On Hamiltonian Perturbations of Hyperbolic Systems of Conservation Laws I: Quasi-Triviality of Bi-Hamiltonian Perturbations

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Abstract

We study the general structure of formal perturbative solutions to the Hamiltonian perturbations of spatially one-dimensional systems of hyperbolic PDEs $\mathbf{v}_t + [\phi(\mathbf{v})]_x = 0$. Under certain genericity assumptions it is proved that any *bi-Hamiltonian* perturbation can be eliminated in all orders of the perturbative expansion by a change of coordinates on the infinite jet space depending *rationally* on the derivatives. The main tool is in constructing the so-called quasi-Miura transformation of jet coordinates, eliminating an arbitrary deformation of a semisimple bi-Hamiltonian structure of hydrodynamic type (the quasi-triviality theorem). We also describe, following [35], the invariants of such bi-Hamiltonian structures with respect to the group of Miura-type transformations depending *polynomially* on the derivatives. © 2005 Wiley Periodicals, Inc.

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1 Introduction

Systems of evolutionary PDEs arising in many physical applications can be written in the form

$$w_t^i + V_j^i(w) w_x^j + \text{perturbation} = 0, \quad i = 1, \dots, n, \tag{1.1}$$

where the perturbation may depend on higher derivatives. The dependent variables of the system

$$w = (w^1(x,t), \dots, w^n(x,t))$$

are functions of one spatial variable x and the time t; summation over repeated upper and lower indices will be assumed unless otherwise specified. $V_j^i(w)$ is a matrix of functions having *real distinct* eigenvalues. Therefore system (1.1) can be considered as a perturbation of the hyperbolic system of first-order quasi-linear PDEs

$$v_t^i + V_j^i(v) v_x^j = 0, \quad i = 1, \dots, n \tag{1.2}$$

(it will be convenient to denote differently the dependent variables of the unperturbed system (1.2) and the perturbed one (1.1)). Recall (see, e.g., [9]) that system (1.2) is called *hyperbolic* if the eigenvalues of the matrix $V_j^i(v)$ are all real and all n eigenvectors are linearly independent. In particular, *strictly hyperbolic* systems are those for which the eigenvalues are all real and pairwise distinct. An important particular class is the so-called systems of conservation laws

$$v_t^i + \partial_x \phi^i(v) = 0, \quad i = 1, \dots, n, \tag{1.3}$$

where the dependent variables are chosen to be densities of conserved quantities, and the functions $\phi^i(v)$ are the corresponding densities of fluxes (see, e.g., [9] regarding the physical applications of such systems). The relationships between solutions of the perturbed and unperturbed systems have been extensively studied for the case of *dissipative* perturbations of spatially one-dimensional systems of conservation laws (see, e.g., [6] and the references therein). Our strategic goal is the study of *Hamiltonian* perturbations of hyperbolic PDEs. Although many concrete examples of such perturbations have been studied (see, e.g., [45, 26, 19, 11, 34, 15, 32]), the general concepts and results are still missing.

Let us first explain how to recognize Hamiltonian systems among all systems of conservation laws. Recall [9] that the system of conservation laws (1.3) is *symmetrizable* in the sense of Friedrichs and Lax, Godunov, if there exists a constant, symmetric, positive definite matrix $\eta = (\eta_{ij})$ such that the matrix

$$\eta_{is} \frac{\partial \phi^s}{\partial v^j}$$

is symmetric,

$$\eta_{is} \frac{\partial \phi^s}{\partial v^j} = \eta_{js} \frac{\partial \phi^s}{\partial v^i}. \tag{1.4}$$

In case the symmetry (1.4) holds true while the symmetric matrix η is only nondegenerate but not necessarily positive definite, one obtains *weakly symmetrizable* systems of conservation laws.

LEMMA 1.1 *The system of conservation laws* (1.3) *is Hamiltonian if it is weakly symmetrizable.*

PROOF: Choosing the Poisson brackets in the constant form

$$\{v^i(x), v^j(y)\} = \eta^{ij}\delta'(x-y), \quad (\eta^{ij}) = (\eta_{ij})^{-1},$$

and a local Hamiltonian

$$H = \int h(v(x))dx,$$

one obtains the Hamiltonian system in the form (1.3) with

$$\phi^i(v) = \eta^{is}\frac{\partial h(v)}{\partial v^s}.$$

Both sides of (1.4) coincide then with the Hessian of the Hamiltonian density $h(v)$. The lemma is proved. □

Recall that weakly symmetrizable systems of conservation laws enjoy the following important property: they possess two additional conservation laws, namely,

$$\partial_t p(v) + \partial_x q(v) = 0, \quad p = \frac{1}{2}\eta_{ij}v^i v^j, \ \ q = v^i\frac{\partial h}{\partial v^i} - h(v), \tag{1.5}$$

$$\partial_t h(v) + \partial_x f(v) = 0, \quad f(v) = \frac{1}{2}\eta^{ij}\frac{\partial h}{\partial v^i}\frac{\partial h}{\partial v^j}, \tag{1.6}$$

where $h(v)$ is the Hamiltonian density in the formulae above. For symmetrizable systems the function $p(v)$ is nonnegative.

The class of Hamiltonian perturbations to be investigated will be written in the form

$$\begin{aligned} w_t^i + \{w^i(x), H\} = w_t^i + V_j^i(w)w_x^j + \sum_{k\geq 1}\epsilon^k U_k^i(w; w_x, \dots, w^{(k+1)}) \\ = 0, \quad i = 1, \dots, n, \end{aligned} \tag{1.7}$$

where ϵ is the small parameter, and the $U_k^i(w; w_x, \dots, w^{(k+1)})$ are graded homogeneous polynomials[1] in the jet variables

$$w_x = (w_x^1, \dots, w_x^n), \ w_{xx} = (w_{xx}^1, \dots, w_{xx}^n), \ \dots, \\ w^{(k+1)} = (w^{1,k+1}, \ \dots, \ w^{n,k+1}),$$

[1] A different class of perturbations for the particular case of the KdV equation was considered by Y. Kodama [30]. In his theory the terms of the perturbative expansion are polynomials also in w (here $n = 1$). The degree on the algebra of differential polynomials is defined by $\deg u^{(m)} = m + 2$, $m \geq 0$. Also, some nonlocal terms appear in Kodama's perturbation theory. Further developments of this method can be found in [31].

with

$$\deg w^{i,m} = m, \quad i = 1, \dots, n, \quad m > 0. \tag{1.8}$$

They arise, e.g., in the study of solutions slowly varying in the space-time directions [45].

In what follows, we will call a function that depends polynomially on the jet variables $w_x, w_{xx}, \dots$, a *differential polynomial*, and call a differential polynomial that is homogeneous with respect to the above gradation a *homogeneous differential polynomial*.

The Hamiltonians are local functionals

$$\begin{aligned} H &= \int [h^{[0]}(w) + \epsilon\, h^{[1]}(w; w_x) + \epsilon^2 h^{[2]}(w; w_x, w_{xx}) + \cdots]dx, \\ &\deg h^{[k]}(w; w_x, \dots, w^{(k)}) = k, \end{aligned} \tag{1.9}$$

for some differential polynomials $h^{[k]}(w; w_x, \dots, w^{(k)})$. The Poisson brackets are assumed to be *local* in every order in ϵ; i.e., they are represented as follows:

$$\{w^i(x), w^j(y)\} = \sum_{m\geq 0}\sum_{l=0}^{m+1} \epsilon^m A^{ij}_{m,l}(w; w_x, \dots, w^{(m-l+1)})\delta^{(l)}(x-y) \tag{1.10}$$

with coefficients being differential polynomials,

$$\deg A^{ij}_{m,l}(w; w_x, \dots, w^{(m-l+1)}) = m - l + 1. \tag{1.11}$$

We also assume that the coefficients of these differential polynomials are smooth functions on an n-dimensional ball $w \in B \subset \mathbb{R}^n$. It is understood that the antisymmetry and the Jacobi identity for (1.10) hold true as identities for formal power series in ϵ. It can be readily seen that, for an arbitrary local Hamiltonian of the form (1.9), the evolutionary systems (1.7) have the needed form.

The leading term

$$\{w^i(x), w^j(y)\}^{[0]} := A^{ij}_{0,1}(w(x))\delta'(x-y) + A^{ij}_{0,0}(w(x); w_x(x))\delta(x-y) \tag{1.12}$$

is itself a Poisson bracket (the so-called Poisson bracket of hydrodynamic type; see [14]). We will always assume that

$$\det A^{ij}_{0,1}(w) \neq 0 \tag{1.13}$$

for all $w \in B \subset \mathbb{R}^n$. Redenote the coefficients of $\{w^i(x), w^j(y)\}^{[0]}$ as follows:

$$g^{ij}(w) := A^{ij}_{0,1}(w), \quad Q^{ij}_k(w) w^k_x := A^{ij}_{0,0}(w; w_x) \tag{1.14}$$

(see (1.11)). The coefficient $(g^{ij}(w))$ can be considered as a symmetric nondegenerate bilinear form on the cotangent spaces. The inverse matrix defines a metric

$$ds^2 = g_{ij}(w) dw^i dw^j, \quad (g_{ij}(w)) := (g^{ij}(w))^{-1} \tag{1.15}$$

(not necessarily positive definite). Recall [14] that (1.12)–(1.14) defines a Poisson structure if and only if the metric is flat and $Q^{ij}_k(w) = -g^{il}(w)\Gamma^j_{kl}(w)$ where Γ^j_{kl} are the Christoffel symbols of the Levi-Civita connection of the metric (1.15).

Miura-type transformations are defined by

$$
\begin{aligned}
&w^i \mapsto \tilde{w}^i = \Phi_0^i(w) + \sum_{k\geq 1} \epsilon^k \Phi_k^i(w; w_x, \dots, w^{(k)}), \quad i = 1, \dots, n, \\
&\deg \Phi_k^i(w; w_x, \dots, w^{(k)}) = k, \quad \text{satisfying } \det\left(\frac{\partial \Phi_0^i}{\partial w^j}\right) \neq 0.
\end{aligned}
\tag{1.16}
$$

As usual, the coefficients $\Phi_k^i(w; w_x, \dots, w^{(k)})$ are assumed to be differential polynomials. It is easy to see that such transformations form a group. The group of Miura-type transformations is a natural extension of the group of local diffeomorphisms that plays an important role in the geometrical study of hyperbolic systems (see, e.g. [43]).

The class of the Hamiltonians (1.9), Poisson brackets (1.10), and the evolutionary systems (1.7) is invariant with respect to Miura-type transformations. Two Poisson brackets of the form (1.10) are called *equivalent* if they are related by a Miura-type transformation.

An important result of [25] (see also [10, 16]) says that any Poisson bracket of the form (1.10) can be locally reduced by Miura-type transformations to the constant form

$$
\{\tilde{w}^i(x), \tilde{w}^j(y)\} = \eta^{ij}\delta'(x-y), \quad \eta^{ij} = \text{const.}
\tag{1.17}
$$

We will denote the inverse matrix by the same symbol with lower indices

$$
(\eta_{ij}) := (\eta^{ij})^{-1}.
\tag{1.18}
$$

Connection of the theory of Hamiltonian systems (1.7) to the theory of systems of conservation laws is clear from the following statement:

LEMMA 1.2 *By a change of dependent variables of the form* (1.16) *the Hamiltonian system* (1.7) *can be recast into the form of a system of conservation laws*

$$
\begin{aligned}
&\tilde{w}_t^i + \partial_x \psi^i(\tilde{w}; \tilde{w}_x, \dots; \epsilon) = 0, \quad i = 1, \dots, n, \\
&\psi^i(\tilde{w}; \tilde{w}_x, \dots; \epsilon) = \sum_{k\geq 0} \epsilon^k \psi_k^i(\tilde{w}; \tilde{w}_x, \dots, \tilde{w}^{(k)}), \\
&\deg \psi_k^i(\tilde{w}; \tilde{w}_x, \dots, \tilde{w}^{(k)}) = k.
\end{aligned}
\tag{1.19}
$$

The system of conservation laws (1.19) *is Hamiltonian with respect to the Poisson bracket* (1.17) *if and only if*

$$
\psi_i := \eta_{ij}\psi^j(\tilde{w}; \tilde{w}_x, \dots; \epsilon)
\tag{1.20}
$$

satisfy

$$
\frac{\partial \psi_i}{\partial \tilde{w}^{j,s}} = \sum_{t\geq s} (-1)^t \binom{t}{s} \partial_x^{t-s} \frac{\partial \psi_j}{\partial \tilde{w}^{i,t}}
\tag{1.21}
$$

for any $i, j = 1, \dots, n$, $s = 0, 1, \dots$.

In this paper we will investigate the structure of *formal* perturbative expansions of the solutions to (1.7)

$$w^i(x,t;\epsilon) = v^i(x,t) + \epsilon\,\delta_1 v^i(x,t) + \epsilon^2 \delta_2 v^i(x,t) + \cdots . \tag{1.22}$$

The leading term solves (1.2); the coefficients of the expansion $\delta_k v^i(x,t)$ are to be determined from linear PDEs with coefficients depending on $v^i, \delta_1 v^i, \dots, \delta_{k-1} v^i$ and their derivatives. Instead of developing this classical technique, we propose a different approach that conceptually goes back to Poincaré's treatment of perturbative expansions in celestial mechanics. We will look for a transformation of the form

$$w^i = v^i + \sum_{k\geq 1} \epsilon^k \Phi_k^i(v; v_x, \dots, v^{(m_k)}), \quad i = 1, \dots, n, \tag{1.23}$$

that maps *any* generic[2] solution $v^i(x,t)$ of the unperturbed system (1.2) to a solution $w^i(x,t;\epsilon)$ of the perturbed system. An important feature of such an approach to the perturbation theory is *locality*: changing the functions $v(x,t)$ for a given t only within a small neighborhood of the given point $x = x_0$ will keep unchanged the values of $w(x,t;\epsilon)$ outside this neighborhood. We call (1.23) *the reducing transformation* for the perturbed system (1.7).

Clearly, applying to (1.2) any transformation (1.23) *polynomial* in the derivatives (in every order in ϵ; in that case $m_k = k$) one obtains a perturbed system of the form (1.7). This is the case of *trivial* perturbations.

It is clear that solutions of trivial Hamiltonian perturbations share many properties of solutions to the unperturbed hyperbolic PDEs (1.2). In particular, the trivial perturbation cannot balance the nonlinear effects in the hyperbolic system that typically cause gradient catastrophe of the solution.

DEFINITION 1.3 The system of PDEs (1.7) is called *quasi-trivial* if it is not trivial but a reducing transformation (1.23) exists with functions $\Phi_k^i(v; v_x, \dots, v^{(m_k)})$ depending *rationally* on the jet coordinates

$$w^{i,l}, \quad i = 1, \dots, n, \ \ 1 \leq l \leq m_k,$$
$$\deg \Phi_k^i(v; v_x, \dots, v^{(m_k)}) = k, \quad k \geq 1,$$

so that (1.7) is reduced to (1.2).

The first example of such a reducing transformation can be found in [4] (see also [16]) for the KdV equation

$$w_t + w\,w_x + \frac{\epsilon^2}{12} w_{xxx} = 0 \tag{1.24}$$

[2] We will later be more specific in describing the range of applicability of the transformations (1.23). Namely, it turns out that all our formal transformations (1.23) will be well-defined on the class of *monotone* solutions (see the definition of monotone solutions after Corollary 1.10).

(here $n = 1$):

$$w = v + \frac{\epsilon^2}{24}\partial_x^2 \log v_x + \epsilon^4 \partial_x^2\left(\frac{v^{(4)}}{1152\, v_x^2} - \frac{7\, v_{xx} v_{xxx}}{1920\, v_x^3} + \frac{v_{xx}^3}{360\, v_x^4}\right) + O(\epsilon^6). \tag{1.25}$$

It is not an easy task to check cancellation of all the denominators even in this example! Because of the denominators, the reducing deformation is defined only on the monotone solutions.

One of the main results of our paper is in proving quasi-triviality of a large class of Hamiltonian perturbations of hyperbolic systems of conservation laws. The systems in question are *bi-Hamiltonian systems of PDEs*. That means that they can be represented in the Hamiltonian form in two different ways:

$$w_t^i = \{H_1, w^i(x)\}_1 = \{H_2, w^i(x)\}_2, \quad i = 1, \dots, n, \tag{1.26}$$

with two local Hamiltonians (see (1.9) above) and local *compatible* Poisson brackets $\{\,\cdot\,,\cdot\,\}_1$, $\{\,\cdot\,,\cdot\,\}_2$ of the form (1.10). Compatibility means that any linear combination

$$a_1\{\,\cdot\,,\cdot\,\}_1 + a_2\{\,\cdot\,,\cdot\,\}_2$$

with arbitrary constant coefficients a_1 and a_2 must be again a Poisson bracket.

The study of bi-Hamiltonian structures was initiated by F. Magri [38] in his analysis of the so-called Lenard scheme of constructing the KdV integrals. Dorfman and Gelfand [24] and also Fokas and Fuchssteiner [22] discovered the connections between the bi-Hamiltonian scheme and the theory of hereditary symmetries of integrable equations. However, it is not easy to apply these beautiful and simple ideas to the study of general bi-Hamiltonian PDEs (see the discussion of the problems encountered in [16]).

In this paper we will use a different approach, proposed in [16], to the study of bi-Hamiltonian PDEs. It is based on the careful study of the transformation properties of the bi-Hamiltonian structures under the transformations of the form (1.23). Let us now proceed to the precise definitions and formulations of the results.

We will study bi-Hamiltonian structures defined by compatible pairs of *local* Poisson brackets of the form (1.10)

$$\begin{aligned} &\{w^i(x), w^j(y)\}_a \\ &\quad = \{w^i(x), w^j(y)\}_a^{[0]} \\ &\qquad + \sum_{m\ge 1}\sum_{l=0}^{m+1} \epsilon^m A_{m,l;a}^{ij}(w; w_x, \dots, w^{(m-l+1)})\delta^{(l)}(x-y), \quad a = 1, 2. \end{aligned} \tag{1.27}$$

As in (1.10), the differential polynomials $A_{m,l;1}^{ij}$ and $A_{m,l;2}^{ij}$ are homogeneous of degree $m - l + 1$, and their coefficients are smooth functions on an n-dimensional ball $w \in B \subset \mathbb{R}^n$. Equivalence of bi-Hamiltonian structures is defined with respect to simultaneous Miura-type transformations.

The leading terms of the bi-Hamiltonian structure also yield a bi-Hamiltonian structure of the form

$$\begin{aligned} &\{w^i(x), w^j(y)\}_a^{[0]} = g_a^{ij}(w(x))\delta'(x-y) + Q_{a;k}^{ij}(w(x))w_x^k\delta(x-y), \\ &\det(g_a^{kl}) \neq 0 \quad \text{for generic points } w \in B, \ \ i, j = 1, \ldots, n, \ \ a = 1, 2 \end{aligned} \tag{1.28}$$

(a bi-Hamiltonian structure of the hydrodynamic type). We additionally assume that $\det(a_1 g_1^{kl}(w) + a_2 g_2^{kl}(w))$ does not vanish identically for $w \in B$ unless $a_1 = a_2 = 0$.

DEFINITION 1.4 The bi-Hamiltonian structure (1.27) is called *semisimple* if the characteristic polynomial $\det(g_2^{ij}(w) - \lambda g_1^{ij}(w))$ in λ has n pairwise distinct real[3] roots $\lambda_1(w), \ldots, \lambda_n(w)$ for any $w \in B$.

Equivalently, the linear operator $U = (U_j^i(w))$ given by the ratio

$$U_j^i(w) = g_2^{ik}(w) g_{1kj}(w), \quad (g_{1kj}(w)) = (g_1^{kj}(w))^{-1}, \tag{1.29}$$

has pairwise distinct real eigenvalues for any $w \in B$.

