q'} + \frac{19cc'q''^2q''''}{480q'^2} - \frac{11c^2q''^3q''''}{480q'^3} \\
& - \frac{cc'q''^2}{48q'} + \frac{3c^2q''q''^2}{160q'^2} + \frac{19c'^2q^{(4)}}{320} + \frac{67cc'q^{(4)}}{960} - \frac{cc'q''q^{(4)}}{48q'} + \frac{11c^2q''^2q^{(4)}}{960q'^2}
\end{aligned}$$

$$\begin{aligned}
 & -\frac{c^2q'''q^{(4)}}{80q'} + \frac{131cc'q^{(5)}}{2880} - \frac{c^2q''q^{(5)}}{240q'} + \frac{c^2q^{(6)}}{180} \Big) u_x^3 u_{xx} \\
 & + \left(\frac{7c'c''q''}{128} + \frac{7cc'''q''}{384} - \frac{7c'^2q''^2}{960q'} - \frac{7cc''q''^2}{960q'} + \frac{7cc'q''^3}{480q'^2} - \frac{7c^2q''^4}{960q'^3} \right. \\
 & + \frac{59c'^2q'''}{960} + \frac{23cc''q'''}{320} - \frac{cc'q''q'''}{30q'} + \frac{13c^2q''^2q'''}{640q'^2} - \frac{3c^2q''^2}{320q'} + \frac{131cc'q^{(4)}}{1920} \\
 & - \frac{3c^2q''q^{(4)}}{320q'} + \frac{31c^2q^{(5)}}{2880} \Big) u_x u_{xx}^2 + \left(\frac{3c'c''q''}{64} + \frac{cc'''q''}{64} - \frac{c'^2q''^2}{160q'} - \frac{cc''q''^2}{160q'} \right. \\
 & + \frac{cc'q''^3}{80q'^2} - \frac{c^2q''^4}{160q'^3} + \frac{47c'^2q'''}{960} + \frac{13cc''q'''}{240} - \frac{13cc'q''q'''}{480q'} + \frac{c^2q''^2q'''}{60q'^2} \\
 & - \frac{7c^2q''^2}{960q'} + \frac{49cc'q^{(4)}}{960} - \frac{7c^2q''q^{(4)}}{960q'} + \frac{23c^2q^{(5)}}{2880} \Big) u_x^2 u_{xxx} \\
 & + \left(\frac{5c'^2q''}{192} + \frac{5cc''q''}{192} - \frac{cc'q''^2}{96q'} + \frac{c^2q''^3}{192q'^2} + \frac{3cc'q''}{64} \right. \\
 & - \frac{c^2q''q'''}{96q'} + \frac{c^2q^{(4)}}{96} \Big) u_{xx} u_{xxx} + \left(\frac{c'^2q''}{64} + \frac{cc''q''}{64} - \frac{cc'q''^2}{160q'} + \frac{c^2q''^3}{320q'^2} \right. \\
 & + \frac{9cc'q'''}{320} - \frac{c^2q''q'''}{160q'} + \frac{c^2q^{(4)}}{160} \Big) u_x u_{xxxx} \\
 & \left. + \left(\frac{cc'q''}{192} - \frac{c^2q''^2}{960q'} + \frac{c^2q'''}{480} \right) u_{xxxxx} \right] \delta(x - y).
 \end{aligned}$$

To prove the theorem one has to analyze the commutativity conditions

$$\mathcal{E} \left(\frac{\delta H_f}{\delta u(x)} L \frac{\delta H_g}{\delta u(x)} \right) = 0$$

for arbitrary two functions $f(u), g(u)$. Here

$$L = q \partial_x + \frac{1}{2} q' u_x - \frac{\epsilon^2}{8} c q' \partial_x^3 + \dots$$

is the Hamiltonian differential operator associated with the second Hamiltonian structure. To prove validity of Jacobi identity one has to check that the ϵ -terms in the second Hamiltonian structure can be eliminated by the quasitriviality transformation described in Sect. 3. We will omit the calculations.

Observe that the family of bihamiltonian structures given in Theorem A.1 depends on two arbitrary functions $c = c(u), q = q(u)$, in agreement with the results of [26]. It is understood that the Jacobi identity for the Poisson pencil holds true identically in λ modulo terms of the order $O(\epsilon^6)$.

