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ON HAMILTONIAN PROPERTIES OF POWERS OF SPECIAL HAMILTONIAN GRAPHS

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1. Introduction. For a connected graph G and positive integer n, the *n-th power* G^n of G is the graph having the same vertex set as G and such that u and v are adjacent in G^n if and only if the distance between u and v in G is at most n. A *hamiltonian graph* H is one containing a *hamiltonian cycle*, i.e., a cycle containing every vertex of H.

The connection between powers of graphs and hamiltonian graphs came into prominence chiefly because of a conjecture made independently by Nash-Williams and Plummer, namely, the square of every 2-connected graph is hamiltonian. It is easy to find connected graphs (of order 3 or more) whose squares are non-hamiltonian. However, Sekanina [12] (and, independently, Karaganis [11]) proved that the cube of every connected graph of order at least 3 is not only hamiltonian but is hamiltonian-connected. It was proved in [4] that the cube of every connected graph of order at least 4 is 1-hamiltonian. (A graph G is *hamiltonian-connected* if, for every two distinct vertices u and v of G, there exists a hamiltonian path u-v. A graph G is 1-*hamiltonian* if G and $G-v$ are hamiltonian for every vertex v of G.)

In 1971, Fleischner [7] presented a proof of the Nash-Williams-Plummer conjecture. With the aid of this result, it was shown [3] that the square of every 2-connected graph G is hamiltonian-connected and 1-hamiltonian (the latter result requiring a graph G to have order at least 4).

Some of the aforementioned results have been extended. A graph G of order p is *n-hamiltonian-connected* (see [5]), $1 \leqslant n \leqslant p-1$, if, for every two distinct vertices u and v of G, there exist paths u-v of each of the lengths $p-n, p-n+1, \ldots, p-1$. A graph G of order p is *n-hamiltonian* (see [6]),

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$0 \leqslant n \leqslant p-3$, if the removal of any k vertices, $0 \leqslant k \leqslant n$, from G results in a hamiltonian graph. If G is a connected graph of order p and $1 \leqslant n \leqslant p-3$, then G^{n+2} is n-hamiltonian-connected (cf. [1]), while it was proved in [2] that G^{n+2} is n-hamiltonian, and in [9] that G^{n+1} is n-hamiltonian for a 2-connected graph G. Our goal here is to generalize these results still further and present some related information.

All terms not defined here may be found in Harary [8].

2. n-hamiltonian graphs. We have already noted that if G is a connected graph of order p and $1 \leqslant n \leqslant p-3$, then G^{n+2} is n-hamiltonian. It would seem logical, therefore, that if G were not merely connected but rather had properties which were hamiltonian in nature, then the powers of G would be even more hamiltonian. This is spelled out below.

THEOREM 1. *If G is an n-hamiltonian graph of order $p \geqslant 3$, then G^m is $(n+2m-2)$-hamiltonian for all $m \geqslant 1$ and $n+2m \leqslant p-1$.*

Proof. Since the result holds trivially for $m = 1$, let $m \geqslant 2$. We adopt the convention of denoting a set of $k \geqslant 0$ arbitrary vertices of G by $V(k)$, with $V(0) = \emptyset$. In order to verify the theorem, we show that the graph $G^m - V(k)$ is hamiltonian for $0 \leqslant k \leqslant n+2m-2$. Since G is n-hamiltonian, the result follows for $0 \leqslant k \leqslant n$. Let $k = n+t$, where $1 \leqslant t \leqslant 2m-2$. Write $V(k) = V(n) \cup V(t)$. By the hypothesis, the graph $H = G - V(n)$ is hamiltonian and, therefore, contains a hamiltonian cycle C. The deletion of t vertices from H, $1 \leqslant t \leqslant 2m-2$, produces a collection of paths, say, $P_1, P_2, \ldots, P_r$, where the paths are consecutive on C. If no more than $m-1$ consecutive vertices of C are removed, then in H^m the terminal vertex of P_i is adjacent to the initial vertex of P_{i+1}, $i = 1, 2, \ldots, r$ $(P_{r+1} = P_1)$, so that H^m and thus $G^m - V(k)$ is hamiltonian. If, on the other hand, some set $V(m)$ of m (not less than 2) consecutive vertices of C are removed, then $P = C - V(m)$ is a path on $p-(n+m)$ vertices, and we may remove up to $m-2$ vertices from P^m to obtain a hamiltonian graph. This completes the proof.

The above-mentioned result is seen to be best possible for $n = 0$ by taking G to be a cycle. It is also the case that the result is best possible for $n = 1$; however, at this stage it is unknown what the situation is for general n.

We observe that an n-hamiltonian graph has the property that not only G, but certain maximal subgraphs of G are also hamiltonian. If the property "having a hamiltonian cycle" is replaced by "having a hamiltonian path", we may then study a wider class of graphs known as *traceable* graphs (see [10]). In particular, a graph G of order p is defined to be *n-traceable*, $0 \leqslant n \leqslant p-2$, if the graph $G - V(k)$ is traceable for all k satisfying $0 \leqslant k \leqslant n$. Clearly, then, 0-traceable graphs are merely the traceable graphs; and every n-hamiltonian graph is necessarily $(n+1)$-traceable. The traceable graphs are involved with powers of graphs, since the square of every traceable graph of order at least three is hamiltonian. Following the techniques of the preceding theorem, it is possible to prove the next result, which we merely state. Also, by considering G to be a path, we note that this result is best possible for $n = 0$.