The role of the semisimplicity assumption can be illustrated by the following:

LEMMA 1.5 *Given a semisimple bi-Hamiltonian structure $\{\cdot,\cdot\}_{1,2}^{[0]}$ satisfying the above conditions, denote by*

$$\lambda = u^1(w), \ldots, u^n(w)$$

the roots of the characteristic equation

$$\det\left(g_2^{ij}(w) - \lambda\, g_1^{ij}(w)\right) = 0. \tag{1.30}$$

The functions $u^1(w), \ldots, u^n(w)$ satisfy

$$\det\left(\frac{\partial u^i(w)}{\partial w^j}\right) \neq 0.$$

Using these functions as new local coordinates

$$w^i = q^i(u^1, \ldots, u^n), \quad i = 1, \ldots, n, \ \ \det\left(\frac{\partial q^i(u)}{\partial u^j}\right) \neq 0, \tag{1.31}$$

reduces simultaneously the two flat metrics to the diagonal form

$$\begin{aligned} &\sum_{k,l=1}^n \frac{\partial u^i}{\partial w^k}\frac{\partial u^j}{\partial w^l} g_1^{kl}(w) = f^i(u)\,\delta^{ij}, \\ &\sum_{k,l=1}^n \frac{\partial u^i}{\partial w^k}\frac{\partial u^j}{\partial w^l} g_2^{kl}(w) = g^i(u)\,\delta^{ij} = u^i f^i(u)\,\delta^{ij}. \end{aligned} \tag{1.32}$$

[3] One can relax the requirement of reality of the roots working with complex manifolds. In that case the coefficients must be analytic in w.

The coefficients $Q^{ij}_{a;k}$ *in the coordinates* $(u^1, \dots, u^n)$ *read*

$$\sum_{k=1}^{n} Q^{ij}_{1;k} u^k_x = \frac{1}{2}\,\delta^{ij}\partial_x f^i + A^{ij}, \quad \sum_{k=1}^{n} Q^{ij}_{2;k} u^k_x = \frac{1}{2}\,\delta^{ij}\partial_x g^i + B^{ij}, \tag{1.33}$$

$$A^{ij} = \frac{1}{2}\left(\frac{f^i}{f^j} f^j_i u^j_x - \frac{f^j}{f^i} f^i_j u^i_x\right), \quad B^{ij} = \frac{1}{2}\left(\frac{u^i f^i}{f^j} f^j_i u^j_x - \frac{u^j f^j}{f^i} f^i_j u^i_x\right) \tag{1.34}$$

where $f^i_k = \partial f^i/\partial u^k$. *The leading term of any bi-Hamiltonian system becomes diagonal in the coordinates* $u^1, \dots, u^n$,

$$u^i_t + V^i(u) u^i_x + \mathcal{O}(\epsilon) = 0, \quad i = 1, \dots, n. \tag{1.35}$$

Such coordinates are called the *canonical coordinates* of the semisimple bi-Hamiltonian structure. In what follows we will *never* use the convention of summation over repeated indices when working with the canonical coordinates.

The canonical coordinates are defined up to a permutation. The functions $f_1(u), \dots, f_n(u)$ satisfy a complicated system of nonlinear differential equations. The general solution to this system depends on n^2 arbitrary functions of one variable. Integrability of this system has recently been proved in [17, 40]. For convenience, we give a brief account of these results, following [17], in the appendix.

The following statement gives a simple criterion to determine whether a Hamiltonian system of conservation laws is bi-Hamiltonian.

LEMMA 1.6 *Let us consider a strictly hyperbolic system* (1.2) *Hamiltonian with respect to the Poisson bracket* $\{\cdot\,,\cdot\}^{[0]}_1$. *This system is bi-Hamiltonian with respect to the semisimple Poisson pencil* $\{\cdot\,,\cdot\}^{[0]}_{1,2}$ *if and only if the coefficient matrix* $V = (V^i_j(w))$ *commutes with the matrix* $U = (U^i_j(w))$ *of the form* (1.29)

$$[U, V] = 0. \tag{1.36}$$

Due to the commutativity (1.36) the matrix V becomes diagonal in the canonical coordinates for the Poisson pencil:

$$\sum_{k,l=1}^{n} \frac{\partial u^i}{\partial v^k}\frac{\partial v^l}{\partial u^j} V^k_l(v) = V^i(u)\delta^i_j. \tag{1.37}$$

Observe that the canonical coordinates are *Riemann invariants* (see, e.g., [45]) for the leading term of the system of PDEs (1.35). The coefficients $V^i(u)$ in the gas dynamics are called *characteristic velocities* [45]. In particular, the semisimplicity assumption implies hyperbolicity of the leading term of the bi-Hamiltonian systems.

DEFINITION 1.7 ([16]) The bi-Hamiltonian structure (1.27) is said to be *trivial* if it can be obtained from the leading term

$$\{v^i(x), v^j(y)\}^{[0]}_a = g^{ij}_a(v(x))\delta'(x-y) + Q^{ij}_{a;k}(v(x))v^k_x\delta(x-y), \quad a = 1, 2, \tag{1.38}$$

by a Miura-type transformation

$$w^i = v^i + \sum_{k\geq 1} \epsilon^k F_k^i(v; v_x, \dots, v^{(k)}),$$
$$\deg F_k^i(v; v_x, \dots, v^{(k)}) = k, \quad i = 1, \dots, n, \tag{1.39}$$

where the coefficients $F_k^i(v; v_x, \dots, v^{(k)})$ are homogeneous differential polynomials. It is called *quasi-trivial* if it is not trivial and there exists a transformation

$$w^i = v^i + \sum_{k\geq 1} \epsilon^k F_k^i(v; v_x, \dots, v^{(m_k)}) \tag{1.40}$$

reducing (1.27) to (1.38) but the functions F_k depend *rationally* on the jet coordinates $v^{i,m}$, $m \geq 1$, with

$$\deg F_k = k, \quad k \geq 1, \tag{1.41}$$

and m_k are some positive integers. If such a transformation (1.39) or (1.40) exists, it is called a *reducing transformation* of the bi-Hamiltonian structure (1.27).

A transformation of the form (1.40) is called a *quasi-Miura transformation*.

We are now in a position to formulate the main result of the present paper.

THEOREM 1.8 (Quasi-Triviality Theorem) *For any semisimple bi-Hamiltonian structure* (1.27) *there exists a reducing transformation of the form* (1.40). *The coefficients F_k^i have the form*

$$F_k^i(v; v_x, \dots, v^{(m_k)}) \in C^\infty(B)[v_x, \dots, v^{(m_k)}]\,[(u_x^1 u_x^2 \dots u_x^n)^{-1}],$$
$$m_k \leq \left[\frac{3\,k}{2}\right]. \tag{1.42}$$

Here $u^i = u^i(v)$ are the canonical coordinates (see Lemma 1.5).

Using this theorem we achieve the goal of constructing the reducing transformation for a bi-Hamiltonian system (1.26), (1.9):

COROLLARY 1.9 *The reducing transformation for the bi-Hamiltonian structure* (1.27) *is also a reducing transformation for any bi-Hamiltonian system* (1.26).

Another corollary says that the solution of any system of bi-Hamiltonian PDEs of the above form can be reduced to solving *linear PDEs*. Let us first rewrite the reducing transformation in the canonical coordinates

$$\tilde{u}^i = u^i + \sum_{k\geq 1} \epsilon^k G_k^i(u; u_x, \dots, u^{(m_k)}), \quad i = 1, \dots, n. \tag{1.43}$$

Let $W^i(u; \epsilon)$, $i = 1, \dots, n$, be an arbitrary solution to the linear system

$$\frac{\partial W^i}{\partial u^j} = \frac{\partial V^i / \partial u^j}{V^i - V^j}(W^i - W^j), \quad i \neq j, \tag{1.44}$$

in the class of formal power series in ϵ. In this system the functions $V^i(u)$ are eigenvalues of the matrix $V^i_j(w)$; cf. (1.35). Let us assume that the system of equations

$$x = V^i(u)t + W^i(u; \epsilon = 0), \quad i = 1, \dots, n, \tag{1.45}$$

has a solution $(x, t, u) = (x_0, t_0, u_0)$ such that

$$\det\left(t\frac{\partial V^i(u)}{\partial u^j} + \frac{\partial W^i(u;\epsilon)}{\partial u^j}\right)_{t=t_0,\, u=u_0,\, \epsilon=0} \neq 0. \tag{1.46}$$

For (x, t) sufficiently close to (x_0, t_0), denote $u(x, t) = (u^1(x, t), \dots, u^n(x, t))$ the unique solution to the equations (1.45) such that

$$u(x_0, t_0) = u_0.$$

Applying the transformation (1.43) to the vector function $u(x, t)$, we obtain a vector function $\tilde{u}(x, t; \epsilon)$. Finally, the substitution

$$q^i(\tilde{u}(x, t; \epsilon)) =: w^i(x, t; \epsilon), \quad i = 1, 2, \dots, \tag{1.47}$$

yields n functions $w^1(x, t; \epsilon), \dots, w^n(x, t; \epsilon)$. Here the functions $q^i(u)$ are defined as in (1.31).

COROLLARY 1.10 *The functions* (1.47) *satisfy* (1.26). *Conversely, any solution to* (1.26) monotone *at at $x = x_0$, $t = t_0$, can be obtained by this procedure.*

By definition, the solution $w(x, t; \epsilon)$ is called monotone at $x = x_0$ and $t = t_0$ if all the x-derivatives

$$\partial_x u^1(w(x, t; \epsilon)), \dots, \partial_x u^n(w(x, t; \epsilon))$$

do not vanish for $x = x_0$, $t = t_0$, $\epsilon = 0$.

Finally, we can combine the quasi-triviality theorem with the main result of the recent paper [35] in order to describe the complete set of invariants with respect to the group of Miura-type transformations of bi-Hamiltonian structures of the above form with the *given* leading term $\{w^i(x), w^j(y)\}^{[0]}_a$.

Introduce the following combinations of the coefficients of $\epsilon\delta''(x-y)$ and $\epsilon^2\delta'''(x-y)$ of the bi-Hamiltonian structure

$$\begin{gathered} P^{ij}_a(u) = \frac{\partial u^i}{\partial w^k}\frac{\partial u^j}{\partial w^l}A^{kl}_{1,2;a}(w), \qquad Q^{ij}_a(u) = \frac{\partial u^i}{\partial w^k}\frac{\partial u^j}{\partial w^l}A^{kl}_{2,3;a}(w), \\ i, j = 1, \dots, n, \quad a = 1, 2. \end{gathered} \tag{1.48}$$

Define the functions

$$c_i(u) = \frac{1}{3(f^i(u))^2}\left(Q^{ii}_2 - u^i Q^{ii}_1 + \sum_{k\neq i}\frac{(P^{ki}_2 - u^i P^{ki}_1)^2}{f^k(u)(u^k - u^i)}\right), \quad i = 1, \dots, n. \tag{1.49}$$

The functions $c^i(u)$ are called *central invariants* of the bi-Hamiltonian structure (1.27). The main result of [35] on the classification of infinitesimal deformations of bi-Hamiltonian structures of hydrodynamic type can be reformulated as follows:

COROLLARY 1.11 *Each function $c_i(u)$ defined in* (1.49) *depends only on u^i. In addition, two semisimple bi-Hamiltonian structures* (1.27) *with the same leading terms $\{\cdot,\cdot\}_a^{[0]}$, $a=1,2$, are equivalent if and only if they have the same set of central invariants $c_i(u^i)$, $i=1,\dots,n$. In particular, a polynomial in the derivatives-reducing transformation exists if and only if all the central invariants vanish:*

$$c_1 = c_2 = \cdots = c_n = 0.$$

The paper is organized as follows. In Section 2 we recall some basic notions of the theory of Poisson structures for PDEs and prove Lemmas 1.2, 1.5, and 1.6. In Section 3 and Section 4, we give the proofs of the quasi-triviality theorem and Corollary 1.9. In Section 5 we reformulate the main result of [35] on the classification of infinitesimal deformations of a semisimple bi-Hamiltonian structure of hydrodynamic type and prove Corollary 1.11. In the final section we give some examples of bi-Hamiltonian structures of the class studied in this paper and formulate some open problems. Last, in the appendix we briefly present, following [17], the theory of semisimple bi-Hamiltonian structures of hydrodynamic type.

2 Some Basic Notions about Poisson Structures for PDEs

An infinite-dimensional Poisson structure of the form (1.27) or (1.28) can be represented, as in finite-dimensional Poisson geometry, by a local bivector on the formal loop space of the manifold M. Recall that in our considerations the manifold M will always be a n-dimensional ball B. In general, let $w^1,\dots,w^n$ be a local coordinate system of a chart of the manifold M. A *local translation-invariant k-vector* [16] is a formal infinite sum of the form

$$\alpha = \sum \frac{1}{k!}\partial_{x_1}^{s_1}\cdots\partial_{x_k}^{s_k} A^{i_1\dots i_k}\frac{\partial}{\partial w^{i_1,s_1}(x_1)}\wedge\cdots\wedge\frac{\partial}{\partial w^{i_k,s_k}(x_k)}. \tag{2.1}$$

Here the coefficients A's have the form

$$A^{i_1\cdots i_k} = \sum_{p_2,\dots,p_k\ge 0} B^{i_1\cdots i_k}_{p_2\cdots p_k}(w(x_1);w_x(x_1),\dots)\,\delta^{(p_2)}(x_1-x_2)\cdots\delta^{(p_k)}(x_1-x_k) \tag{2.2}$$

with only a finite number of nonzero terms in the summation; for a certain integer N that depends on the indices $i_1,\dots,i_k$ and $p_2,\dots,p_k$, the $B^{i_1\dots i_k}_{p_2\dots p_k}(w;w_x,\dots)$ are smooth functions on a domain in the jet space $J^N(M)$. The delta function and its derivatives are defined formally by

$$\int f(w(y);w_y(y),w_{yy}(y),\dots)\,\delta^{(k)}(x-y)\,dy = \partial_x^k f(w(x);w_x(x),w_{xx}(x),\dots). \tag{2.3}$$

In this formula the operator of the total derivative ∂_x is defined by

$$\partial_x f(w;w_x,w_{xx},\dots) = \sum w^{i,s+1}\frac{\partial f}{\partial w^{i,s}}. \tag{2.4}$$

Note the useful identity

$$f(w(y); w_y(y), w_{yy}(y), \dots)\delta^{(k)}(x-y) = \sum_{m=0}^{k} \binom{k}{m} \partial_x^m f(w(x); w_x(x), w_{xx}(x), \dots)\, \delta^{(k-m)}(x-y). \tag{2.5}$$

The distributions

$$A^{i_1\dots i_k} = A^{i_1\dots i_k}(x_1, \dots, x_k; w(x_1), \dots, w(x_k), \dots) \tag{2.6}$$

are antisymmetric with respect to the simultaneous permutations $i_p, x_p \leftrightarrow i_q, x_q$. They are called the *components* of the local k-vector α. Note that in the definition of local k-vectors given in [16] it is required that the functions $B^{i_1\cdots i_k}_{p_2\cdots p_k}$ be differential polynomials. Here we drop this requirement for the convenience of our use of these notations during our proof of the theorem. The space of all such local k-vectors is still denoted by Λ^k_{loc} as done in [16]. For $k = 0$ by definition Λ^0_{loc} is the space of local functionals of the form

$$I = \int f(w, w_x, \dots, w^{(m)})\, dx. \tag{2.7}$$

The Schouten-Nijenhuis bracket is defined on the space of local multivectors

$$[\,\cdot\,,\cdot\,] : \Lambda^k_{\mathrm{loc}} \times \Lambda^l_{\mathrm{loc}} \to \Lambda^{k+l-1}_{\mathrm{loc}}, \quad k, l \geq 0. \tag{2.8}$$

It generalizes the usual commutator of two local vector fields and possesses the following properties:

$$[\alpha, \beta] = (-1)^{kl}[\beta, \alpha], \tag{2.9}$$

$$(-1)^{km}[[\alpha, \beta], \gamma] + (-1)^{kl}[[\beta, \gamma], \alpha] + (-1)^{lm}[[\gamma, \alpha], \beta] = 0, \tag{2.10}$$

for any $\alpha \in \Lambda^k_{\mathrm{loc}}$, $\beta \in \Lambda^l_{\mathrm{loc}}$, and $\gamma \in \Lambda^m_{\mathrm{loc}}$. For the definition of the Schouten-Nijenhuis bracket, see [16] and references therein. Here we write down the formulae, used below, for the bracket of a local bivector with a local functional and with a local vector field.

Let a local vector field ξ and a local bivector ϖ have the representation

$$\xi = \sum_{s\geq 0} \partial_x^s \xi^i(w(x); w_x(x), \dots, w^{(m_i)}) \frac{\partial}{\partial w^{i,s}(x)}, \tag{2.11}$$

$$\varpi = \frac{1}{2} \sum_{s,t\geq 0} \partial_x^s \partial_y^t \varpi^{ij} \frac{\partial}{\partial w^{i,s}(x)} \wedge \frac{\partial}{\partial w^{j,t}(y)}. \tag{2.12}$$

Here we assume that

$$\varpi^{ij} = \sum_{k\geq 0} A^{ij}_k(w(x), w_x(x), \dots, w^{(m_k)}(x))\, \delta^{(k)}(x-y). \tag{2.13}$$

Then the components of $[\varpi, I]$ and of $[\varpi, \xi]$ are given, respectively, by

$$[\varpi, I]^i = \sum_{k\geq 0} A_k^{ij} \partial_x^k \frac{\delta I}{\delta w^j(x)}, \tag{2.14}$$

$$[\varpi, \xi]^{ij} = \sum_{t\geq 0} \left(\partial_x^t \xi^k(w(x); \dots) \frac{\partial \varpi^{ij}}{\partial w^{k,t}(x)} - \frac{\partial \xi^i(w(x); \dots)}{\partial w^{k,t}(x)} \partial_x^t \varpi^{kj} - \frac{\partial \xi^j(w(y); \dots)}{\partial w^{k,t}(y)} \partial_y^t \varpi^{ik} \right). \tag{2.15}$$

In the last formula it is understood that

$$\partial_y^t \varpi^{ij} = \sum_{k\geq 0} (-1)^t A_k^{ij}(w(x), w_x(x), \dots, w^{(m_k)}(x))\, \delta^{(k+t)}(x-y),$$

and identity (2.5) has been used in order to represent the resulting bivector in the normalized form (2.13).

Let us denote by ϖ_1 and ϖ_2 the two bivectors that correspond to the bi-Hamiltonian structure (1.28); the components ϖ_a^{ij}, $a = 1, 2$, are given by the right-hand side of (1.28). The bi-Hamiltonian property is equivalent to the following identity, which is valid for an arbitrary parameter λ:

$$[\varpi_2 - \lambda\, \varpi_1, \varpi_2 - \lambda\, \varpi_1] = 0. \tag{2.16}$$

Denote by ∂_1 and ∂_2 the differentials associated with ϖ_1 and ϖ_2. By definition

$$\partial_a : \Lambda_{\text{loc}}^k \to \Lambda_{\text{loc}}^{k+1}, \quad \partial_a \alpha = [\varpi_a, \alpha], \quad \forall \alpha \in \Lambda_{\text{loc}}^k, \quad a = 1, 2. \tag{2.17}$$

The bi-Hamiltonian property (2.16) can be recast in the form

$$\partial_1^2 = \partial_2^2 = \partial_1 \partial_2 + \partial_2 \partial_1 = 0. \tag{2.18}$$

The important fact that we need to use below is the vanishing of the first and second Poisson cohomologies

$$H^k(\mathcal{L}(M), \varpi_a) = \operatorname{Ker} \partial_a\big|_{\Lambda_{\text{loc}}^k} / \operatorname{Im} \partial_a\big|_{\Lambda_{\text{loc}}^{k-1}}, \quad a = 1, 2, \quad k = 1, 2. \tag{2.19}$$

This fact is proved in [25, 10, 16]. It readily implies, along with the results of [14], the reducibility of any Poisson bracket of the form (1.10)–(1.13) to the constant form (1.17).

Let us now give the following:

PROOF OF LEMMA 1.2: For the Poisson bracket written in the form (1.17), the Hamiltonian system reads

$$\tilde{w}_t^i = \{H, \tilde{w}^i(x)\} = -\eta^{ij} \partial_x \frac{\delta H}{\delta \tilde{w}^j(x)}.$$

This represents the equations as a system of conservation laws $\tilde{w}_t^i + \psi_x^i = 0$ with

$$\psi^i = \eta^{ij} \frac{\delta H}{\delta \tilde{w}^j(x)}.$$

The system of equations (1.21) is nothing but the spelling of the classical Helmholtz criterion [27] for functions ψ_i to be representable in the form of variational derivatives. $\square$

We now pass to the theory of canonical coordinates.