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References

1. Arnold, V.I., Gusein-Zade, S.M., Varchenko, A.N.: Singularities of differentiable maps. Vol. I. The classification of critical points, caustics and wave fronts. Monographs in Mathematics **82**. Boston, MA: Birkhäuser Boston, Inc., 1985
2. Baikov, V.A., Gazizov, R.K., Ibragimov, N.Kh.: Approximate symmetries and formal linearization. *PMTF* **2**, 40–49 (1989) (In Russian)
3. Bressan, A.: One dimensional hyperbolic systems of conservation laws. In: Current developments in mathematics, 2002, Somerville, MA: Int. Press, 2003, pp. 1–37
4. Brézin, É., Marinari, E., Parisi, G.: A nonperturbative ambiguity free solution of a string model. *Phys. Lett. B* **242**, 35–38 (1990)
5. Camassa, R., Holm, D.D.: An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.* **71**, 1661–1664 (1993)
6. Degiovanni, L., Magri, F., Sciacca, V.: On deformation of Poisson manifolds of hydrodynamic type. *Commun. Math. Phys.* **253**, no. 1, 1–24 (2005)
7. Dickey, L.A.: Soliton equations and Hamiltonian systems. Second edition. Advanced Series in Mathematical Physics **26**. River Edge, NJ: World Scientific Publishing Co., Inc., 2003
8. Dobrokhotov, S., Pankrashkin, K., Semenov, E.: On Maslov's conjecture on the structure of weak point singularities of the shallow water equations. *Dokl. Akad. Nauk* **379**, no. 2, 173–176 (2001); English translation: *Doklady Math.* **64**, 127–130 (2001)
9. Dubrovin, B., Zhang, Y.: Normal forms of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants. <http://arxiv.org/list/math.DG/0108160>, 2001
10. Dubrovin, B., Liu, S.-Q., Zhang, Y.: On hamiltonian perturbations of hyperbolic systems of conservation laws, I: quasitriviality of bihamiltonian perturbations. *Comm. Pure and Appl. Math.* **59**, 559–615 (2006)
11. Dubrovin, B., Novikov, S.P.: Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogolyubov-Whitham averaging method. *Dokl. Akad. Nauk SSSR* **270**, no. 4, 781–785 (1983); English translation: *Soviet Math. Dokl.* **27**, 665–669 (1983)
12. El, G.A.: Resolution of a shock in hyperbolic systems modified by weak dispersion. *Chaos* **15**, 037103 (2005)
13. Faddeev, L.D., Takhtajan, L.A.: Hamiltonian methods in the theory of solitons. Springer Series in Soviet Mathematics, Berlin: Springer-Verlag, 1987
14. Fokas, A.S.: On a class of physically important integrable equations. *Physica D* **87**, 145–150 (1995)
15. Getzler, E.: A Darboux theorem for Hamiltonian operators in the formal calculus of variations. *Duke Math. J.* **111**, 535–560 (2002)
16. Grava, T., Klein, C.: Numerical solution of the small dispersion limit of the KdV equation and Whitham equations. <http://arxiv.org/list/math-ph/0511011>, 2005
17. Gurevich, A., Meshcherkin, A.: Expanding self-similar discontinuities and shock waves in dispersive hydrodynamics. *Sov. Phys. JETP* **60**, 732–740 (1984)
18. Gurevich, A., Pitaevski, L.: Nonstationary structure of a collisionless shock wave. *Sov. Phys. JETP Lett.* **38**, 291–297 (1974)
19. Hou, T.Y., Lax, P.D.: Dispersive approximations in fluid dynamics. *Comm. Pure Appl. Math.* **44**, 1–40 (1991)
20. Kapaev, A.A.: Weakly nonlinear solutions of the equation P_1^2 . *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **187** (1991), *Differentsialnaya Geom. Gruppy Li i Mekh.* **12**, 88–109, 172–173, 175; translation in *J. Math. Sci.* **73**, no. 4, 468–481 (1975)
21. Khesin, B., Misiołek, G.: Euler equations on homogeneous spaces and Virasoro orbits. *Adv. Math.* **176**, 116–144 (2003)
22. Kodama, Y., Mikhailov, A.: Obstacles to asymptotic integrability. In: Algebraic aspects of integrable systems, *Progr. Nonlinear Differential Equations Appl.* **26**, Boston, MA: Birkhäuser, 1997, pp. 173–204
23. Kudashev, V., Suleimanov, B.: A soft mechanism for the generation of dissipationless shock waves. *Phys. Lett. A* **221**, 204–208 (1996)
24. Lax, P., Levermore, D.: The small dispersion limit of the Korteweg-de Vries equation. I, II, III. *Comm. Pure Appl. Math.* **36**, 253–290, 571–593, 809–829 (1983)
25. Lax, P. D., Levermore, C.D., Venakides, S.: The generation and propagation of oscillations in dispersive initial value problems and their limiting behavior. In: Important developments in soliton theory, Springer Ser. Nonlinear Dynam., Berlin: Springer, 1993, pp. 205–241

26. Liu, S.Q., Zhang, Y.: Deformations of semisimple bihamiltonian structures of hydrodynamic type. *J. Geom. Phys.* **54**, 427–453 (2005)
27. Liu, S.Q., Zhang, Y.: On quasitriviality of a class of scalar evolutionary PDEs. *J. Geom. Phys.*, 2006, to appear. <http://arxiv.org/list/nlin.SI/0510019>, 2005
28. Lorenzoni, P.: Deformations of bihamiltonian structures of hydrodynamic type. *J. Geom. Phys.* **44**, 331–375 (2002)
29. Potěmin, G.: Algebro-geometric construction of self-similar solutions of the Whitham equations. *Uspekhi Mat. Nauk* **43**, no. 5(263), 211–212 (1988); translation in *Russ. Math. Surv.* **43**, 252–253 (1988)
30. Strachan, I.A.B.: Deformations of the Monge/Riemann hierarchy and approximately integrable systems. *J. Math. Phys.* **44**, 251–262 (2003)
31. Tsarëv, S.P.: The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method, *Izv. Akad. Nauk SSSR Ser. Mat.* **54**, no. 5, 1048–1068 (1990); English translation in *Math. USSR-Izv.* **37**, 397–419 (1991)
32. Zabusky, N.J., Kruskal, M.D.: Interaction of “solitons” in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.* **15**, 240–243 (1965)

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