THEOREM 2. *If G is an n-traceable graph of order $p \geqslant 3$, then G^m is $(n+m-1)$-traceable for all $m \geqslant 1$ and $n+m \leqslant p-1$.*

3. n-hamiltonian connected graphs. For any two distinct vertices u and v of a graph G of order p, let their *path number* $\pi(u, v)$ be the least integer (if one exists) such that there are paths of each of the lengths $\pi(u, v), \pi(u, v)+1, \ldots, p-2$ and $p-1$ between u and v. Then, for $1 \leqslant n \leqslant p-1$, a graph G is n-hamiltonian-connected if and only if $\pi(u, v) \leqslant p-n$ for all pairs of distinct vertices u and v of G.

For a real number x, the symbol $\{x\}$ denotes the greatest integer not less than x.

THEOREM 3. *If G is a connected graph on $p \geqslant 2$ vertices having diameter d^*, then the graph G^m is $(p-\{d^*/m\})$-hamiltonian-connected for all $m \geqslant 3$.*

Proof. Since the cube of a connected graph is hamiltonian-connected, it follows that the graph G^m is n-hamiltonian-connected for some $n \geqslant 1$. Let u and v be any two distinct vertices of G. Let T be a spanning tree in G such that the distance d between u and v is the same in G and T. Now, T has a unique path $P = a_0 a_1 \ldots a_d$, where $u = a_0$ and $v = a_d$. For $1 \leqslant i \leqslant d-1$, let the tree T_i be that component in the forest $T - \{a_{i-1}a_i, a_i a_{i+1}\}$ which contains the point a_i; the trees T_0 and T_d being defined similarly. Let T_i have $p_i \geqslant 1$ vertices so that

$$p = \sum_{i=0}^{d} p_i.$$

Let b_i be any vertex in T_i which is adjacent to a_i (if $p_i = 1$, then, for completeness, we assign a second label b_i to the vertex a_i).

For $p_i \geqslant 2$, consider the tree T_i. By successively removing the end-vertices of T_i (except a_i and b_i) one at a time, we can obtain a strictly monotonic decreasing sequence of trees $T_i(p_i-j)$ on p_i-j vertices such that

$$T_i = T_i(p_i) \supset T_i(p_i-1) \supset \ldots \supset T_i(p_i-j) \supset \ldots \supset T_i(2) = \langle a_i, b_i \rangle.$$

Since $T_i^m(p_i-j)$ is hamiltonian-connected, it contains a path between the vertices a_i and b_i in T_i^m of the length p_i-j-1, $0 \leqslant j \leqslant p_i-2$. Thus, in the graph T_i^m, $\pi(a_i, b_i) = 1$.

We assert that, in T^m, $\pi(a_0, a_d) \leqslant d$. If $p_i = 1$ for each i, then we are done. Otherwise, note that each b_i, $0 \leqslant i \leqslant d-1$, is adjacent to a_{i+1}, and b_{d-1} is adjacent to b_d in T^m. For any i satisfying $0 \leqslant i \leqslant d-1$ and $p_i \geqslant 2$, we can replace the edge between a_i and a_{i+1} in P by a path $P_i(p_i - j - 1)$ which starts at a_i, lies in T_i^m, ends at b_i, and has the length $p_i - j - 1$, $0 \leqslant j \leqslant p_i - 2$, followed by the edge $b_i a_{i+1}$. If $p_d \geqslant 2$, we replace the edge $a_{d-1}a_d$ of P by the edge $a_{d-1}b_d$, followed by the path $P_d(p_d - j - 1)$ in T_d^m, starting at b_d and ending at a_d and of the length $p_d - j - 1$, $0 \leqslant j \leqslant p_d - 2$. Clearly, when this extension of P is carried out through successive values of i, in the graph T^m there will exist paths of each of the lengths

$$d,\ d+1,\ \ldots,\ d+\sum_{i=0}^{d}(p_i - 1) = p - 1$$

between a_0 and a_d, i.e., $\pi(a_0, a_d) \leqslant d$.

If $d \leqslant m$, then the vertices of the path P induce a complete subgraph H in T^m, resulting in paths of each of the lengths $1, 2, \ldots, d$ between a_0 and a_d in H. This, together with the earlier result, shows that in T^m we must have $\pi(a_0, a_d) = 1$.

For $d > m$, the distance between a_0 and a_d in T^m is at most $\{d/m\}$. So in the graph H induced by the vertices of P in T^m there exist paths of each of the lengths $\{d/m\}, \{d/m\}+1, \ldots, d$ between the vertices a_0 and a_d. Thus $\pi(a_0, a_d) = \{d/m\}$ in T^m.

In fact, now we can make a uniform statement. If u and v are any two distinct vertices of the graph G such that the distance between them is d, then $\pi(u, v) = \{d/m\}$ in G^m. And, if d^* is the diameter of G, then, in the graph G^m, $\pi(u, v) \leqslant \{d^*/m\}$ for all pairs of distinct vertices u and v. Consequently, G^m is $(p - \{d^*/m\})$-hamiltonian-connected. This completes the proof.

That the preceding result cannot be improved is illustrated by a path P on p vertices with end-vertices u and v, where we note that $\pi(u, v)$ is not less than $\{(p-1)/m\}$ in P^m, $m \geqslant 3$.

THEOREM 4. *Let G be an n-hamiltonian-connected graph. Then, for $m \geqslant 3$, the graph G^m is $\big(p - \{(p-n)/m\}\big)$-hamiltonian-connected.*

Proof. By the hypothesis, between any two vertices of G there exists a path of length $p - n$. Therefore, $d^* \leqslant p - n$, where d^* is the diameter of G. So

$$(p - \{d^*/m\}) \geqslant \big(p - \{(p-n)/m\}\big),$$

and the desired result follows from Theorem 3.

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