PROOF OF LEMMA 1.5: From the results of [17, 40], it follows that there exists a system of local coordinates $\hat{u}^1, \dots, \hat{u}^n$ such that both of the metrics become diagonal and $g_1^{ij} = \delta_{ij} h^i(\hat{u})$, $g_2^{ij} = \delta_{ij}\lambda_i(\hat{u}^i)h^i(\hat{u})$. Since by our assumption $\det(a_1 g_1^{kl}(w) + a_2 g_2^{kl}(w))$ does not vanish identically for $w \in M$ unless $a_1 = a_2 = 0$, we can choose $u^1 = \lambda_1(\hat{u}^1), \dots, u^n = \lambda_n(\hat{u}^n)$ as a system of local coordinates that are just the canonical coordinates. $\square$

We now proceed to the next proof:

PROOF OF LEMMA 1.6: In the canonical coordinates, a bi-Hamiltonian system

$$u_t^i = \{H_1, u^i(x)\}_1^{[0]} = \{H_2, u^i(x)\}_2^{[0]}, \quad H_a = \int h_a^{[0]}(u)dx, \;\; a = 1, 2,$$

has the expression

$$u_t^i = -\sum_{j=1}^n V_j^i(u) u_x^j, \quad i = 1, \dots, n, \tag{2.20}$$

where

$$V_j^i(u) = f^i(u)\mathcal{A}_{ij} h_1^{[0]}(u) = u^i f^i(u) \mathcal{A}_{ij} h_2^{[0]}(u) \quad \text{for } i \neq j. \tag{2.21}$$

(Recall that no summation over repeated indices is assumed when working in the canonical coordinates; all summation signs will be written explicitly.) Here the linear differential operators $\mathcal{A}_{ij}$ are defined by

$$\mathcal{A}_{ij} = \frac{\partial^2}{\partial u^i \partial u^j} + \frac{1}{2}\frac{\partial(\log f^i(u))}{\partial u^j}\frac{\partial}{\partial u^i} + \frac{1}{2}\frac{\partial(\log f^j(u))}{\partial u^i}\frac{\partial}{\partial u^j}. \tag{2.22}$$

Symmetry with respect to the indices i and j implies

$$(u^i - u^j)\mathcal{A}_{ij}\, h_2^{[0]}(u) = 0, \quad i \neq j. \tag{2.23}$$

Thus $V_j^i(u) = 0$ when $i \neq j$. This proves the first part of the lemma.

To prove the converse statement, we use the following result of [44]: the diagonal system

$$u_t^i + V^i(u) u_x^i = 0, \quad i = 1, \dots, n,$$

is Hamiltonian with respect to the Poisson bracket associated with the diagonal metric of zero curvature

$$ds^2 = \sum_{i=1}^n g_{ii}(u)(du^i)^2$$

if and only if the following equations hold true:

$$\partial_k V^i(u) = (V^k(u) - V^i(u))\partial_k \log\sqrt{g_{ii}(u)}, \quad i \neq k. \tag{2.24}$$

By assumption these equations hold true for the first metric

$$g_{ii}(u) = \frac{1}{f_i(u)}.$$

For the pencil $\{\cdot\,,\cdot\}_2^{[0]} - \lambda\,\{\cdot\,,\cdot\}_1^{[0]}$, one has to replace

$$g_{ii}(u) = \frac{1}{(u^i - \lambda)\,f^i(u)}.$$

Such a replacement does not change equations (2.24). The lemma is proved. □

3 Proof of the Quasi-Triviality Theorem

In this and the following sections, we assume that the bi-Hamiltonian structure (ϖ_1, ϖ_2) defined by (1.28) is semisimple, and we work in the canonical coordinates $u^1, \dots, u^n$.

Due to the triviality of the Poisson cohomology $H^2(\mathcal{L}(M), \varpi_1)$, the bi-Hamiltonian structure (1.27) can always be assumed, if necessary by performing a usual Miura-type transformation, to have the following form:

$$\{u^i(x), u^j(y)\}_1 = \{u^i(x), u^j(y)\}_1^{[0]}, \tag{3.1}$$

$$\{u^i(x), u^j(y)\}_2 = \{u^i(x), u^j(y)\}_2^{[0]} + \sum_{k\ge 1} \epsilon^k Q_k^{ij}. \tag{3.2}$$

Here Q_k^{ij} are the components of the bivectors Q_k and have the expressions

$$Q_k^{ij} = \sum_{l=0}^{k+1} Q_{k,l}^{ij}(u; u_x, \dots, u^{(k+1-l)})\,\delta^{(l)}(x-y).$$

We also denote by Q_0 the bivector corresponding to the undeformed second Poisson structure $\{\cdot\,,\cdot\}_2^{[0]}$. The coefficients $Q_{k,l}^{ij}$ are homogeneous differential polynomials of degree $k+1-l$. The compatibility of the above two Poisson brackets implies the existence of vector fields X_k, $k \ge 1$, such that

$$Q_k = \partial_1 X_k, \quad k \ge 0, \tag{3.3}$$

and the components of X_k are homogeneous differential polynomials of degree k.

The strategy of our proof of the quasi-triviality theorem is to construct a series of quasi-Miura transformations that keep the first Poisson structure (3.1) unchanged while removing the perturbation terms of (3.2) in a successive way.

The key property of the bi-Hamiltonian structure (3.1)–(3.2) that we use to construct the first of such a series of quasi-Miura transformations is given by

$$\partial_1 \partial_2 X_1 = 0. \tag{3.4}$$

It results from the vanishing of the Schouten-Nijenhuis bracket

$$\left[\sum_{k\ge 0} \epsilon^k Q_k, \sum_{k\ge 0} \epsilon^k Q_k\right].$$

Indeed, by using the result of Theorem 3.2 that will be proved below, we know that identity (3.4) implies the existence of two local functionals

$$I_1 = \int h_{1,1}(u(x))dx, \quad J_1 = \int h_{2,1}(u(x))dx, \tag{3.5}$$

such that $X_1 = \partial_1 I_1 - \partial_2 J_1$.

The first quasi-Miura transformation that we are looking for is given by

$$u^i \mapsto \exp(\epsilon \partial_1 J_1)\, u^i. \tag{3.6}$$

After its action, the first Poisson structure (3.1) remains unchanged while the second Poisson structure (3.2) is transformed to

$$\exp(-\epsilon \mathrm{ad}_{\partial_1 J_1})\Bigl(Q_0 + \sum_{k\ge 1} \epsilon^k \partial_1 X_k\Bigr) = Q_0 + \sum_{k\ge 2} \epsilon^k \partial_1 \tilde{X}_k. \tag{3.7}$$

Here the vector fields $\tilde{X}_k$ have the expressions

$$\tilde{X}_k = \sum_{l=0}^{k} (-1)^l \frac{(\mathrm{ad}_{\partial_1 J_1})^l}{l!} X_{k-l}, \quad k \ge 2. \tag{3.8}$$

Let us note that (3.6) is in fact a usual Miura-type transformation, so the components of the new vector fields $\tilde{X}_k$ are homogeneous differential polynomials of degree k.

Now we proceed to construct the second quasi-Miura transformation in order to remove the first perturbation term $\epsilon^2 \partial_1 \tilde{X}_2$ of the Poisson structure (3.7). We need to use the property of the vector field $\tilde{X}_2$ that is the analogue of (3.4),

$$\partial_1 \partial_2 \tilde{X}_2 = 0. \tag{3.9}$$

Due to the result of Theorem 3.9 that will be given below, we can find two local functionals I_2 and J_2 such that $\tilde{X}_2 = \partial_1 I_2 - \partial_2 J_2$. Then our second quasi-Miura transformation is given by

$$u^i \mapsto \exp(\epsilon^2 \partial_1 J_2)\, u^i, \tag{3.10}$$

which leaves unchanged the form of the first Poisson structure while transforming the second one (3.7) to the form

$$\exp(-\epsilon^2 \mathrm{ad}_{\partial_1 J_2})\Bigl(Q_0 + \sum_{k\ge 2} \epsilon^k \partial_1 \tilde{X}_k\Bigr) = Q_0 + \sum_{k\ge 3} \epsilon^k \partial_1 \bar{X}_k. \tag{3.11}$$

Here the vector fields $\bar{X}_k$ have the expressions

$$\bar{X}_k = \sum_{l=0}^{[k/2]} (-1)^l \frac{(\mathrm{ad}_{\partial_1 J_2})^l}{l!} \tilde{X}_{k-2l}, \quad \tilde{X}_1 = 0, \;\; k \ge 3. \tag{3.12}$$

From (3.108)–(3.110) we see that (3.10) is no longer a usual Miura-type transformation, since in general the components of the vector field $\partial_1 J_2$ are not differential polynomials; instead, they belong to the ring $\mathcal{A}$ of functions that can be represented as a finite sum of rational functions of the form

$$\frac{P^i_{j_1,\dots,j_m}(u; u_x, \dots)}{u_x^{j_1}\cdots u_x^{j_m}}, \quad m \geq 0 \tag{3.13}$$

(no denominator for $m = 0$). Here $P^i_{j_1,\dots,j_m}$ are homogeneous differential polynomials.

Define a gradation on the ring $\mathcal{A}$ by

$$\deg u^{i,m} = m, \quad i = 1, \dots, n, \quad m > 0. \tag{3.14}$$

DEFINITION 3.1 We call elements of $\mathcal{A}$ *almost differential polynomials*.

Below we will also encounter functions that belong to the ring

$$\tilde{\mathcal{A}} = \mathcal{A}[\log u_x^1, \dots, \log u_x^n].$$

It is also a graded ring with the definition of degree (3.14) and

$$\deg(\log u_x^i) = 0, \quad i = 1, \dots, n. \tag{3.15}$$

The density of the above functionals I_2, J_2 can be chosen as homogeneous almost differential polynomials of degree 1; thus the resulting vector fields $\bar{X}_k$ have components that are homogeneous almost differential polynomials of degree k.

By continuing the above procedure, with the help of the result of Theorem 3.3 to be given below, we can construct in a successive way the series of quasi-Miura transformations that reduce the bi-Hamiltonian structure (3.1)–(3.2) to the one given by its leading terms, and the final reducing transformation is the composition of this series of quasi-Miura transformations. We thus prove the quasi-triviality theorem with the help of Theorems 3.2, 3.3, and 3.9. We now start to formulate and prove these theorems.

THEOREM 3.2 *Assume that a vector field X has components of the form*

$$X^i = \sum_j X^i_j(u)\, u^j_x, \quad i = 1, \dots, n, \tag{3.16}$$

and satisfies

$$\partial_1\partial_2 X = 0. \tag{3.17}$$

Then there exist two local functionals I and J of the form

$$I = \int G(u(x))dx, \quad J = \int \tilde{G}(u(x))dx, \tag{3.18}$$

such that X has the representation $X = \partial_1 I - \partial_2 J$.

PROOF: In this proof summations over repeated Greek indices are assumed. Let us redenote the components of the two bivectors that correspond to the bi-Hamiltonian structure (1.28) in the form

$$\begin{aligned}\varpi_1^{ij} &= g^{ij}\delta'(x-y) + \Gamma_\alpha^{ij} u_x^\alpha \delta(x-y),\\ \varpi_2^{ij} &= \tilde g^{ij}\delta'(x-y) + \tilde\Gamma_\alpha^{ij} u_x^\alpha \delta(x-y).\end{aligned}$$

The Levi-Civita connections of the two metrics g^{ij} and $\tilde g^{ij}$ are denoted by ∇ and $\tilde\nabla$, respectively. Denote ∇_i and $\tilde\nabla_i$ the covariant derivatives of these two connections along $\partial/\partial u^i$. We also introduce the notation $\nabla^i = g^{i\alpha}\nabla_\alpha$ and $\tilde\nabla^i = \tilde g^{i\alpha}\tilde\nabla_\alpha$.

Condition (3.17) implies existence of a vector field Y with components of the form

$$Y^i = \sum_j Y_j^i(u) u_x^j, \quad i = 1, \dots, n, \tag{3.19}$$

such that $\partial_1 X = \partial_2 Y$. Denote by

$$Z^{ij} = \sum_{p\ge 0} Z_p^{ij}(u(x), u_x(x), \dots)\, \delta^{(p)}(x-y) := (\partial_1 X - \partial_2 Y)^{ij} \tag{3.20}$$

the components of the bivector $\partial_1 X - \partial_2 Y$, and denote by $Z^{ij}_{p,(k,m)}$ the derivatives $\partial Z_p^{ij}/\partial u^{k,m}$. Then we have

$$Z_2^{ij} = (X^{ij} - X^{ji}) - (Y^{ij} - Y^{ji}) = 0, \tag{3.21}$$

$$\begin{aligned} Z^{ij}_{0,(k,2)} = {} & (\nabla_k X^{ij} - \nabla^i X_k^j + \Gamma^j_{k\alpha}(X^{\alpha i} - X^{i\alpha}))\\ & - (\tilde\nabla_k Y^{ij} - \tilde\nabla^i Y_k^j + \tilde\Gamma^j_{k\alpha}(Y^{\alpha i} - Y^{i\alpha})) = 0 \end{aligned} \tag{3.22}$$

where $X^{ij} = g^{i\alpha}X_\alpha^j$ and $Y^{ij} = \tilde g^{i\alpha} Y_\alpha^j$. From the above two equations we obtain

$$\tilde\nabla^k Y^{ij} - \tilde\nabla^i Y^{kj} = \tilde g^{k\alpha}(\nabla_\alpha X^{ij} - \nabla^i X_\alpha^j + T^j_{\alpha\beta}(X^{i\beta} - X^{\beta i})). \tag{3.23}$$

Here the components of the $(1,2)$–tensor T are defined by $T^j_{\alpha\beta} = \tilde\Gamma^j_{\alpha\beta} - \Gamma^j_{\alpha\beta}$. Since the left-hand side of the above equation is antisymmetric with respect to k and i, we have

$$\begin{aligned} & \tilde g^{k\alpha}(\nabla_\alpha X^{ij} - \nabla^i X_\alpha^j + T^j_{\alpha\beta}(X^{i\beta} - X^{\beta i}))\\ & \qquad + \tilde g^{i\alpha}(\nabla_\alpha X^{kj} - \nabla^k X_\alpha^j + T^j_{\alpha\beta}(X^{k\beta} - X^{\beta k})) = 0. \end{aligned} \tag{3.24}$$

The trivial identity

$$\begin{aligned} & (\tilde\nabla^k Y^{ij} - \tilde\nabla^i Y^{kj}) + (\tilde\nabla^i Y^{jk} - \tilde\nabla^j Y^{ik}) + (\tilde\nabla^j Y^{ki} - \tilde\nabla^k Y^{ji})\\ & \qquad = \tilde\nabla^k(Y^{ij} - Y^{ji}) + \tilde\nabla^i(Y^{jk} - Y^{kj}) + \tilde\nabla^j(Y^{ki} - Y^{ik}) \end{aligned}$$

implies that

$$
\begin{aligned}
&\tilde\nabla^k(X^{ij}-X^{ji})+\tilde\nabla^i(X^{jk}-X^{kj})+\tilde\nabla^j(X^{ki}-X^{ik})\\
&\quad=\tilde g^{k\alpha}(\nabla_\alpha X^{ij}-\nabla^i X^j_\alpha+T^j_{\alpha\beta}(X^{i\beta}-X^{\beta i}))\\
&\qquad+\tilde g^{i\alpha}(\nabla_\alpha X^{jk}-\nabla^j X^k_\alpha+T^k_{\alpha\beta}(X^{j\beta}-X^{\beta j}))\\
&\qquad+\tilde g^{j\alpha}(\nabla_\alpha X^{ki}-\nabla^k X^i_\alpha+T^i_{\alpha\beta}(X^{k\beta}-X^{\beta k})).
\end{aligned}
\tag{3.25}
$$

By using the formula

$$\tilde\nabla_k A^{ij}=\nabla_k A^{ij}+T^i_{k\alpha}A^{\alpha j}+T^j_{k\alpha}A^{i\alpha}$$

we can simplify equations (3.25) to the form

$$
\begin{aligned}
&\tilde g^{k\alpha}(\nabla_\alpha X^{ji}-\nabla^i X^j_\alpha+T^i_{\alpha\beta}(X^{j\beta}-X^{\beta j}))\\
&\quad+\tilde g^{i\alpha}(\nabla_\alpha X^{kj}-\nabla^j X^k_\alpha+T^j_{\alpha\beta}(X^{k\beta}-X^{\beta k}))\\
&\quad+\tilde g^{j\alpha}(\nabla_\alpha X^{ik}-\nabla^k X^i_\alpha+T^k_{\alpha\beta}(X^{i\beta}-X^{\beta i}))=0.
\end{aligned}
\tag{3.26}
$$

Let us employ equations (3.24) and (3.26) to prove the existence of two local functionals I and J of the form (3.18) such that $X=\partial_1 I-\partial_2 J$. Equivalently, we need to find functions G and $\tilde G$ that satisfy the conditions

$$X^i_j=g_{j\alpha}\nabla^i\nabla^\alpha G-\tilde g_{j\alpha}\tilde\nabla^i\tilde\nabla^\alpha\tilde G. \tag{3.27}$$

To this end, we first define two symmetric (2, 0)–tensors A and $\tilde A$ such that

$$X^i_j=g_{j\alpha}A^{\alpha i}-\tilde g_{j\alpha}\tilde A^{\alpha i}. \tag{3.28}$$

In the canonical coordinates, the off-diagonal components of A and $\tilde A$ are uniquely determined by the above relations and have the explicit forms

$$A^{ij}=\frac{g^iX^j_i-g^jX^i_j}{u^i-u^j},\quad \tilde A^{ij}=\frac{u^jg^iX^j_i-u^ig^jX^i_j}{u^i-u^j},\quad \text{for } i\neq j. \tag{3.29}$$

Here we use the fact that in the canonical coordinates the two metrics have components of the form $g^{ij}=\delta_{ij}f^i$ and $\tilde g^{ij}=\delta_{ij}g^i=\delta_{ij}u^if^i$. For an arbitrary choice of the diagonal components A^{ii}, the above relation uniquely determines the diagonal components $\tilde A^{ii}$ by

$$\tilde A^{ii}=u^i\big(A^{ii}-f^iX^i_i\big). \tag{3.30}$$

We will specify the choice of A^{ii}, $i=1,\dots,n$, in a moment.

Let us now express equations (3.24) and (3.26) in terms of the components of the tensors A and $\tilde A$. By substituting the expression (3.28) of X^i_j into (3.24) and (3.26) and by using the fact that $g_{\alpha\beta}$, $\tilde g_{\alpha\beta}$, and $T^i_{\alpha\beta}$ are all diagonal with respect to α and β in the canonical coordinates, we arrive at

$$(u^k-u^i)(\nabla^kA^{ij}-\nabla^iA^{kj})+\left(\frac{1}{u^k}-\frac{1}{u^i}\right)(\tilde\nabla^k\tilde A^{ij}-\tilde\nabla^i\tilde A^{kj})=0, \tag{3.31}$$

$$
\begin{aligned}
&u^k(\nabla^k A^{ij} - \nabla^i A^{kj}) - \left(\frac{1}{u^j}\tilde{\nabla}^k \tilde{A}^{ij} - \frac{1}{u^i}\tilde{\nabla}^i \tilde{A}^{kj}\right) \\
&\quad + u^i(\nabla^i A^{jk} - \nabla^j A^{ik}) - \left(\frac{1}{u^k}\tilde{\nabla}^i \tilde{A}^{jk} - \frac{1}{u^j}\tilde{\nabla}^j \tilde{A}^{ik}\right) \\
&\quad + u^j(\nabla^j A^{ki} - \nabla^k A^{ji}) - \left(\frac{1}{u^i}\tilde{\nabla}^j \tilde{A}^{ki} - \frac{1}{u^k}\tilde{\nabla}^k \tilde{A}^{ji}\right) = 0.
\end{aligned}
\tag{3.32}
$$

Rewrite (3.32) in the form

$$
\begin{aligned}
&u^k(\nabla^k A^{ij} - \nabla^i A^{kj}) + \frac{1}{u^k}(\tilde{\nabla}^k \tilde{A}^{ij} - \tilde{\nabla}^i \tilde{A}^{kj}) \\
&\qquad + \text{terms obtained by cyclic permutations of } (i, j, k) = 0.
\end{aligned}
\tag{3.33}
$$

By using (3.31), we can replace u^k by u^i in the first two terms of (3.33). Then after the cancellation of some terms we arrive at the simplification of (3.32),

$$
(u^i - u^j)(\nabla^j A^{ki} - \nabla^k A^{ij}) + \left(\frac{1}{u^i} - \frac{1}{u^j}\right)(\tilde{\nabla}^j \tilde{A}^{ki} - \tilde{\nabla}^k \tilde{A}^{ij}) = 0. \tag{3.34}
$$

Changing the indices $(i, j, k) \mapsto (j, k, i)$, we obtain

$$
(u^j - u^k)(\nabla^k A^{ij} - \nabla^i A^{kj}) + \left(\frac{1}{u^j} - \frac{1}{u^k}\right)(\tilde{\nabla}^k \tilde{A}^{ij} - \tilde{\nabla}^i \tilde{A}^{kj}) = 0. \tag{3.35}
$$

From equations (3.31) and (3.35) it readily follows that

$$
\nabla^k A^{ij} = \nabla^i A^{kj}, \quad \tilde{\nabla}^k \tilde{A}^{ij} = \tilde{\nabla}^i \tilde{A}^{kj}, \quad \text{for } i \neq j,\ k \neq j. \tag{3.36}
$$

Now let us proceed to choose the diagonal components A^{ii} in such a way to ensure that the components of the tensor $\nabla^k A^{ij}$ are totally symmetric in i, j, and k. This amounts to requiring that A^{ii} satisfy

$$
\nabla^k A^{ii} = \nabla^i A^{ki}, \quad i, k = 1, \ldots, n, \quad k \neq i. \tag{3.37}
$$

The existence of solutions A^{ii} is guaranteed by the compatibility of the above systems due to the equalities

$$
\nabla^j(\nabla^i A^{ki}) = \nabla^k(\nabla^i A^{ji}) \quad \text{for distinct } i, j, k. \tag{3.38}
$$

Fix a solution A^{ii}, $i = 1, \ldots, n$, of system (3.37). From the validity of equations (3.31) and (3.36), we know that tensor $\tilde{A}$ with components $\tilde{A}^{ij}$ determined by (3.29) and (3.30) also has the property of symmetry of $\tilde{\nabla}^k \tilde{A}^{ij}$ in i, j, and k. Thus we can find functions $G(u)$ and $\tilde{G}(u)$ such that

$$
A^{ij} = \nabla^i \nabla^j G, \quad \tilde{A}^{ij} = \tilde{\nabla}^i \tilde{\nabla}^j \tilde{G}.
$$

The theorem is proved. □

The above theorem implies that the linear-in-ϵ terms of the bi-Hamiltonian structure (1.27) can be eliminated by a Miura-type transformation.

THEOREM 3.3 *Let $X \in \Lambda^1_{\mathrm{loc}}$ be a local vector field with components*

$$X^i(u, u_x, \dots, u^{(N)}), \quad i = 1, \dots, n, \quad N \ge 1, \tag{3.39}$$

where X^i are homogeneous almost differential polynomials of degree $d \ge 3$. If X satisfies condition (3.17)*, then there exist local functionals*

$$I = \int f(u, u_x, \dots, u^{([N/2])})dx, \quad J = \int g(u, u_x, \dots, u^{([N/2])})dx, \tag{3.40}$$

with densities that are homogeneous almost differential polynomials of degree $d-1$ such that $X = \partial_1 I - \partial_2 J$.

Let us first sketch an outline of the proof. We will prove the theorem by induction on the highest order of the x-derivatives of u^k on which the components X^i of the vector field X depend. Due to the triviality of Poisson cohomology of the Poisson structure given by the leading terms of (3.2) and the result of Lemma 3.7 that will be given below, we are led to the following problem:

Let X, Y be two local vector fields with components

$$X^i(u, \dots, u^{(N)}), \quad Y^i(u, \dots, u^{(N)}), \quad i = 1, \dots, n, \quad N \ge 1,$$

where X^i and Y^i are homogeneous almost differential polynomials of degree $d \ge 3$. Assuming that X and Y satisfy the relation $\partial_1 X = \partial_2 Y$, prove existence of two local functionals I and J such that the components of the local vector field $X - (\partial_1 I - \partial_2 J)$ depend only on $u, \dots, u^{(N-1)}$.

We are to solve this problem in several steps following Lemmas 3.4, 3.5, 3.6, and 3.8 below. Lemma 3.4 separates the terms containing $u^{(N)}$ (which will be called the leading terms) from X^i and Y^i. Lemma 3.5 proves some combinatorial identities. By using these identities, Lemma 3.6 reduces the leading terms of X^i and Y^i to the simplest form. The last lemma, Lemma 3.8, gives the explicit conditions for the leading terms of X^i and Y^i to ensure the existence of the needed local functionals I and J. Finally, in the proof of Theorem 3.3, we fill the gap between Lemma 3.6 and Lemma 3.8 to finish the proof of the theorem.

In what follows, for a function $A = A(u, u_x, \dots)$, we will use the subscript (k, m) to indicate the derivative of A with respect to $u^{k,m}$, i.e., $A_{(k,m)} = \partial A / \partial u^{k,m}$.

LEMMA 3.4 *For any two vector fields X and Y with components of the form*

$$X^i = X^i(u, u_x, \dots, u^{(N)}), \quad Y^i = Y^i(u, u_x, \dots, u^{(N)}), \quad N \ge 1, \tag{3.41}$$

the conditions that

$$\begin{aligned} &Z^{ij}_{0,(k,2N+1)} = 0, \quad Z^{ij}_{0,(k,2N)} = 0, && \textit{when } N \ge 2, \\ &Z^{ij}_{0,(k,3)} = 0, \quad Z^{ij}_{0,(k,2)} = 0, \quad Z^{ij}_2 = 0, && \textit{when } N = 1, \end{aligned}$$

where the coefficients Z_p^{ij} of $Z^{ij} = (\partial_1 X - \partial_2 Y)^{ij}$ are defined as in (3.20)*, imply that the components of the vector fields X and Y must take the form*

$$
\begin{aligned}
X^i &= \sum_{j=1}^{n} \left(u^j G_j^i(u, \dots, u^{(N-1)}) + F_j^i(u, \dots, u^{(N-1)})\right) u^{j,N} \\
&\quad + Q^i(u, \dots, u^{(N-1)}), \\
Y^i &= \sum_{j=1}^{n} G_j^i(u, \dots, u^{(N-1)})\, u^{j,N} + R^i(u, \dots, u^{(N-1)}).
\end{aligned} \tag{3.42}
$$

Moreover, when $N \geq 2$ the functions F_j^i and G_j^i satisfy the following equations:

$$
F^j_{i,(k,N-1)} - F^j_{k,(i,N-1)} + (u^i - u^k) G^j_{k,(i,N-1)} = 0. \tag{3.43}
$$

PROOF: By definition, we have

$$
Z^{ij}_{0,(k,2N+1)} = (-1)^{N+1}(f^i X^j_{(i,N)(k,N)} - g^i Y^j_{(i,N)(k,N)}).
$$

So the vanishing of $Z^{ij}_{0,(k,2N+1)}$ implies that the functions X^i and Y^i can be represented as

$$
\begin{aligned}
X^i &= \sum_{j=1}^{n} (u^j \tilde{G}_j^i(u, \dots, u^{(N-1)}; u^{j,N}) \\
&\qquad + \tilde{F}_j^i(u, \dots, u^{(N-1)}) u^{j,N}) + \tilde{Q}^i(u, \dots, u^{(N-1)}), \\
Y^i &= \sum_{j=1}^{n} \tilde{G}_j^i(u, \dots, u^{(N-1)}; u^{j,N}).
\end{aligned}
$$

When $N \geq 2$ the equations (3.42) of X^i and Y^i follow from the vanishing of

$$
Z^{ij}_{0,(i,2N)} = (-1)^{N+1}\left(N + \frac{1}{2}\right) f^i u^i_x \tilde{G}^j_{i,(i,N)(i,N)}.
$$

In the case of $N = 1$, formulae (3.42) follow from the vanishing of $Z^{ij}_{0,(i,2N)}$ and of Z^{ij}_2. Finally, for $N \geq 2$, equation (3.43) is derived from the vanishing of

$$
Z^{ij}_{0,(k,2N)} = (-1)^{N+1} f^i (F^j_{i,(k,N-1)} - F^j_{k,(i,N-1)} + (u^i - u^k) G^j_{k,(i,N-1)}).
$$

The lemma is proved. □

LEMMA 3.5

(i) *Assume that the vector fields X and Y have components of the form* (3.42) *and $N \geq 2$. Then for any $m = 1, 2, \dots, [\frac{N}{2}]$, the following identity holds true:*

$$
\sum_{l \geq 0} (-1)^{m-l} \binom{N-l}{m-l} Z^{ij}_{l,(k,2N+1-m-l)} = (-1)^{N+1} f^i F^j_{i,(k,N-m)}. \tag{3.44}
$$

Here and below we denote Z^{ij} as in (3.20)*.*

(ii) *Assume that the vector fields X and Y have components of the following form*:

$$\begin{aligned} X^i &= \sum_{j=1}^n \bigl(u^j G^i_j(u,\dots,u^{(N-m-1)}; u^{j,N-m}) + F^i_j(u,\dots,u^{(N-m-1)})\bigr)u^{j,N} \\ &\quad + Q^i(u,\dots,u^{(N-1)}), \\ Y^i &= \sum_{j=1}^n G^i_j(u,\dots,u^{(N-m-1)}; u^{j,N-m})\, u^{j,N} + R^i(u,\dots,u^{(N-1)}) \end{aligned} \tag{3.45}$$

with $N \geq 3$ and $1 \leq m \leq [\frac{N-1}{2}]$. Then we have

$$\begin{aligned} &\sum_{l\geq 0}(-1)^{m-l}\binom{N-l}{m-l} Z^{ij}_{l,(k,2N-m-l)} \\ &\quad = (-1)^{N+1}\Bigl[f^i(F^j_{i,(k,N-m-1)} - F^j_{k,(i,N-m-1)}) + f^i(u^i-u^k)G^j_{k,(i,N-m-1)} \\ &\qquad\quad + \left(N-m+\frac{1}{2}\right) f^i u^i_x G^j_{k,(i,N-m)}\delta^{ik} \\ &\qquad\quad + (A^{ik} - u^k B^{ik}) G^j_{k,(k,N-m)}\Bigr]. \end{aligned} \tag{3.46}$$

Here A^{ik} and B^{ik} are defined by (1.34).

PROOF: By using formula (2.15) the components Z^{ij} of the bivector $\partial_1 X - \partial_2 Y$ have the explicit expressions

$$\begin{aligned} Z^{ij} = \sum_{s\geq 0}(-1)^{s+1}\biggl(& f^i \partial_x^{s+1}(X^j_{(i,s)}\delta) + \frac{\partial_x f^i}{2}\partial_x^s(X^j_{(i,s)}\delta) + \sum_{k=1}^n A^{ik}\partial_x^s(X^j_{(k,s)}\delta) \\ & - g^i\partial_x^{s+1}(Y^j_{(i,s)}\delta) - \frac{\partial_x g^i}{2}\partial_x^s(Y^j_{(i,s)}\delta) - \sum_{k=1}^n B^{ik}\partial_x^s(Y^j_{(k,s)}\delta)\biggr) + \cdots . \end{aligned}$$

Here $\delta = \delta(x-y)$. It is easy to see that when $m \leq [\frac{N}{2}]$, the first two terms in formula (2.15) don't affect identity (3.44). So we denote their contributions by an ellipsis in the above expressions of Z^{ij} and will omit them in the calculations below. Then Z_p^{ij} reads

$$Z_p^{ij} = \sum_{s\ge 0}(-1)^{s+1}\Bigg(f^i\binom{s+1}{p}\partial_x^{s+1-p}X^j_{(i,s)} + \frac{\partial_x f^i}{2}\binom{s}{p}\partial_x^{s-p}X^j_{(i,s)} + \sum_{k=1}^n A^{ik}\binom{s}{p}\partial_x^{s-p}X^j_{(k,s)} - g^i\binom{s+1}{p}\partial_x^{s+1-p}Y^j_{(i,s)} - \frac{\partial_x g^i}{2}\binom{s}{p}\partial_x^{s-p}Y^j_{(i,s)} - \sum_{k=1}^n B^{ik}\binom{s}{p}\partial_x^{s-p}Y^j_{(k,s)}\Bigg).$$

Denote by LHS the left-hand side of the identity (3.44). We obtain

$$\begin{aligned}
&\text{LHS}\\
&= \sum_{p,s\ge 0}(-1)^{m-p+s+1}\binom{N-p}{m-p}\\
&\quad\cdot\Bigg(f^i\binom{s+1}{p}\sum_{t\ge 0}\binom{s+1-p}{t}\partial_x^t X^j_{(i,s)(k,2N-m-s+t)}\\
&\quad + \frac{\partial_x f^i}{2}\binom{s}{p}\sum_{t\ge 0}\binom{s-p}{t}\partial_x^t X^j_{(i,s)(k,2N+1-m-s+t)}\\
&\quad + \sum_{l=1}^n A^{il}\binom{s}{p}\sum_{t\ge 0}\binom{s-p}{t}\partial_x^t X^j_{(l,s)(k,2N+1-m-s+t)}\\
&\quad - g^i\binom{s+1}{p}\sum_{t\ge 0}\binom{s+1-p}{t}\partial_x^t Y^j_{(i,s)(k,2N-m-s+t)}\\
&\quad - \frac{\partial_x g^i}{2}\binom{s}{p}\sum_{t\ge 0}\binom{s-p}{t}\partial_x^t Y^j_{(i,s)(k,2N+1-m-s+t)}\\
&\quad - \sum_{l=1}^n B^{il}\binom{s}{p}\sum_{t\ge 0}\binom{s-p}{t}\partial_x^t Y^j_{(l,s)(k,2N+1-m-s+t)}\Bigg).
\end{aligned}$$

Here we used the commutation relations

$$\frac{\partial}{\partial u^{i,q}}\partial_x^m = \sum_{t\ge 0}\binom{m}{t}\partial_x^t\frac{\partial}{\partial u^{i,q-m+t}}.$$

By using the identity

$$\sum_{p\ge 0}(-1)^p\binom{N-p}{m-p}\binom{s}{p}\binom{s-p}{t} = \binom{s}{t}\binom{N-s+t}{m}$$

and by changing the order of summation, we can rewrite LHS as follows:

$$
\begin{aligned}
&\mathrm{LHS}\\
&= \sum_{s,t\ge 0}(-1)^{m+s+1}\Bigg(\binom{s+1}{t}\binom{N-s+t-1}{m} f^i\,\partial_x^t X^j_{(i,s)(k,2N-m-s+t)}\\
&\qquad + \binom{s}{t}\binom{N-s+t}{m}\frac{\partial_x f^i}{2}\partial_x^t X^j_{(i,s)(k,2N+1-m-s+t)}\\
&\qquad + \binom{s}{t}\binom{N-s+t}{m}\sum_{l=1}^{n} A^{il}\partial_x^t X^j_{(l,s)(k,2N+1-m-s+t)}\\
&\qquad - \binom{s+1}{t}\binom{N-s+t-1}{m} g^i\,\partial_x^t Y^j_{(i,s)(k,2N-m-s+t)}\\
&\qquad - \binom{s}{t}\binom{N-s+t}{m}\frac{\partial_x g^i}{2}\partial_x^t Y^j_{(i,s)(k,2N+1-m-s+t)}\\
&\qquad - \binom{s}{t}\binom{N-s+t}{m}\sum_{l=1}^{n} B^{il}\partial_x^t Y^j_{(l,s)(k,2N+1-m-s+t)}\Bigg).
\end{aligned}
\tag{3.47}
$$

Now we substitute the expression (3.42) of X and Y into the right-hand side of the above formula. By using the properties of binomial coefficients, it is easy to see that all terms in the above summation vanish except for the terms with $s = N$, $t = 0$, so the above formula can be simplified to

$$
\begin{aligned}
\mathrm{LHS} &= (-1)^{m+N+1}((-1)^m f^i X^j_{(i,N)(k,N-m)} - (-1)^m g^i Y^j_{(i,N)(k,N-m)})\\
&= (-1)^{N+1} f^i F^j_{i,(k,N-m)}\,.
\end{aligned}
$$

So part (i) of the lemma is proved.

Similarly to the derivation of (3.47), we can prove the following formula:

$$
\begin{aligned}
&\sum_{l\ge 0}(-1)^{m-l}\binom{N-l}{m-l} Z^{ij}_{l,(k,2N-m-l)}\\
&= \sum_{s,t\ge 0}(-1)^{m+s+1}\Bigg(\binom{s+1}{t}\binom{N-s+t-1}{m} f^i\,\partial_x^t X^j_{(i,s)(k,2N-1-m-s+t)}\\
&\quad + \binom{s}{t}\binom{N-s+t}{m}\frac{\partial_x f^i}{2}\partial_x^t X^j_{(i,s)(k,2N-m-s+t)}\\
&\quad + \binom{s}{t}\binom{N-s+t}{m}\sum_{l=1}^{n} A^{il}\partial_x^t X^j_{(l,s)(k,2N-m-s+t)}\\
&\quad - \binom{s+1}{t}\binom{N-s+t-1}{m} g^i\,\partial_x^t Y^j_{(i,s)(k,2N-1-m-s+t)}
\end{aligned}
\tag{3.48}
$$

$$- \binom{s}{t}\binom{N-s+t}{m}\frac{\partial_x g^i}{2}\partial_x^t Y^j_{(i,s)(k,2N-m-s+t)}$$
$$- \binom{s}{t}\binom{N-s+t}{m}\sum_{l=1}^n B^{il}\partial_x^t Y^j_{(l,s)(k,2N-m-s+t)}\Bigg).$$

The summands of the right-hand side vanish except for the terms with $(s,t) = (N,0)$, $(N-m-1,0)$, $(N-m,1)$, $(N-m,0)$. Then LHS of identity (3.46) reads

$$\begin{aligned} \text{LHS} = {} & (-1)^{N+1}(f^i X^j_{(i,N)(k,N-m-1)} - g^i Y^j_{(i,N)(k,N-m-1)}) \\ & + (-1)^N (f^i X^j_{(k,N)(i,N-m-1)} - g^i Y^j_{(k,N)(i,N-m-1)}) \\ & + (-1)^{N+1}(N-m+1)(f^i \partial_x X^j_{(k,N)(i,N-m)} - g^i \partial_x Y^j_{(k,N)(i,N-m)}) \\ & + (-1)^{N+1}\Bigg(\frac{\partial_x f^i}{2} X^j_{(k,N)(i,N-m)} - \frac{\partial_x g^i}{2} Y^j_{(k,N)(i,N-m)} \\ & \qquad + \sum_{l=1}^n \big(A^{il} X^j_{(k,N)(l,N-m)} - B^{il} Y^j_{(k,N)(l,N-m)}\big)\Bigg). \end{aligned} \tag{3.49}$$

Note that our X and Y have the properties

$$X^j_{(k,N)(i,N-m)} = u^k G^j_{k,(i,N-m)}\delta^{ki}, \quad Y^j_{(k,N)(i,N-m)} = G^j_{k,(i,N-m)}\delta^{ki}.$$

Then identity (3.46) follows from (3.49) immediately. Part (ii) of the lemma is proved. □

LEMMA 3.6 *Let X and Y be two local vector fields that have components of the form* (3.41) *and satisfy the relation*

$$\partial_1 X = \partial_2 Y. \tag{3.50}$$

Then the following statements hold true:

(i) *When $N = 2M+1$, the components of these vector fields have the expressions*

$$X^i = \sum_{j=1}^n X^i_j(u, u_x, \dots, u^{(M)})u^{j,2M+1} + Q_i(u, u_x, \dots, u^{(2M)}), \tag{3.51}$$

$$Y^i = \sum_{j=1}^n Y^i_j(u, u_x, \dots, u^{(M)})u^{j,2M+1} + R_i(u, u_x, \dots, u^{(2M)}). \tag{3.52}$$

(ii) *When $N = 2M$, there exist local functionals I_a, $a = 1,2,3$, such that the components of the vector fields X and Y have, after the modification,*

$$X \mapsto X - (\partial_1 I_1 - \partial_2 I_2), \quad Y \mapsto Y - (\partial_1 I_2 - \partial_2 I_3) \tag{3.53}$$

(if necessary), the expressions

$$X^i = \sum_{j\neq i} X^i_j(u, u_x, \dots, u^{(M-1)})u^{j,2M} + Q_i(u, u_x, \dots, u^{(2M-1)}), \tag{3.54}$$

$$Y^i = \sum_{j\neq i} Y^i_j(u, u_x, \dots, u^{(M-1)})u^{j,2M} + R_i(u, u_x, \dots, u^{(2M-1)}). \tag{3.55}$$

In the case when the components X^i of the vector field X are homogeneous almost differential polynomials of degree $d \geq 3$, we can choose the densities of the local functionals I_a such that they are homogeneous almost differential polynomials of degree $d-1$.

PROOF: For the case when $N = 1$, the result of the lemma follows from Lemma 3.4, so we assume that $N \geq 2$. It follows from Lemma 3.4 that the components of X and Y must take the form (3.42). The result of part (i) of Lemma 3.5 then shows that the F^i_j are independent of $u^{(N-m)}$ for $m = 1, \dots, [\frac{N}{2}]$, so the identities in (3.43) read

$$(u^i - u^k)G^j_{k,(i,N-1)} = 0;$$

thus the G^j_k are independent of $u^{i,N-1}$ when $i \neq k$. When $N \geq 3$ we use the identity (3.46) of Lemma 3.5 with $m = 1$ to obtain, by putting $i = k$,

$$G^j_{i,(i,N-1)} = 0.$$

So the G^j_k are in fact independent of $u^{(N-1)}$. Now identity (3.46) shows that

$$G^j_{k,(i,N-2)} = 0, \quad i \neq k.$$

By repeatedly using identity (3.46), we know that G^i_j is independent of $u^{(N-m)}$ for $m = 1, \dots, [\frac{N-1}{2}]$, and $G^j_{i,(k,M)} = 0$ for $N = 2M \geq 2$ and $i \neq k$.

For the case of $N = 2M + 1$, $M \geq 1$, the above argument shows that the components of the vector fields X and Y have the form (3.51)–(3.52) where X^i_j and Y^i_j are given by the expressions

$$\begin{aligned} X^i_j &= u^j G^i_j(u, u_x, \dots, u^{(M)}) + F^i_j(u, u_x, \dots, u^{(M)}), \\ Y^i_j &= G^i_j(u, u_x, \dots, u^{(M)}), \end{aligned} \tag{3.56}$$

and for the case of $N = 2M$, $M \geq 1$, we have

$$\begin{aligned} X^i &= \sum_{j=1}^n \big(u^j G^i_j(u, \dots, u^{(M-1)}, u^{j,M}) + F^i_j(u, \dots, u^{(M-1)})\big)u^{j,N} \\ &\quad + Q^i(u, \dots, u^{(N-1)}), \end{aligned} \tag{3.57}$$

$$Y^i = \sum_{j=1}^n G^i_j(u, \dots, u^{(M-1)}, u^{j,M})\, u^{j,N} + R^i(u, \dots, u^{(N-1)}). \tag{3.58}$$

From the vanishing of the coefficients of $\delta^{(2M+1)}(x-y)$ in the expression of $(\partial_1 X - \partial_2 Y)^{ij}$, it follows that

$$f^i(u)F_i^j + f^j(u)F_j^i = 0. \tag{3.59}$$

In particular, we have

$$F_i^i = 0, \quad i = 1, \dots, n. \tag{3.60}$$

Define the functionals

$$I_k = (-1)^M \int \sum_{i=1}^n \partial_{u^{i,M}}^{-2} \frac{(u^i)^{3-k}\, G_i^i}{(M+\frac{1}{2}) f^i u_x^i}\, dx, \quad k = 1, 2, 3. \tag{3.61}$$

Then the components of the vector fields $\tilde{X} = X - (\partial_1 I_1 - \partial_2 I_2)$ and $\tilde{Y} = Y - (\partial_1 I_2 - \partial_2 I_3)$ have the form of (3.57)–(3.58) with $G_i^i = 0$. Since the vector fields $\tilde{X}$ and $\tilde{Y}$ still satisfy the relation $\partial_1 \tilde{X} = \partial_2 \tilde{Y}$, we can assume without loss of generality that the components of X and Y have the form (3.57)–(3.58) with vanishing G_i^i. By using equations (3.59), we check that identity (3.46) is still valid for $i = k$ when $m = M$ and $N \geq 2$. This leads to the fact that the functions G_j^i for $i \neq j$ do not depend on $u^{j,M}$. Thus we have proved that the components of X and Y have the form (3.54)–(3.55) after the modification (3.53) if necessary.

When the components of the vector field X are homogeneous almost differential polynomials of degree $d \geq 3$, equations (3.60) and the expression (3.57) imply that the functions G_i^i are also homogeneous almost differential polynomials of degree $d - 2M$. So, when $M \geq 2$ we can choose the densities of the functionals I_a defined in (3.61) to be homogeneous almost differential polynomials of degree $d-1$. In the case when $M = 1$, since the functions $G_i^i = G_i^i(u; u_x^i)$ are homogeneous of degree $d - 2 \geq 1$ (recall our assumption $d \geq 3$), the function $G_i^i(u; u_x^i)/u_x^i$ is in fact a polynomial in u_x^i. Thus in this case we can still choose the densities of the functionals I_a defined in (3.61) to be homogeneous almost differential polynomials of degree $d - 1$. The lemma is proved. □

LEMMA 3.7 *Let the vector fields X and Y have components of the form* (3.41) *with $N \geq 2$ and satisfy relation* (3.50). *If the functions X^i, $i = 1, \dots, n$, do not depend on $u^{(N)}$, then we can modify the vector field Y by*

$$Y \mapsto Y - \partial_2 J \tag{3.62}$$

for a certain local functional J such that the components of this modified vector field Y depend at most on $u, \dots, u^{(N-1)}$, and relation (3.50) *still holds true.*

PROOF: We first assume that $N = 2M+1$. From the assumption of the lemma and the result of Lemma 3.6 we know that the components of the vector fields X and Y have the form (3.51)–(3.52) with $X_j^i = 0$. To prove the lemma, we need to find a local functional J with density $h(u, u_x, \dots, u^{(M)})$ that satisfies the conditions

$$\frac{\partial^2 h}{\partial u^{i,M}\partial u^{j,M}} = (-1)^M \frac{1}{g^i(u)} Y^i_j, \quad i, j = 1, \dots, n. \tag{3.63}$$

Denote by A_{ij} the right-hand side of the above formulae. Then from the vanishing of the coefficients of $\delta^{(2M+2)}(x-y)$ in the expression of the components of the bivector $\partial_1 X - \partial_2 Y$, it follows that the functions A_{ij} are symmetric with respect to the indices i and j. From equation (3.46) with $m = M$, we also know that

$$\frac{\partial Y^j_i}{\partial u^{k,M}} = \frac{\partial Y^j_k}{\partial u^{i,M}}. \tag{3.64}$$

So the functions $\partial A_{ij}/\partial u^{k,M}$ are symmetric with respect to the indices i, j, and k, which implies the existence of a function $h(u, u_x, \dots, u^{(M)})$ satisfying the requirement (3.63).

Next let us assume that $N = 2M$. As we did in the proof of Lemma 3.6, we can show that the components of X and Y have the form (3.57)–(3.58) with

$$F^i_j = -u^j G^i_j. \tag{3.65}$$

Since the functions F^i_j do not depend on $u^{(M)}$, we deduce that the functions X^i and Y^i must have the expressions (3.54) with $X^i_j = 0$. From (3.60) and the independence of X^i on $u^{(N)}$, it also follows that $G^i_i = 0$ for $i = 1, \dots, n$. Now by using the vanishing of the coefficients of $\delta^{(2M+1)}$ of the components of $\partial_1 X - \partial_2 Y$ and that of the left-hand side of (3.46) with $m = M$, we obtain

$$\hat{Y}_{ij} + \hat{Y}_{ji} = 0, \tag{3.66}$$

$$\hat{Y}_{jk,(i,M-1)} - \hat{Y}_{ik,(j,M-1)} - \hat{Y}_{ji,(k,M-1)} = 0. \tag{3.67}$$

Here $\hat{Y}_{ij} = (1/g^i) Y^i_j$. The above two equations ensure the existence of a 1-form $\alpha = \sum_{i=1}^n h_i(u, \dots, u^{(M-1)}) du^{i,M-1}$ such that

$$d\alpha = \frac{1}{2} \sum_{i,j} \hat{Y}_{ij}\, du^{i,M-1} \wedge du^{j,M-1}. \tag{3.68}$$

Now the functional J defined by

$$J = \int \sum_{i=1}^n h_i(u(x), \dots, u^{(M-1)}(x))\, u^{i,M}(x) dx \tag{3.69}$$

meets the requirement of the lemma, and we have finished the proof. □

LEMMA 3.8 *Let X be a local vector field with components*

$$X^i = X^i(u; u_x, \dots, u^{(N)}), \quad i = 1, \dots, n, \quad N \ge 4,$$

that are homogeneous almost differential polynomials of degree $d \ge 3$. Assume X also has the following properties:

(i) *When $N = 2M + 2$, the components of X have the form*

$$X^i = \sum_{j\neq i} X^i_j(u, \dots, u^{(M)}) u^{j,2M+2} + Q^i(u, \dots, u^{(2M+1)}) \tag{3.70}$$

and satisfy the conditions

$$(u^k - u^j)(\hat{X}_{ij,(k,M)} - \hat{X}_{ik,(j,M)}) + (u^k - u^i)\hat{X}_{jk,(i,M)} + (u^j - u^i)\hat{X}_{kj,(i,M)} = 0 \tag{3.71}$$

for

$$\hat{X}_{ij} = \frac{1}{f^i(u)} X^i_j(u, \dots, u^{(M)}). \tag{3.72}$$

(ii) *When $N = 2M + 1$, X has components of the form*

$$X^i = \sum_{j=1}^{n} X^i_j(u, \dots, u^{(M)}) u^{j,2M+1} + Q^i(u, \dots, u^{(2M)}) \tag{3.73}$$

and satisfies the conditions

$$X^i_{j,(k,M)} - X^i_{k,(j,M)} = 0, \tag{3.74}$$

$$(u^k - u^j)\hat{X}_{ij,(k,M)} + (u^i - u^k)\hat{X}_{jk,(i,M)} + (u^j - u^i)\hat{X}_{ki,(j,M)} = 0. \tag{3.75}$$

Then there exist two local functionals I_1 and I_2 with densities that are homogeneous almost differential polynomials of degree $d - 1$ such that the components of the vector field $X - (\partial_1 I_1 - \partial_2 I_2)$ depend at most on $u, u_x, \dots, u^{(N-1)}$.

PROOF: We first prove the lemma for the case when $d \geq 3$, $N \geq 5$. Assume $N = 2M + 2$ and the vector field X satisfies conditions (3.70)–(3.71). We want to find two local functionals I_1 and I_2 with densities of the form

$$h_a = \sum_{j=1}^{n} h_{a;j}(u, u_x, \dots, u^{(M)}) u^{j,M+1}, \quad a = 1, 2, \tag{3.76}$$

such that they meet the requirements of the lemma. For this we need to find the functions $h_{a;j}$, $a = 1, 2$, $j = 1, \dots, n$, satisfying the following equations:

$$(-1)^{M+1} X^i_j = f^i \left(\frac{\partial h_{1;i}}{\partial u^{j,M}} - \frac{\partial h_{1;j}}{\partial u^{i,M}} \right) - g^i \left(\frac{\partial h_{2;i}}{\partial u^{j,M}} - \frac{\partial h_{2;j}}{\partial u^{i,M}} \right). \tag{3.77}$$

Denote

$$P_{ij} = (-1)^M \frac{g^j X^i_j + g^i X^j_i}{f^i g^j - f^j g^i}, \quad Q_{ij} = (-1)^M \frac{f^j X^i_j + f^i X^j_i}{f^i g^j - f^j g^i}. \tag{3.78}$$

Then it follows from (3.71) that the 2-forms

$$\varpi_1 = \frac{1}{2} \sum_{i,j} P_{ij} du^{i,M} \wedge du^{j,M}, \quad \varpi_2 = \frac{1}{2} \sum_{i,j} Q_{ij} du^{i,M} \wedge du^{j,M}, \tag{3.79}$$

are closed. So there exist 1-forms

$$\alpha_a = \sum_j h_{a;j}(u, u_x, \dots, u^{(M)})\, du^{j,M}, \quad a = 1, 2, \tag{3.80}$$

such that $d\alpha_a = \varpi_a$ and the functions $h_{a;j}$ are homogeneous almost differential polynomials of degree $d - M - 2$. Now it's easy to see that the functions $h_{a;j}$ satisfy (3.77). So we have proved the lemma for $N = 2M + 2 > 4$.

Next we assume that $N = 2M+1 > 4$ and the vector field X satisfies conditions (3.73)–(3.75). Let the two local functionals I_1 and I_2 that we are looking for have densities of the form

$$h_a(u, u_x, \dots, u^{(M)}), \quad a = 1, 2. \tag{3.81}$$

Since the i^{th} component of the vector field $\partial_1 I_1 - \partial_2 I_2$ depends at most on $u, \dots, u^{(2M+1)}$, and, moreover, it depends linearly on $u^{k,2M+1}$, we only need to find functions h_1 and h_2 such that

$$X^i_j = \frac{\partial(\partial_1 I_1 - \partial_2 I_2)^i}{\partial u^{j,2M+1}} = (-1)^M (f^i h_{1,(i,M)(j,M)} - g^i h_{2,(i,M)(j,M)}). \tag{3.82}$$

To this end, let us define P_{ii} and Q_{ii}, $i = 1, \dots, n$, by solving the following systems:

$$\begin{aligned} \frac{\partial P_{ii}}{\partial u^{j,M}} &= \frac{(-1)^{M+1}}{u^i - u^j}\left(u^j \frac{X^i_{j,(i,M)}}{f^i} - u^i \frac{X^j_{i,(i,M)}}{f^j}\right), \quad j \neq i, \\ \frac{\partial Q_{ii}}{\partial u^{j,M}} &= \frac{(-1)^{M+1}}{u^i - u^j}\left(\frac{X^i_{j,(i,M)}}{f^i} - \frac{X^j_{i,(i,M)}}{f^j}\right), \qquad j \neq i. \end{aligned} \tag{3.83}$$

Conditions (3.74) and (3.75) imply the compatibility of the above systems, i.e.,

$$\frac{\partial}{\partial u^{k,M}}\left(\frac{\partial P_{ii}}{\partial u^{j,M}}\right) = \frac{\partial}{\partial u^{j,M}}\left(\frac{\partial P_{ii}}{\partial u^{k,M}}\right), \quad j, k \neq i. \tag{3.84}$$

So we have a set of functions

$$P_{ii} = P_{ii}(u, u_x, \dots, u^{(M)}), \quad Q_{ii} = Q_{ii}(u, u_x, \dots, u^{(M)}),$$

satisfying conditions (3.83). The ambiguity in the definition of these functions lies in the following shifts:

$$\begin{aligned} P_{ii} &\mapsto P_{ii} + W_{ii}(u, u_x, \dots, u^{(M-1)}, u^{i,M}), \\ Q_{ii} &\mapsto Q_{ii} + R_{ii}(u, u_x, \dots, u^{(M-1)}, u^{i,M}). \end{aligned} \tag{3.85}$$

Here W_{ii} and R_{ii} are arbitrary functions to be specified later.

We also define functions P_{ij} and Q_{ij}, $i \neq j$, by the following formulae:

$$P_{ij} = \frac{(-1)^{M+1}}{u^i - u^j}\left(u^j \frac{X^i_j}{f^i} - u^i \frac{X^j_i}{f^j}\right), \quad Q_{ij} = \frac{(-1)^{M+1}}{u^i - u^j}\left(\frac{X^i_j}{f^i} - \frac{X^j_i}{f^j}\right). \tag{3.86}$$

By using conditions (3.74)–(3.75), we easily verify that

$$\frac{\partial P_{ij}}{\partial u^{k,M}}, \quad \frac{\partial Q_{ij}}{\partial u^{k,M}},$$

are symmetric with respect to their indices i, j, and k. So there exist functions $h_a(u, u_x, \dots, u^{(M)})$, $a = 1, 2$, such that

$$\frac{\partial^2 h_1}{\partial u^{i,M} \partial u^{j,M}} = P_{ij}, \quad \frac{\partial^2 h_2}{\partial u^{i,M} \partial u^{j,M}} = Q_{ij}. \tag{3.87}$$

Now it's easy to verify that, when $i \neq j$, functions h_1 and h_2 satisfy the conditions given in (3.82). When $i = j$ we have

$$\begin{aligned} X_i^i - (-1)^M (f^i\, h_{1,(i,M)(i,M)} - g^i\, h_{2,(i,M)(i,M)}) \\ = X_i^i - (-1)^M (f^i\, P_{ii} - g^i\, Q_{ii}). \end{aligned} \tag{3.88}$$

It follows from the definition of P_{ii} and Q_{ii} and the conditions given in (3.74) that the right-hand side of the above formula does not depend on $u^{k,M}$ for any $k \neq i$, so we can make it 0 by adjusting functions P_{ii} and Q_{ii} as in (3.85). By the above construction, functions h_1 and h_2 can be chosen to be homogeneous almost differential polynomials of degree $d - 1$. So the lemma is proved for the case mentioned above.

Now let us consider the case $d \geq 3$, $N = 4$. Proceeding in the same way as for the case of $N = 2M + 2$, $M \geq 2$, we can find the 1-forms (3.80) such that the 2-forms ϖ_1 and ϖ_2 that are defined as in (3.79) can be represented by $\varpi_a = d\alpha_a$, $a = 1, 2$. The pecularity of this case $M = 1$ lies in the fact that the functions $h_{a,j}$ that we constructed above are in general no longer rational functions of the jet coordinates $u^{i,k}$, $k \geq 1$; they can be chosen to have the form

$$h_{a,j} = \sum_{k=1}^{n} W_{a,j;k}(u; u_x) \log u_x^k + U_{a,j}(u; u_x), \quad a = 1, 2, \quad j = 1, \dots, n. \tag{3.89}$$

Here $W_{a,j;k}, U_{a,j} \in \mathcal{A}$ and are homogeneous of degree $d - 1$. Since

$$\frac{\partial h_{a,j}}{\partial u_x^i} - \frac{\partial h_{a,i}}{\partial u_x^j} \in \mathcal{A}, \tag{3.90}$$

we must have

$$\frac{\partial W_{a,j;k}}{\partial u_x^i} = \frac{\partial W_{a,i;k}}{\partial u_x^j}, \quad i, j = 1, \dots, n. \tag{3.91}$$

This implies the existence of functions $A_{a,k}(u; u_x) \in \tilde{\mathcal{A}}$ of degree $d - 2$ such that

$$W_{a,j;k} = \frac{\partial A_{a,k}(u; u_x)}{\partial u_x^j}, \quad a = 1, 2, \quad j, k = 1, \dots, n. \tag{3.92}$$

Since $W_{a,j;k}$ are almost differential polynomials, the functions $A_{a,k}$ can also be chosen as homogeneous almost differential polynomials of degree $d - 2 \geq 1$ up to the addition of terms of the form

$$\sum_{l=1}^{n} B_{a,k,l}(u) \log u_x^l. \tag{3.93}$$

However, such functions have degree 0, so they are not allowed to appear in the expressions of $A_{a,k}$. Now the needed functionals I and J that satisfy the requirements of the lemma can be chosen to have densities $\tilde{h}_1$ and $\tilde{h}_2$ of the form

$$\tilde{h}_a = \sum_{k=1}^{n} \left(U_{a,k}(u, u_x) - A_{a,k}(u, u_x) \frac{1}{u_x^k} \right) u_{xx}^k, \quad a = 1, 2. \tag{3.94}$$

The lemma is proved. □

PROOF OF THEOREM 3.3: Assume that the components X^i of the vector field X have the form

$$X^i = X^i(u, \dots, u^{(N)}), \quad N \geq 4, \ \ i = 1, \dots, n. \tag{3.95}$$

The relation (3.17) implies the existence of a local vector field Y such that $\partial_1 X = \partial_2 Y$. By using Lemma 3.7 we can choose the vector field Y such that its components depend at most on the coordinates $u, \dots, u^{(N)}$. Then it follows from the results of Lemma 3.6 that the components of X and Y have the expressions (3.51)–(3.52) when N is odd and those of (3.54)–(3.55) when N is even (after a modification of (3.53) that does not affect our result).

We now proceed to employ the result of Lemma 3.8 in order to find two local functionals I and J with densities that are homogeneous almost differential polynomials of degree $d-1$ such that $X - (\partial_1 I - \partial_2 J)$ depends at most on $u, \dots, u^{(N-1)}$. To this end we need to verify that the X^i satisfy equations (3.71) when N is even and (3.74)–(3.75) when N is odd.

Let N be an even integer $2M + 2$. Then by using the vanishing of the left-hand side of (3.46) with $m = M$, we obtain

$$\hat{X}_{jk,(i,M)} - \hat{X}_{ik,(j,M)} - \hat{X}_{ji,(k,M)} = u^i u^j (\hat{Y}_{jk,(i,M)} - \hat{Y}_{ik,(j,M)} - \hat{Y}_{ji,(k,M)}). \tag{3.96}$$

Here the $\hat{Y}_{ij}$ are defined as in the (3.67) and the $\hat{X}_{ij}$ are defined by (3.72). We also have the equation

$$\hat{X}_i^j + \hat{X}_j^i = u^i u^j (\hat{Y}_{ji} + \hat{Y}_{ij}) \tag{3.97}$$

due to the vanishing of the coefficients of $\delta^{(2M+1)}$ in the components of $\partial_1 X - \partial_2 Y$. Denote by $L_{i,j,k}$ and $R_{i,j,k}$ the expressions of the left-hand side and right-hand side of (3.96), respectively, multiplied by u^k. Then we have

$$\begin{aligned} R_{i,j,k} - R_{j,k,i} - R_{k,i,j} - R_{i,k,j} &= u^i u^j u^k (\hat{Y}_{jk,(i,M)} + \hat{Y}_{kj,(i,M)}) \\ &= u^i (\hat{X}_{jk,(i,M)} + \hat{X}_{kj,(i,M)}). \end{aligned}$$

Here in the last equality we used the equations in (3.97). By equating the last expression with $L_{i,j,k} - L_{j,k,i} - L_{k,i,j} - L_{i,k,j}$, we arrive at the proof that the vector field X satisfies the conditions given in (3.71).

Now let us assume that $N = 2M + 1$. From (3.46) with $m = M$, we know that

$$f^i (X^j_{i,(k,M)} - X^j_{k,(i,M)}) = g^i (Y^j_{i,(k,M)} - Y^j_{k,(i,M)}). \tag{3.98}$$

This equation together with the one that is obtained from it by exchanging the indices i and k implies that

$$X^j_{i,(k,M)} = X^j_{k,(i,M)}, \quad Y^j_{i,(k,M)} = Y^j_{k,(i,M)}. \tag{3.99}$$

So the components X^i and Y^i satisfy the conditions of form (3.74). We are left to prove that they also satisfy the conditions of form (3.75). For this, let us consider the coefficients of $\delta^{(2M+2)}(x - y)$ in the expressions of the components of the bivector $\partial_1 X - \partial_2 Y$. Vanishing of these coefficients leads to the equations

$$\hat{X}_{ji} - \hat{X}_{ij} = u^i u^j (\hat{Y}_{ji} - \hat{Y}_{ij}). \tag{3.100}$$

By taking the derivative with respect to $u^{k,M}$ and multiplying by u^k on both sides of the above equation, we obtain

$$u^k (\hat{X}_{ji,(k,M)} - \hat{X}_{ij,(k,M)}) = u^k u^i u^j (\hat{Y}_{ji,(k,M)} - \hat{Y}_{ij,(k,M)}). \tag{3.101}$$

Denote the right-hand side of the last equation by $W_{i,j,k}$. Then condition (3.75) follows from $W_{i,j,k} + W_{j,k,i} + W_{k,i,j} = 0$.

Above we showed that the vector field X satisfies the requirements of Lemma 3.8. So we can find local functionals I and J with densities that are homogeneous almost differential polynomials of degree $d-1$ such that $X-(\partial_1 I-\partial_2 J)$ depends at most on $u, \dots, u^{(N-1)}$. Repeating this procedure by subtracting terms of the form $\partial_1 I - \partial_2 J$, we reduce the proof of Theorem 3.3 to the case when the components X^i of the vector field X have the form (3.95) with $N = 1, 2, 3$.

Note that in the case when $N = 3$ or $N = 2$, the components of the vector field X and the accompanying one Y also have the forms (3.51)–(3.52) and (3.54)–(3.55) with $M = 1$, and the above equations (3.96)–(3.97) and (3.98), (3.100) still hold true. Thus when $N = 3$, the vector field X fulfills the requirements of Lemma 3.8, and we can find local functionals I_1 and I_2 such that $X - (\partial_1 I_1 - \partial_2 I_2)$ depend at most on u, u_x, and u_{xx}. The difference of this special case from the general one lies in the fact that now it is not obvious that we can choose the densities $h_a(u, u_x)$, $a = 1, 2$, to be almost differential polynomials. What can easily be seen from our construction is that they can be chosen to have the form

$$\begin{aligned} h_a(u, u_x) = & \sum_{i \neq j} V_{a;i,j}(u, u_x) \log u^i_x \log u^j_x \\ & + \sum_{i=1}^{n} V_{a;i}(u, u_x) \log u^i_x + U_a(u, u_x). \end{aligned} \tag{3.102}$$

Here the functions $V_{a;i,j}$, $V_{a;i}$, and U_a are homogeneous almost differential polynomials of degree $d-1$. The vector field $\tilde{X} = X - (\partial_1 I_1 - \partial_2 I_2)$ still has the property $\partial_1\partial_2\tilde{X} = 0$, so the same argument as above shows that the components of $\tilde{X}$ have the form (3.70) with $M = 0$ and satisfy equations (3.71). By using the construction of Lemma 3.8 we can find local functionals I_3 and I_4 such that the vector field $\tilde{X} - (\partial_1 I_3 - \partial_2 I_4)$ depends at most on u and u_x. A careful analysis of this construction shows that the densities of these local functionals can also be chosen to have the form of (3.102).

Let us denote the densities $I = I_1 + I_3$ and $J = I_2 + I_4$ also by h_1 and h_2, respectively, which have the expression (3.102). A simple calculation shows that

$$\frac{\partial(\partial_1 I - \partial_2 J)^i}{\partial u^j_{xxx}} = g^i \frac{\partial^2 h_2}{\partial u^i_x \partial u^j_x} - f^i \frac{\partial^2 h_1}{\partial u^i_x \partial u^j_x}, \tag{3.103}$$

$$\frac{\partial(\partial_1 I - \partial_2 J)^i}{\partial u^i_{xx}} = -\frac{3}{2} f^i(u) u^i_x \frac{\partial^2 h_2}{\partial u^i_x \partial u^i_x}. \tag{3.104}$$

The right-hand side of the above two identities equal, respectively,

$$\frac{\partial X^i}{\partial u^j_{xxx}} \quad \text{and} \quad \frac{\partial X^i}{\partial u^j_{xx}}.$$

We deduce that the functions

$$\frac{\partial^2 h_a}{\partial u^i_x \partial u^j_x}, \quad a = 1, 2, \quad i, j = 1, \ldots, n, \tag{3.105}$$

are homogeneous almost differential polynomials of degree $d-3$. This fact yields the restriction on the coefficients $V_{a;i,j}$ and $V_{a;i}$ in the expression of the densities (3.102) that they depend on u_x at most linearly. Since h_1 and h_2 have degree $d - 1 \geq 2$ (recall that we assume $d \geq 3$), it follows that the functions $V_{a;i,j}$ and $V_{a;i}$ must vanish, and as a result the densities h_1 and h_2 of the local functionals I and J are homogeneous almost differential polynomials of degree $d-1$.

Now let us prove that we have in fact

$$X = \partial_1 I - \partial_2 J. \tag{3.106}$$

This is due to the fact that the vector field $\bar{X} = X - (\partial_1 I - \partial_2 J)$ still satisfies the property $\partial_1\partial_2\bar{X} = 0$. So, by using Lemma 3.7 we can find a vector field $\bar{Y}$ that depends at most on u and u_x such that $\partial_1\bar{X} = \partial_2\bar{Y}$. Then by using Lemma 3.4 we know that the components of $\bar{X}$ depend at most linearly on u_x; since they are homogeneous almost differential polynomials of degree $d \geq 3$, we must have $\bar{X} = 0$. Thus we have proved the theorem. □

THEOREM 3.9 *Let the vector field X satisfying condition* (3.17) *have components of the form*

$$X^i = \sum_{j=1}^n X^i_j(u) u^j_{xx} + \sum_{k,l} Q^i_{kl}(u) u^k_x u^l_x. \tag{3.107}$$

Then each function $\hat{X}_{ii} = (f^i(u))^{-1} X^i_i(u)$ *depends only on* u^i*, and there exist local functionals* $\tilde{I}_1$ *and* $\tilde{I}_2$ *with densities that are homogeneous differential polynomials of degree* 1 *such that*

$$X = \partial_1(I_1 + \tilde{I}_1) - \partial_2(I_2 + \tilde{I}_2) \tag{3.108}$$

where

$$I_a = -\frac{2}{3} \sum_{i=1}^{n} \int (u^i)^{2-a} \hat{X}_{ii}(u^i) u^i_x \log u^i_x \, dx, \quad a = 1, 2. \tag{3.109}$$

Note that the densities h_1 and h_2 of the functionals I_1 and I_2 can be chosen as

$$h_a = \sum_{i=1}^{n} V_{a,i}(u^i) \frac{u^i_{xx}}{u^i_x}, \quad a = 1, 2, \tag{3.110}$$

which are homogeneous almost differential polynomials of degree $d - 1 = 1$. Here the functions $V_{a,i}$ are defined by $V_{a,i}(u^i)' = \frac{2}{3}(u^i)^{2-a} \hat{X}_{ii}(u^i)$.

PROOF: Since the X^i are differential polynomials, condition (3.17) implies the existence of a vector field Y with components Y^i of the same form (3.107) of X^i such that $\partial_1 X = \partial_2 Y$. By using the vanishing of the left-hand side of (3.46) with $k = i \neq j$, we deduce that

$$\frac{\partial \hat{X}_{ii}}{\partial u^j} - u^i u^j \frac{\partial \hat{Y}_{ii}}{\partial u^j} = 0, \quad i \neq j = 1, \dots, n. \tag{3.111}$$

Since (3.97) also holds true in this case, we obtain

$$\hat{Y}_{ii} = \frac{1}{(u^i)^2} \hat{X}_{ii}, \tag{3.112}$$

which yields, together with (3.111), the first result of the theorem

$$\frac{\partial \hat{X}_{ii}}{\partial u^j} = 0, \quad i \neq j = 1, \dots, n. \tag{3.113}$$

From definition (3.51) it is easy to see that the components of the vector field $\tilde{X} = X - (d_1 I_1 - d_2 I_2)$ are still homogeneous differential polynomials of the form (3.60) with $X^i_i(u) = 0,\ i = 1, \dots, n$. Then by using the same construction as we give in the proof of Lemma 3.8 for the local functionals with densities (3.76), we can find functionals $\tilde{I}_1$ and $\tilde{I}_2$ with homogeneous differential polynomial densities of degree 1 such that the vector field $\bar{X} = \tilde{X} - (\partial_1 \tilde{I}_1 - \partial_2 \tilde{I}_2)$ depends at most on u and u_x. The equation $\partial_1 \partial_2 \bar{X} = 0$ then implies that $\bar{X}$ depends at most linearly on u_x. Since the components of $\bar{X}$ are homogeneous differential polynomials of degree 2, we arrive at the equalities

$$X = \partial_1(I_1 + \tilde{I}_1) - \partial_2(I_2 + \tilde{I}_2). \tag{3.114}$$

The theorem is proved. $\square$

From the proof of Theorems 3.2, 3.3, and 3.9 and the argument given at the beginning of this section for the proof of the quasi-triviality theorem, we see that the reducing transformation

$$u^i \mapsto u^i + \sum_{k\geq 1} \epsilon^k F_k(u, \dots, u^{(m_k)}) \tag{3.115}$$

of the bi-Hamiltonian structure (1.27) has the properties that the F_k are homogeneous almost differential polynomials of degree k and that $m_k \leq \frac{3}{2}k$.

4 Reducing Bi-Hamiltonian PDEs

In this section we study properties of bi-Hamiltonian systems (1.26).

LEMMA 4.1 *Let I and J be two local functionals*

$$I = \int p(u, u_x, \dots, u^{(N)})\, dx, \quad J = \int q(u, u_x, \dots, u^{(N)})\, dx, \tag{4.1}$$

that satisfy the relation

$$\partial_1 I = \partial_2 J. \tag{4.2}$$

Then up to additions of total x-derivatives, the densities p and q do not depend on the jet coordinates $u_x, \dots, u^{(N)}$.

PROOF: Denote by X the vector field $\partial_1 I - \partial_2 J$ and by X^i its components. Then from the vanishing of

$$\frac{\partial X^i}{\partial u^{j,2N+1}} = (-1)^N \left(f^i \frac{\partial^2 p}{\partial u^{i,N} \partial u^{j,N}} - g^i \frac{\partial^2 q}{\partial u^{i,N} \partial u^{j,N}} \right), \tag{4.3}$$

we see that the functions p and q satisfy

$$(u^i - u^j) \frac{\partial^2 q}{\partial u^{i,N} \partial u^{j,N}} = 0, \quad \frac{\partial^2 p}{\partial u^{i,N} \partial u^{j,N}} - u^i \frac{\partial^2 q}{\partial u^{i,N} \partial u^{j,N}} = 0. \tag{4.4}$$

So the functions p and q can be represented by some functions p_i, r_i, and s as

$$p = \sum_{i=1}^n p_i(u, \dots, u^{(N-1)}, u^{i,N}),$$
$$q = \sum_{i=1}^n \left[u^i p_i(u, \dots, u^{(N-1)}, u^{i,N}) + r_i(u, \dots, u^{(N-1)}) u^{i,N} \right] + s(u, \dots, u^{(N-1)}).$$

By substituting these expressions of p and q into the equations

$$\frac{\partial X^i}{\partial u^{i,2N}} = 0, \tag{4.5}$$

we deduce that

$$f^i u^i \frac{\partial^2 p_i}{\partial u^{i,N} \partial u^{i,N}} = 0. \tag{4.6}$$

Thus we can rewrite the functions p and q in the forms

$$
\begin{aligned}
p &= \sum_{i=1}^{n} a_i(u,\ldots,u^{(N-1)})u^{i,N} + c(u,\ldots,u^{(N-1)}),\\
q &= \sum_{i=1}^{n} b_i(u,\ldots,u^{(N-1)})u^{i,N} + d(u,\ldots,u^{(N-1)}).
\end{aligned} \tag{4.7}
$$

From the identity

$$\frac{\partial X^i}{\partial u^{j,2N}} = 0 \quad \text{for } i \neq j$$

we have

$$\left(\frac{\partial^2 p}{\partial u^{i,N}\partial u^{j,N-1}} - \frac{\partial^2 p}{\partial u^{j,N}\partial u^{i,N-1}}\right) = u^i\left(\frac{\partial^2 q}{\partial u^{i,N}\partial u^{j,N-1}} - \frac{\partial^2 q}{\partial u^{j,N}\partial u^{i,N-1}}\right). \tag{4.8}$$

These equations imply that the functions a_i and b_i satisfy

$$\frac{\partial a_i}{\partial u^{j,N-1}} = \frac{\partial a_j}{\partial u^{i,N-1}}, \qquad \frac{\partial b_i}{\partial u^{j,N-1}} = \frac{\partial b_j}{\partial u^{i,N-1}}. \tag{4.9}$$

So there exist functions $A(u,\ldots,u^{(N-1)})$ and $B(u,\ldots,u^{(N-1)})$ such that

$$a_i = \frac{\partial A}{\partial u^{i,N-1}}, \quad b_i = \frac{\partial B}{\partial u^{i,N-1}}, \quad i = 1,\ldots,n. \tag{4.10}$$

Now we can replace the densities p and q of the Hamiltonians I and J, respectively, with

$$\tilde{p} = p - \partial_x A, \quad \tilde{q} = q - \partial_x B. \tag{4.11}$$

Then the new densities become independent of the jet variables $u^{i,N}$, $i = 1,\ldots,n$. Repeating the above procedure successively, we arrive at the result of the lemma. □

PROOF OF COROLLARY 1.9: Let us assume that after the quasi-Miura transformation the Hamiltonians of system (1.26) have the expansions

$$H_a = \sum_{k\geq 0} \epsilon^k H_a^{[k]} = \sum_{k\geq 0} \epsilon^k \int \tilde{h}_a^{[k]}(u,u_x,\ldots,u^{(m_k)})dx, \quad a = 1,2. \tag{4.12}$$

Here m_k are some positive integers that may also depend on the index a, and the functions $h_a^{[k]}$ have degrees k. Due to the bi-Hamiltonian property

$$\partial_1 H_1 = \partial_2 H_2, \tag{4.13}$$

we know in particular that

$$\partial_1 H_1^{[1]} = \partial_2 H_2^{[1]}. \tag{4.14}$$

Then the result of the above lemma implies that $H_1^{[1]} = H_2^{[1]} = 0$. Similarly, we prove that all other Hamiltonians $H_a^{[k]}$, $k \geq 2$, are trivial. The theorem is proved. □

COROLLARY 4.2 *Any two bi-Hamiltonian flows of the form* (1.26) *that correspond to the same bi-Hamiltonian structure* (1.27) *mutually commute.*

PROOF: Denote by X and Y the vector fields corresponding to the given bi-Hamiltonian systems; then their commutator $[X, Y]$ is also a bi-Hamiltonian vector field of degree greater than 1. From Lemma 4.1 it follows that under the quasi-Miura transformation reducing the bi-Hamiltonian structure (1.27) to (1.28) this vector field must vanish. Thus we have proved the corollary. □

5 Central Invariants of Bi-Hamiltonian Structures

One of the important applications of the property of quasi-triviality is the classification of deformations of a given bi-Hamiltonian structure of hydrodynamic type. The problem of classification of quasi-trivial infinitesimal deformations was solved in [35]. It was also conjectured that all deformations of the form (1.27) have reducing transformation. The quasi-triviality theorem proves this conjecture. In this section we reformulate the main result of [35] in order to describe the complete list of invariants, with respect to Miura-type transformation (1.39), of a bihamiltonian structure with a given leading-order term $\{\cdot\,,\,\cdot\}^{[0]}_{1,2}$. Recall that these transformations must depend polynomially on the derivatives in every order in ϵ.

Let us rewrite the bi-Hamiltonian structure (1.27) in terms of the canonical coordinates $u^i = u^i(w)$, $i = 1, \dots, n$,

$$
\begin{aligned}
&\{u^i(x), u^j(y)\}_a \\
&\quad = \{u^i(x), u^j(y)\}^{[0]}_a \\
&\qquad + \sum_{m\geq 1}\sum_{l=0}^{m+1} \epsilon^m A^{ij}_{m,l;a}(u; u_x, \dots, u^{(m-l+1)})\delta^{(l)}(x-y), \quad a = 1, 2.
\end{aligned}
\tag{5.1}
$$

Then the functions P^{ij}_a and Q^{ij}_a defined in (1.48) have the expressions

$$
P^{ij}_a(u) = A^{ij}_{1,2;a}(u), \quad Q^{ij}_a(u) = A^{ij}_{2,3;a}(u), \quad i, j = 1, \dots, n, \;\; a = 1, 2. \tag{5.2}
$$

PROOF OF COROLLARY 1.11: First we assume that the bi-Hamiltonian structure (5.1) has the following special form:

$$
(\varpi_1, \varpi_2 + \epsilon^2\gamma + \mathcal{O}(\epsilon^3)). \tag{5.3}
$$

Here (ϖ_1, ϖ_2) denotes the bi-Hamiltonian structure given by the leading terms of (5.1), and γ is a bivector that can be represented as $\gamma = \partial_1 X$ through a vector field with components that are homogenous differential polynomials of degree 2. Due to Theorem 3.9, the vector field X can be represented up to a Miura-type transformation in the form

$$
X = \partial_1 I - \partial_2 J \tag{5.4}
$$

where the functionals I and J are defined by

$$I = \int \sum_{i=1}^{n} u^i \hat{c}_i(u^i) u_x^i \log u_x^i \, dx, \quad J = \int \sum_{i=1}^{n} \hat{c}_i(u^i) u_x^i \log u_x^i \, dx, \tag{5.5}$$

with

$$\hat{c}_i(u^i) = \frac{1}{3(f^i(u))^2} (\gamma)_3^{ii}, \quad i = 1, \ldots, n. \tag{5.6}$$

Here $(\gamma)_3^{ii}$ denotes the coefficients of $\delta'''(x-y)$ in the components $(\gamma)^{ii}$ of the bivector γ. The main result of [35] together with the quasi-triviality theorem shows that any two bi-Hamiltonian structures of the form (5.3) are equivalent if and only if they correspond to the same set of functions $\hat{c}_i$, $i = 1, \ldots, n$. In the case that the two bi-Hamiltonian structures of the present theorem have the form (5.3), it is easy to see that $\hat{c}_i(u^i) = c_i(u)$, and the result of the theorem follows.

Now return to the general form (5.1) of a bi-Hamiltonian structure. We redenote it as

$$(\varpi_1 + \epsilon\alpha_1 + \epsilon^2\beta_1, \varpi_2 + \epsilon\alpha_2 + \epsilon^2\beta_2) + \mathcal{O}(\epsilon^3). \tag{5.7}$$

By using the result of Theorem 3.2, we can eliminate the linear-in-ϵ terms by a Miura-type transformation

$$u^i \mapsto \exp(-\epsilon X) u^i, \quad i = 1, \ldots, n, \tag{5.8}$$

given by a local vector field X with components of the form

$$X^i = \sum_{j=1}^{n} X_j^i(u) u_x^j. \tag{5.9}$$

This implies that $P_1 = \partial_1 X$ and $P_2 = \partial_2 X$, and they in turn yield

$$\begin{aligned} P_1^{ik} &= -f^k(u) X_k^i(u) + f^i(u) X_i^k(u), \\ P_2^{ik} &= -g^k(u) X_k^i(u) + g^i(u) X_i^k(u), \end{aligned} \tag{5.10}$$

where the functions P_1^{ij} and P_2^{ij} are defined by (5.2). Solving the above system, we obtain

$$X_k^i(u) = \frac{P_2^{ki} - u^i P_1^{ki}}{f^k(u)(u^k - u^i)}, \quad k \neq i. \tag{5.11}$$

After the Miura-type transformation (5.8), the bi-Hamiltonian structure (5.7) becomes

$$\left(\varpi_1 + \epsilon^2\left(\beta_1 - \frac{1}{2}[X, \alpha_1]\right), \varpi_2 + \epsilon^2\left(\beta_2 - \frac{1}{2}[X, \alpha_2]\right)\right) + \mathcal{O}(\epsilon^3). \tag{5.12}$$

Then there exists a local vector field Y such that

$$\partial_1 Y = \beta_1 - \frac{1}{2}[X, \alpha_1]. \tag{5.13}$$

So the Miura-type transformation

$$u^i \mapsto \exp(-\epsilon Y)u^i, \quad i = 1, \dots, n, \tag{5.14}$$

reduces the bi-Hamiltonian structure (5.12) to the form of (5.8)

$$\left(\varpi_1, \varpi_2 + \epsilon^2\left(\beta_2 - \frac{1}{2}[X, \alpha_2] - \partial_2 Y\right)\right) + \mathcal{O}(\epsilon^3), \tag{5.15}$$

and we need to compute the coefficients χ_3^{ij} of $\delta'''(x-y)$ in the components of the bivector $\chi = \beta_2 - \frac{1}{2}[X, \alpha_2] - \partial_2 Y$. By using the notation introduced in (5.2), we have

$$[X, \alpha_1]_3^{ii} = \sum_k X_k^i (P_1^{ik} - P_1^{ki}) = -2 \sum_{k \neq i} X_k^i P_1^{ki},$$

$$[X, \alpha_2]_3^{ii} = \sum_k X_k^i (P_2^{ik} - P_2^{ki}) = -2 \sum_{k \neq i} X_k^i P_2^{ki},$$

$$(\partial_2 Y)_3^{ii} = u^i (\partial_1 Y)_3^{ii} = u^i \left(\beta_1 - \frac{1}{2}[X, \alpha_1]\right)_3^{ii} = u^i Q_1^{ii} - \frac{1}{2} u^i [X, \alpha_1]_3^{ii}.$$

Here as above, for any bivector η we denote by η_3^{ij} the coefficient of $\delta'''(x-y)$ in the components η^{ij}. These formulae together with the expressions (5.11) for X_k^i, $k \neq i$, show that the functions $\hat{c}_i(u^i)$ that are defined by (5.6) with γ replaced by χ coincide with the functions $c_i(u)$ introduced in (1.49). Thus we have proved the theorem. □

From this theorem, the following corollary easily follows:

COROLLARY 5.1 *Any deformation* (1.27) *of the bi-Hamiltonian structure* (1.28) *is equivalent, under an appropriate Miura-type transformation, to a deformation in which only even powers of* ϵ *appear.*

This result can also be seen from the construction of the functionals I and J in the proof of Theorem 3.3 and the argument given in the proof of the quasi-triviality theorem.

THEOREM 5.2 *If we choose another representative*

$$\widetilde{\{\cdot\,,\cdot\}}_1 = c\{\cdot\,,\cdot\}_2 + d\{\cdot\,,\cdot\}_1, \quad \widetilde{\{\cdot\,,\cdot\}}_2 = a\{\cdot\,,\cdot\}_2 + b\{\cdot\,,\cdot\}_1, \quad ad - bc \neq 0, \tag{5.16}$$

of the bi-Hamiltonian structure (5.1)*, then the functions* $c_i(u)$ *that we define in* (1.49) *are changed to*

$$\tilde{c}_i(\tilde{u}^i) = \frac{cu^i + d}{ad - bc} c_i(u^i), \quad i = 1, \dots, n, \tag{5.17}$$

where

$$\tilde{u}^i = \frac{au^i + b}{cu^i + d}, \quad i = 1, \dots, n, \tag{5.18}$$

are the canonical coordinates of the bi-Hamiltonian structure (5.1) *with respect to the new representative* (5.16)*.*

PROOF: The result of the theorem is obtained by a straightforward calculation with the help of formula (5.18) and the tensor rule abided by P_a^{ij} and Q_a^{ij} under the change of coordinates $u^i \mapsto \tilde{u}^i(u)$. □

From the above theorem we see that, for the bi-Hamiltonian structure (5.1), the following $\frac{1}{2}$-forms

$$\Omega_i = c_i(u^i)(du^i)^{1/2} \tag{5.19}$$

are invariant, up to permutations, under transformation (5.16) with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

6 Examples and Concluding Remarks

Let us give some examples of bi-Hamiltonian structures, their central invariants and reducing transformations.

Example 1. The bi-Hamiltonian structure of the KdV hierarchy [23, 38, 47] has the form

$$\{w(x), w(y)\}_1 = \delta'(x-y), \tag{6.1}$$

$$\begin{aligned}\{w(x), w(y)\}_2 = w(x)\delta'(x-y) + \frac{1}{2}\, w'(x)\delta(x-y) \\ + 3\, c\, \epsilon^2\, \delta'''(x-y).\end{aligned} \tag{6.2}$$

The canonical coordinate is $u = w$, and the constant c is the central invariant. Up to terms of order ϵ^6, the reducing transformation [16] is given by[4]

$$w = v + \epsilon^2 \partial_x^2 \left[c\, \log v_x + \frac{c^2\epsilon^2}{10}\left(5\frac{v^{(4)}}{v_x^2} - 21\frac{v_{xx}\, v_{xxx}}{v_x^3} + 16\frac{v_{xx}^3}{v_x^4}\right) + O(\epsilon^4)\right]. \tag{6.3}$$

Observe that the inverse transformation can be written in a similar form,

$$v = w + \epsilon^2 \partial_x^2 \left[-c\, \log w_x + \frac{c^2\epsilon^2}{10}\left(5\frac{w^{(4)}}{w_x^2} - 9\frac{w_{xx}\, w_{xxx}}{w_x^3} + 4\frac{w_{xx}^3}{w_x^4}\right) + O(\epsilon^4)\right]. \tag{6.4}$$

The transformations of the form

$$u \mapsto u + \epsilon^2 \partial_x^2 F(u; u_x, u_{xx}, \dots; \epsilon)$$

form a subgroup of the group of all Miura-type (or quasi-Miura-type) transformations. The appearance of this subgroup is closely related to the existence of a tau-function in the theory of the KdV hierarchy (see details in [16]).

[4] The (inverse to the) reducing transformation for the KdV equation was constructed in [4]. However, the action of this transformation on the Poisson pencil was not studied.

The Poisson pencils (6.1)–(6.2) with different values of c are inequivalent with respect to Miura-type transformations.

Example 2. The bi-Hamiltonian structure related to the Camassa-Holm hierarchy [7, 8, 20, 21, 22] has the expression

$$\{w(x), w(y)\}_1 = \delta'(x-y) - \frac{\epsilon^2}{8}\delta'''(x-y), \tag{6.5}$$

$$\{w(x), w(y)\}_2 = w(x)\delta'(x-y) + \frac{1}{2}\,w'(x)\delta(x-y). \tag{6.6}$$

The canonical coordinate u also coincides with w, and the central invariant $c(u) = \frac{1}{24}u$. As shown in [37], the reducing transformation is, up to ϵ^4, given by

$$\begin{aligned} w = v &+ \epsilon^2\partial_x\left(\frac{v\,v_{xx}}{24\,v_x} - \frac{v_x}{48}\right) \\ &+ \epsilon^4\partial_x\left(\frac{7\,v_{xx}^2}{2880\,v_x} + \frac{v\,v_{xx}^3}{180\,v_x^3} - \frac{v^2\,v_{xx}^4}{90\,v_x^5} - \frac{v_{xxx}}{512} - \frac{59\,v\,v_{xx}\,v_{xxx}}{5760\,v_x^2}\right. \\ &\qquad + \frac{37\,v^2\,v_{xx}^2\,v_{xxx}}{1920\,v_x^4} - \frac{7\,v^2\,v_{xxx}^2}{1920\,v_x^3} + \frac{5\,v\,v^{(4)}}{1152\,v_x} \\ &\qquad \left. - \frac{31\,v^2\,v_{xx}\,v^{(4)}}{5760\,v_x^3} + \frac{v^2\,v^{(5)}}{1152\,v_x^2}\right). \end{aligned} \tag{6.7}$$

Example 3. Let us consider the bi-Hamiltonian structure related to the multicomponent KdV-CH (Camassa-Holm) hierarchy. Define

$$\mathcal{D}_i = w^i\delta'(x-y) + \frac{w_x^i}{2}\delta(x-y) + a_i\frac{\epsilon^2}{8}\delta'''(x-y), \qquad i = 0, 1, \dots, n. \tag{6.8}$$

Here $w^0 = 1$ and $a_0, \dots, a_n$ are given constants with at least one nonzero. Define also the numbers

$$f_m^{ij} = \begin{cases} -1 & i, j \le m \\ +1 & i, j \ge m+1 \\ \;\;0 & \text{otherwise.} \end{cases} \tag{6.9}$$

Then we have the following $n+1$ compatible Hamiltonian structures:

$$\{w^i(x), w^j(y)\}_m = (-1)^m f_m^{ij}\mathcal{D}_{i+j-m-1}, \qquad 1 \le i, j \le n,\ m = 0, 1, \dots, n. \tag{6.10}$$

When $i < 0$ or $i > n$, we assume that $\mathcal{D}_i = 0$. These Hamiltonian structures were introduced[5] in the study of the hierarchies of integrable systems (called the

[5] To the best of our knowledge, connections of these bi-Hamiltonian structures with the Camassa-Holm equation and its multicomponent generalizations were never considered in the literature.

coupled KdV hierarchies) associated with the compatibility conditions of the linear systems of the form [1, 2, 3, 39]

$$\left(\frac{1}{2}(\epsilon\partial_x)^2 + A(w;\lambda)\right)\psi = 0, \tag{6.11}$$

$$\psi_t = \frac{1}{2}B\psi_x - \frac{1}{4}B_x\psi. \tag{6.12}$$

Here $A(w;\lambda)$ has the expression

$$A(w;\lambda) = \frac{\sum_{i=0}^{n}(-1)^i w^i \lambda^{n-i}}{\sum_{i=0}^{n}(-1)^i a_i \lambda^{n-i}}, \tag{6.13}$$

and B is a certain polynomial or Laurent polynomial in λ with coefficients that are differential polynomials of $w^1, \dots, w^n$ which can be chosen according to the equation

$$A_t = AB_x + \frac{A_x}{2}B + \frac{\epsilon^2}{8}B_{xxx}. \tag{6.14}$$

As shown by Ferapontov in [18], if a system of hydrodynamic type with n dependent variables possesses $n+1$ compatible Hamiltonian structures of hydrodynamic type, then this $(n+1)$–Hamiltonian structure must be equivalent to the one obtained from the leading terms of (6.10).

From (6.10) we readily have the following bi-Hamiltonian structures:

$$\mathcal{B}_{k,l} = (\{\cdot\,,\cdot\}_k, \{\cdot\,,\cdot\}_l), \quad k \neq l. \tag{6.15}$$

Denote by $\lambda_1(w), \dots, \lambda_n(w)$ the roots of the polynomial $P(\lambda) = \lambda^n - w^1\lambda^{n-1} + \cdots + (-1)^n w^n$. Then the canonical coordinates for the bi-Hamiltonian structure $\mathcal{B}_{k,l}$ are given by $u^i = (\lambda_i)^{k-l}$, $i = 1, \dots, n$, and the central invariants, $c_1, \dots, c_n$ have the expressions

$$c_i(u^i) = \frac{\sum_{j=0}^{n}(-1)^j a_j \lambda_i^{n-j}}{24(l-k)\lambda_i^{n-1-l}}, \quad i = 1, \dots, n. \tag{6.16}$$

In particular, for the one-component case $n = 1$, choosing $a_0 = 0$ and $a_1 = 1$, we get the bi-Hamiltonian structure $\mathcal{B}_{1,0}$, which coincides with (6.1)–(6.2) for the KdV hierarchy. The choice $a_0 = -1$ and $a_1 = 0$ yields the bi-Hamiltonian structure (6.5)–(6.6) of the Camassa-Holm hierarchy. In general, we call the hierarchy generated by the bi-Hamiltonian structure $\mathcal{B}_{k,l}$ the *multi-component KdV-CH hierarchy*.

For the case when $n = 2$, the above defined bi-Hamiltonian structure $\mathcal{B}_{2,1}$ yields, with different choices of the constants a_0, a_1, and a_2 and up to certain Miura-type transformations and rescaling of the Poisson structures, the following four bi-Hamiltonian structures that appeared in the literature. They have the same leading terms

$$\begin{gathered}\{\varphi(x),\varphi(y)\}_1^{[0]}=0,\quad \{\rho(x),\varphi(y)\}_1^{[0]}=\delta'(x-y),\\ \{\rho(x),\rho(y)\}_1^{[0]}=0,\\ \{\varphi(x),\varphi(y)\}_2^{[0]}=2\delta'(x-y),\quad \{\rho(x),\varphi(y)\}_2^{[0]}=\varphi(x)\delta'(x-y),\\ \{\rho(x),\rho(y)\}_2^{[0]}=2\rho(x)\delta'(x-y)+\rho'(x)\,\delta(x-y).\end{gathered}\tag{6.17}$$

A bi-Hamiltonian structure related to the nonlinear Schrödinger hierarchy is given by the above brackets with the only difference [5, 35]

$$\{\rho(x),\varphi(y)\}_2=\{\rho(x),\varphi(y)\}_2^{[0]}+\epsilon\delta''(x-y).\tag{6.18}$$

After the Miura-type transformation

$$w^1=2\varphi,\quad w^2=\varphi^2-4\rho+2\epsilon\varphi_x,\tag{6.19}$$

it is transformed to the bi-Hamiltonian structure $8\mathcal{B}_{2,1}$ with the choice of constants $a_0=a_1=0$ and $a_2=-8$. Here $8\mathcal{B}_{2,1}$ denotes the bi-Hamiltonian structure obtained from $\mathcal{B}_{2,1}$ by the multiplication of an overall factor 8.

In [35] a generalization of the Camassa-Holm hierarchy is introduced that is called the 2-component Camassa-Holm hierarchy. It is reduced to the usual Camassa-Holm hierarchy under a natural constraint on its two dependent variables. The related bi-Hamiltonian structure is defined by the brackets (6.17) except

$$\{\rho(x),\varphi(y)\}_1=\{\rho(x),\varphi(y)\}_1^{[0]}+\epsilon\delta''(x-y).\tag{6.20}$$

After the Miura-type transformation

$$w^1=2\varphi+2\epsilon\varphi_x,\quad w^2=\varphi^2-4\rho,\tag{6.21}$$

it is converted, up to the approximation to ϵ^2, to the bi-Hamiltonian structure $8\mathcal{B}_{2,1}$ with the choice of constants $a_0=-8$, $a_1=0$, and $a_2=0$.

In [29] the bi-Hamiltonian structure for the so-called classical Boussinesq hierarchy is given. It is defined by the brackets (6.17) except for

$$\{\rho(x),\rho(y)\}_2=\{\rho(x),\rho(y)\}_2^{[0]}+\frac{1}{2}\epsilon^2\delta'''(x-y).\tag{6.22}$$

After the Miura-type transformation

$$w^1=2\varphi,\quad w^2=\varphi^2-4\rho,\tag{6.23}$$

it is transformed to the bi-Hamiltonian structure $8\mathcal{B}_{2,1}$ with the choice of constants $a_0=0$, $a_1=0$, $a_2=-8$.

Note that, for the bi-Hamiltonian structure related to the nonlinear Schrödinger hierarchy, by moving the perturbation term from the second Poisson bracket to the first one we obtain the bi-Hamiltonian structure of the 2-component Camassa-Holm hierarchy. Doing precisely the same procedure we obtain from the above bi-Hamiltonian structure of the classical Boussinesq hierarchy the one that is defined by (6.17) except for the bracket

$$\{\rho(x), \rho(y)\}_1 = \{\rho(x), \rho(y)\}_1^{[0]} - \frac{1}{2}\epsilon^2\delta'''(x-y). \tag{6.24}$$

After the change of dependent variables

$$w^1 = 2\varphi, \quad w^2 = \varphi^2 - 4\rho, \tag{6.25}$$

it is transformed to the bi-Hamiltonian structure $8\mathcal{B}_{2,1}$ with the choice of constants $a_0 = 0$, $a_1 = 8$, and $a_2 = 0$. This bi-Hamiltonian structure is related to the Ito-type equations [28, 33].

The bi-Hamiltonian structures related to the nonlinear Schrödinger hierarchy and the classical Boussinesq hierarchy are equivalent. Indeed, their central invariants are given by $c_1 = c_2 = \frac{1}{24}$. The central invariants for the bi-Hamiltonian structure related to the 2-component Camassa-Holm hierarchy are given by $c_1 = (u^1)^2/24$ and $c_2 = (u^2)^2/24$, and those for the bi-Hamiltonian structure defined by (6.17) and (6.24) have the form $c_1 = u^1/24$ and $c_2 = u^2/24$.

We omit here the presentation of the reducing transformations of the above bi-Hamiltonian structures due to their cumbersome expressions.

Example 4. The equations of motion of one-dimensional isentropic gas with the equation of state $p = \frac{\kappa}{\kappa+1}\rho^{\kappa+1}$ read

$$u_t + \left(\frac{u^2}{2} + \rho^\kappa\right)_x = 0, \quad \rho_t + (\rho\, u)_x = 0. \tag{6.26}$$

Here κ is an arbitrary parameter, $\kappa \neq 0, -1$. For a gas with m degrees of freedom one has

$$\kappa = \frac{2}{m}$$

(see, e.g., [9]). This is a weakly symmetrizable system with

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This gives the first Poisson structure of the equations with the Poisson brackets

$$\left\{u(x), \rho(y)\right\}_1^{[0]} = \delta'(x-y); \tag{6.27}$$

other brackets vanish.

The second Hamiltonian structure

$$\begin{aligned} \left\{u(x), u(y)\right\}_2^{[0]} &= 2\rho^{\kappa-1}(x)\delta'(x-y) + \rho_x^{\kappa-1}\delta(x-y), \\ \left\{u(x), \rho(y)\right\}_2^{[0]} &= u(x)\delta'(x-y) + \frac{1}{\kappa}u'(x)\delta(x-y), \\ \left\{\rho(x), \rho(y)\right\}_2^{[0]} &= \frac{1}{\kappa}(2\rho(x)\delta'(x-y) + \rho'(x)\delta(x-y)), \end{aligned} \tag{6.28}$$

was found in [41, 42].

As shown in [16] the isentropic gas equations have the following deformation, which preserves the bi-Hamiltonian property (up to corrections[6] of order ϵ^6):

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \partial_x\Big\{\frac{u^2}{2} + \rho^\kappa + \epsilon^2\Big[\frac{\kappa(\kappa-2)}{8}\rho^{\kappa-3}\rho_x^2 + \frac{\kappa^2}{12}\rho^{\kappa-2}\rho_{xx}\Big] \\
&\quad + \epsilon^4(\kappa-2)(\kappa-3)\big[a_1\rho^{-4}u_x^2\rho_x^2 + a_2\rho^{\kappa-6}\rho_x^4 + a_3\rho^{-3}u_{xx}u_x\rho_x \\
&\quad + a_4\rho^{-2}u_{xx}^2 + a_5\rho^{-3}u_x^2\rho_{xx} + a_6\rho^{\kappa-5}\rho_x^2\rho_{xx} + a_7\rho^{\kappa-4}\rho_{xx}^2 \\
&\quad + a_8\rho^{-2}u_xu_{xxx} + a_9\rho^{\kappa-4}\rho_x\rho_{xxx} + a_{10}\rho^{-\kappa-2}u_x^4\big] + \epsilon^4\frac{\kappa(\kappa^2-4)}{360}\rho^{\kappa-3}\rho_{xxxx}\Big\} \\
&\qquad = \mathcal{O}(\epsilon^6), \\
&\qquad \frac{\partial \rho}{\partial t} + \partial_x\Big\{\rho u + \epsilon^2\Big(\frac{(2-\kappa)(\kappa-3)}{12\kappa\rho}u_x\rho_x + \frac{1}{6}u_{xx}\Big) \\
&\qquad\quad + \epsilon^4(\kappa-2)(\kappa-3)\big[b_1\rho^{-4}u_x\rho_x^3 + b_2\rho^{-3}\rho_x^2u_{xx} + b_3\rho^{-3}u_x\rho_x\rho_{xx} \\
&\qquad\quad + b_4\rho^{-2}u_{xx}\rho_{xx} + b_5\rho^{-2}u_{xxx}\rho_x + b_6\rho^{-2}u_x\rho_{xxx} + b_7\rho^{-1}u_{xxxx} \\
&\qquad\quad + b_8\rho^{-\kappa-1}u_x^2u_{xx} + b_9\rho^{-\kappa-2}u_x^3\rho_x\big]\Big\} = \mathcal{O}(\epsilon^6).
\end{aligned}$$

The coefficients are given by

$$a_1 = \frac{18+75\kappa-15\kappa^2+20\kappa^3+2\kappa^4}{2880\kappa^3},$$
$$a_2 = \frac{6+113\kappa+409\kappa^2-185\kappa^3+17\kappa^4}{5760\kappa^2},$$
$$a_3 = -\frac{18+11\kappa+3\kappa^2}{720\kappa^2}, \quad a_4 = \frac{7}{720\kappa}, \quad a_5 = \frac{-6+3\kappa-\kappa^2}{480\kappa^2},$$
$$a_6 = \frac{-6-39\kappa-10\kappa^2+5\kappa^3}{480\kappa}, \quad a_7 = \frac{14+5\kappa+5\kappa^2}{1440}, \quad a_8 = \frac{1}{120\kappa},$$
$$a_9 = \frac{2+5\kappa}{240}, \quad a_{10} = -\frac{(\kappa+2)(\kappa+3)(\kappa^2-1)}{5760\kappa^4},$$

$$b_1 = \frac{42+83\kappa-53\kappa^2+8\kappa^3}{1440\kappa^3}, \quad b_2 = -\frac{6+35\kappa-24\kappa^2+5\kappa^3}{720\kappa^3},$$
$$b_3 = -\frac{12+40\kappa-13\kappa^2+5\kappa^3}{720\kappa^3}, \quad b_4 = \frac{6-4\kappa+\kappa^2}{180\kappa^2}, \quad b_5 = \frac{6+\kappa+\kappa^2}{720\kappa^2},$$

[6] In principle one can continue the expansions until an arbitrary order in ϵ. However, the computations become very involved.

$$b_6 = \frac{6+\kappa+\kappa^2}{720\kappa^2}, \quad b_7 = -\frac{1}{360\kappa}, \quad b_8 = -\frac{(\kappa+2)(\kappa+3)}{720\kappa^4},$$
$$b_9 = \frac{(\kappa+1)(\kappa+2)(\kappa+3)}{1440\kappa^4}.$$

The corresponding bi-Hamiltonian structure (at the approximation up to ϵ^4) is given in section 4.2.3 of [16]; the central invariants are $c_1 = c_2 = \frac{1}{24}$. The above system can be represented as

$$\begin{aligned} \frac{\partial u}{\partial t} &= \{H_1, u(x)\}_1 = \frac{\kappa}{\kappa+1}\{H_2, u(x)\}_2, \\ \frac{\partial \rho}{\partial t} &= \{H_1, \rho(x)\}_1 = \frac{\kappa}{\kappa+1}\{H_2, \rho(x)\}_2. \end{aligned}$$

Here the densities h_1 and h_2 of the Hamiltonians H_1 and H_2 have the expressions

$$\begin{aligned} h_1 &= \frac{1}{2}\rho u^2 + \frac{\rho^{\kappa+1}}{\kappa+1} + \Delta h_1, \\ h_2 &= \rho u + \Delta h_2, \end{aligned}$$

where

$$\begin{aligned} \Delta h_1 = {} & u\Delta h_2 - \frac{\epsilon^2}{24\kappa}\big((\kappa^2-3\kappa+6)u_x^2 + \kappa(2\kappa^2-5\kappa+6)\rho^{\kappa-2}\rho_x^2\big) \\ & + \epsilon^4\frac{(\kappa-2)(\kappa-3)}{240\kappa^3}\bigg(-\frac{1}{3}\kappa(\kappa^2-4\kappa+6)\rho^{-1}u_x u_{xxx} \\ & + \frac{1}{3}\kappa(2\kappa^2-13\kappa+12)\rho^{-2}u_x u_{xx}\rho_x \\ & + \frac{1}{72}\frac{(3\kappa+5)(\kappa+3)(\kappa+2)}{\kappa}\rho^{-\kappa-1}u_x^4 \\ & - \frac{1}{12}(2\kappa-3)(\kappa+3)(\kappa+2)\rho^{-3}u_x^2\rho_x^2 \\ & + \frac{\kappa^2(\kappa-1)(3\kappa^2-8\kappa+12)}{2(\kappa-3)}\rho^{\kappa-3}\rho_{xx}^2 \\ & - \frac{1}{72}\kappa(\kappa-1)(16\kappa^4-100\kappa^3+229\kappa^2-211\kappa+6)\rho^{\kappa-5}\rho_x^4\bigg), \end{aligned}$$

$$\begin{aligned} \Delta h_2 = {} & -\epsilon^2\frac{(\kappa-2)(\kappa-3)}{12\kappa}\rho^{-1}u_x\rho_x \\ & + \epsilon^4\frac{(\kappa-2)(\kappa-3)}{720\kappa^3}\Big[-2\kappa(\kappa^2-8\kappa+6)\rho^{-2}u_x\rho_{xxx} \\ & + \kappa(7\kappa^2-61\kappa+42)\rho^{-3}u_x\rho_x\rho_{xx} \end{aligned}$$

$$+\left(-5\kappa^3+\frac{79}{2}\kappa^2-\frac{55}{2}\kappa+3\right)\rho^{-4}u_x\rho_x^3$$
$$\left.+\frac{1}{6\kappa}(\kappa+3)(\kappa+2)(\kappa+1)\rho^{-k-2}u_x^3\rho_x\right].$$

To write down the reducing transformation of the perturbed system of the one-dimensional isentropic gas and its bi-Hamiltonian structure, we introduce the operators $\mathcal{T}_1$ and $\mathcal{T}_2$

$$\begin{array}{ll} \mathcal{T}_1 u^{(m)}=u^{(m+1)}, & \mathcal{T}_1\rho^{(m)}=\rho^{(m+1)}, \\ \mathcal{T}_2 u^{(m)}=\partial_x^m(\kappa\rho^{\kappa-2}\rho_x), & \mathcal{T}_2\rho^{(m)}=u^{(m+1)}, \end{array}\quad m\geq 0.$$

We will use Greek subscripts for the result of the operators acting on the functions ρ and u, i.e.,

$$\rho_{\alpha_1\alpha_2\cdots}:=\mathcal{T}_{\alpha_1}\mathcal{T}_{\alpha_2}\cdots\rho,\quad u_{\alpha_1\alpha_2\cdots}:=\mathcal{T}_{\alpha_1}\mathcal{T}_{\alpha_2}\cdots u.$$

Define the functions

$$\mathcal{F}_1=\frac{1}{24}\log(\kappa\rho^{\kappa-2}\rho_x^2-u_x^2)-\frac{1}{24}\frac{(\kappa-2)(\kappa-3)}{\kappa}\log\rho,$$
$$\mathcal{F}_2=\frac{1}{1152}\rho_{\alpha_1\alpha_2\alpha_3\alpha_4}M^{\alpha_1\alpha_2}M^{\alpha_3\alpha_4}-\frac{1}{360}\rho_{\alpha_1\alpha_2\alpha_3}\rho_{\alpha_4\alpha_5\alpha_6}M^{\alpha_1\alpha_4}M^{\alpha_2\alpha_5}M^{\alpha_3\alpha_6}$$
$$-\frac{1}{1152}\rho_{\alpha_1\alpha_2}\rho_{\alpha_3\alpha_4\alpha_5\alpha_6}M^{\alpha_1\alpha_3}M^{\alpha_2\alpha_4}M^{\alpha_5\alpha_6}$$
$$+\frac{1}{360}\rho_{\alpha_1\alpha_2}\rho_{\alpha_3\alpha_4\alpha_5}\rho_{\alpha_6\alpha_7\alpha_8}M^{\alpha_1\alpha_3}M^{\alpha_2\alpha_6}M^{\alpha_4\alpha_7}M^{\alpha_5\alpha_8}$$
$$+(\kappa-2)(\kappa-3)D^{-2}\left[-\frac{1}{240}\kappa\rho^{2k-5}\rho_{xxx}\rho_x^3\right.$$
$$+\frac{11}{2880}\kappa\rho^{2\kappa-5}\rho_{xx}^2\rho_x^2$$
$$+\left(-\frac{7}{5760}\kappa^2+\frac{19}{5760}\kappa+\frac{7}{960}\right)\rho^{2\kappa-6}\rho_{xx}\rho_x^4+\frac{11}{2880}\rho^{\kappa-3}\rho_x^2u_{xx}^2$$
$$-\frac{1}{5760\kappa}(\kappa^4-9\kappa^3+\kappa^2+53\kappa+6)\rho^{2\kappa-7}\rho_x^6$$
$$+\frac{1}{240}\rho^{\kappa-3}\rho_x^2u_{xxx}u_x+\frac{1}{240}\rho^{\kappa-3}\rho_x\rho_{xxx}u_x^2-\frac{11}{720}\rho^{\kappa-3}\rho_xu_xu_{xx}\rho_{xx}$$
$$+\frac{11}{2880}\rho^{\kappa-3}u_x^2\rho_{xx}^2-\frac{1}{1440}(11\kappa-21)\rho^{\kappa-4}\rho_x^3u_{xx}u_x$$
$$+\frac{1}{2880\kappa}(22\kappa^2-47\kappa-42)\rho^{\kappa-4}\rho_x^2u_x^2\rho_{xx}$$
$$+\frac{1}{5760\kappa^2}(12\kappa^4-45\kappa^3+15\kappa^2+101\kappa+6)\rho^{\kappa-5}u_x^2\rho_x^4$$
$$-\frac{1}{240\kappa}\rho^{-1}u_{xxx}u_x^3+\frac{11}{2880\kappa}\rho^{-1}u_x^2u_{xx}^2+\frac{1}{1440\kappa}\rho^{-2}u_{xx}u_x^3\rho_x$$

$$+\frac{1}{5760\kappa^2}(7\kappa^2-13\kappa+42)\rho^{-2}\rho_{xx}u_x^4$$
$$-\frac{1}{5760\kappa^3}(8\kappa^3-31\kappa^2+43\kappa-6)\rho^{-3}u_x^4\rho_x^2$$
$$\left.-\frac{1}{5760\kappa^4}(\kappa+3)(\kappa+2)\rho^{-\kappa-1}u_x^6\right]$$

Here the matrix $M=(M^{\alpha\beta})$ and differential polynomial D read

$$M=D^{-1}\begin{pmatrix}-\kappa\rho^{\kappa-2}\rho_x & u_x\\ u_x & -\rho_x\end{pmatrix},\quad D=u_x^2-\kappa\rho^{\kappa-2}\rho_x^2.$$

Then the reducing transformation is given by the formula

$$u\mapsto u+\mathcal{T}_1\mathcal{T}_2(\epsilon^2\mathcal{F}_1+\epsilon^4\mathcal{F}_2),\quad \rho\mapsto\rho+\mathcal{T}_1\mathcal{T}_1\Big(\epsilon^2\mathcal{F}_1+\epsilon^4\mathcal{F}_2\Big).$$

We leave as an exercise for the reader to check that the denominator $D\neq 0$ on the monotone solutions.

In conclusion, let us formulate some open problems.

Problem 1. Study the convergence (or at least the asymptotic nature) of the reducing transformations for the case of analytic-in-ϵ perturbations (1.7) (e.g., for the case of polynomial dependence on ϵ).

Problem 2. Near the point of gradient catastrophe of the hyperbolic system (1.2) one expects to have

$$u_x^i\sim\epsilon^{-1},\quad u_{xx}^i\sim\epsilon^{-2},\quad \text{etc.}$$

So, all the terms of the reducing transformation become of the same order and the formal series in ϵ diverge. How does this expected divergence influence the qualitative properties of the solutions to bi-Hamiltonian PDEs near the point of gradient catastrophe? The arguments of [13] suggest that, at least in the case $n=1$, the generic solutions must locally behave in a universal way near the point of gradient catastrophe. This behavior is rather different from the shock formation present in the solutions to the dissipative perturbations of hyperbolic systems. We postpone discussion of this behavior for subsequent publications.

Problem 3. Are there more wide classes of perturbations of systems of hyperbolic PDEs admitting reducing transformations? The natural candidate to be considered is the perturbation of the so-called semi-Hamiltonian systems in Tsarev's sense [44], i.e., hyperbolic systems written in the diagonal form and possessing a complete family of commuting flows. First results in this direction were obtained in the recent paper [36].

Problem 4. According to our results, classes of equivalence of semisimple bi-Hamiltonian structures depend at most on $n(n+1)$ arbitrary functions of one variable. Prove existence of such bi-Hamiltonian structures for an arbitrary choice of these functional parameters.

Appendix: Bi-Hamiltonian Structures of Hydrodynamic Type

In this appendix we will describe in more detail, following [17], the defining equations for semisimple bi-Hamiltonian structures of hydrodynamic type as well as their Lax pair representation. We will work in the canonical coordinates $u^1, \dots, u^n$ (see Lemma 1.5 above). Introduce the classical *Lamé coefficients*

$$H_i(u) := f_i^{-1/2}(u), \quad i = 1, \dots, n,$$

and the *rotation coefficients*

$$\gamma_{ij}(u) := H_i^{-1} \partial_i H_j, \quad i \neq j. \tag{A.1}$$

Here, as usual,

$$\partial_i = \frac{\partial}{\partial u^i};$$

no summation over repeated indices will be assumed within this section. The classical *Lamé equations*

$$\partial_k \gamma_{ij} = \gamma_{ik} \gamma_{kj}, \quad i, j, k \text{ distinct}, \tag{A.2}$$

$$\partial_i \gamma_{ij} + \partial_j \gamma_{ji} + \sum_{k \neq i, j} \gamma_{ki} \gamma_{kj} = 0, \quad i \neq j, \tag{A.3}$$

describe diagonal metrics of curvature zero.[7] Adding the equations

$$u^i \partial_i \gamma_{ij} + u^j \partial_j \gamma_{ji} + \sum_{k \neq i, j} u^k \gamma_{ki} \gamma_{kj} + \frac{1}{2}(\gamma_{ij} + \gamma_{ji}) = 0, \quad i \neq j, \tag{A.4}$$

one obtains the defining relations for semisimple Poisson pencils of hydrodynamic type. The solutions to the system (A.2)–(A.4) are parametrized by $n(n-1)$ arbitrary functions of one variable. Indeed, one can freely choose the functions

$$\gamma_{ij}(u_0^1, \dots, u^j, \dots, u_0^n)$$

[7] Integrability of the system (A.2)–(A.3) was discovered by Zakharov [46].

near a given point

$$u_0 = (u_0^1, \dots, u_0^n), \quad u_0^i \neq u_0^j, \ \ u_0^i \neq 0. \tag{A.5}$$

Equations (A.2)–(A.4) can be represented as the compatibility conditions of the linear system

$$\begin{gathered} \partial_i \psi_j = \gamma_{ji} \psi_i, \quad i \neq j, \\ \partial_i \psi_i + \sum_{k \neq i} \gamma_{ki} \frac{u^k - \lambda}{u^i - \lambda} \psi_k + \frac{1}{2\,(u^i - \lambda)} \psi_i = 0 \end{gathered} \tag{A.6}$$

("Lax pair" with the spectral parameter λ for (A.2)–(A.4)). The solutions to the linear system (A.6) are closely related to the common first integrals of the bi-Hamiltonian systems of hydrodynamic type, i.e., with the Casimirs of the Poisson pencil

$$\begin{gathered} \{\,\cdot\,, I\}_2^{[0]} - \lambda \{\,\cdot\,, I\}_1^{[0]} = 0, \quad I = \int P(u) dx, \\ \partial_i P(u) = \psi_i H_i, \quad i = 1, \dots, n. \end{gathered} \tag{A.7}$$

As we already know (see Lemma 1.6 above) the bi-Hamiltonian systems are all diagonal in the canonical coordinates

$$u_t^i + V^i(u) u_x^i = 0, \quad i = 1, \dots, n. \tag{A.8}$$

The characteristic velocities $V^i(u)$ are determined from the following linear system:

$$\partial_k \chi_i = \gamma_{ki} \chi_k, \quad i \neq k, \tag{A.9}$$

$$\chi_i = H_i V^i, \quad i = 1, \dots, n. \tag{A.10}$$

For the given rotation coefficients $\gamma_{ij}(u)$ satisfying (A.2)–(A.4), the general solution to (A.9) depends on n arbitrary functions of one variable. In particular, the Lamé coefficients $\chi_i = H_i(u)$ give a solution to (A.9). They correspond to the spatial translations $V^i(u) \equiv 1$, $i = 1, \dots, n$. Finally, to reconstruct the flat pencil of metrics starting from a given solution to (A.2)–(A.4) near a given point (A.5), one has to choose a solution $\chi_1(u), \dots, \chi_n(u)$ such that

$$\chi_i(u_0) \neq 0, \quad i = 1, \dots, n.$$

Then we put

$$g_1^{ij}(u) = \chi_i^{-2}(u) \delta_{ij}, \quad g_2^{ij}(u) = u^i \chi_i^{-2}(u) \delta_{ij}. \tag{A.11}$$

The flat coordinates of the metrics correspond to particular solutions of the system (A.6). Namely, to find flat coordinates for the first metric, one has to choose a fundamental system of solutions

$$\psi_i^\alpha(u), \quad \alpha = 1, \dots, n, \quad \det(\psi_i^\alpha(u_0)) \neq 0,$$

to the linear overdetermined system

$$\begin{aligned} &\partial_i \psi_j = \gamma_{ji}\psi_i, \quad i \neq j, \\ &\partial_i \psi_i + \sum_{k\neq i} \gamma_{ki}\psi_k = \frac{1}{2}\psi_i, \end{aligned} \tag{A.12}$$

obtained from (A.6) at $\lambda = \infty$. Then the flat coordinates v^α are defined by quadratures

$$dv^\alpha = \sum_{i=1}^n \chi_i \psi_i^\alpha du^i, \quad \alpha = 1, \dots, n. \tag{A.13}$$

Flat coordinates for the second metric are constructed in a similar way by using a fundamental system of solutions to (A.6) at $\lambda = 0$. We deduce that semisimple bi-Hamiltonian structures of hydrodynamic type with n dependent variables are parametrized by n^2 arbitrary functions of one variable. For $n \leq 2$ equations (A.2)–(A.4) are linear. So an explicit parametrization of the Poisson pencils is available [40]. The equations become nonlinear starting from $n \geq 3$. All nontrivial solutions known so far are obtained within the theory of Frobenius manifolds. In this case the rotation coefficients are symmetric,

$$\gamma_{ji} = \gamma_{ij}$$

(the so-called Egoroff metrics), and equations (A.2), (A.3), and (A.4) are reduced to isomonodromy deformations [12]. We will study a more general case in subsequent publications.

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