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Calculus of Variations

On Harnack inequalities and singularities of admissible metrics in the Yamabe problem

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Abstract In this paper we study the local behaviour of admissible metrics in the k-Yamabe problem on compact Riemannian manifolds (M, g_0) of dimension $n \geq 3$. For $n/2 < k < n$, we prove a sharp Harnack inequality for admissible metrics when (M, g_0) is not conformally equivalent to the unit sphere S^n and that the set of all such metrics is compact. When (M, g_0) is the unit sphere we prove there is a unique admissible metric with singularity. As a consequence we prove an existence theorem for equations of Yamabe type, thereby recovering as a special case, a recent result of Gursky and Viaclovsky on the solvability of the k-Yamabe problem for $k > n/2$.

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1 Introduction

Let $(\mathcal{M}, g_0)$ be a compact Riemannian manifold of dimension $n \geq 3$ and $[g_0]$ the set of metrics conformal to g_0. For $g \in [g_0]$ we denote by

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)}g\right) \tag{1.1}$$

the Schouten tensor and by $\lambda(A_g) = (\lambda_1, \ldots, \lambda_n)$ the eigenvalues of A_g with respect to g (so one can also write $\lambda = \lambda(g^{-1}A_g)$), where Ric and R are respectively the Ricci tensor and the scalar curvature. We also denote as usual

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$$\sigma_k(\lambda) = \sum_{i_1<\cdots<i_k} \lambda_{i_1}\cdots\lambda_{i_k} \tag{1.2}$$

the kth elementary symmetric polynomial and

$$\Gamma_k = \{\lambda \in \mathbf{R}^n \mid \quad \sigma_j(\lambda) > 0 \quad \text{for } j = 1, \ldots, k\} \tag{1.3}$$

the corresponding open, convex cone in $\mathbf{R}^n$. Denote

$$[g_0]_k = \{g \in [g_0] \mid \quad \lambda(A_g) \in \bar{\Gamma}_k\}. \tag{1.4}$$

Following [3], we call a metric in $[g_0]_k$ *k-admissible*. In this paper we prove three main theorems pertaining to the cases $k > \frac{n}{2}$.

Theorem A *If $(\mathcal{M}, g_0)$ is not conformally equivalent to the unit sphere S^n and $\frac{n}{2} < k \leq n$, then $[g_0]_k$ is compact in C^0 and satisfies the following Harnack inequality, namely for any $g = \chi g_0 \in [g_0]_k$,*

$$\max_{x,y\in\mathcal{M}} \frac{\chi(x)}{\chi(y)} \leq \exp\left(C|x-y|^{2-\frac{n}{k}}\right) \tag{1.5}$$

for some fixed constant C depending only on n, k and $(\mathcal{M}, g_0)$, where $|x-y|$ denotes the geodesic distance in the metric g_0 between x and y.

By compactness we mean in the sense of restriction to conformal metrics with fixed volume. Equivalently the compactness can be understood as the set of all conformal factors χ with $\sup \chi = 1$. See also (3.28) below. When the manifold $(\mathcal{M}, g_0)$ is the unit sphere, the compactness is no longer true. In this case $(\mathcal{M}, g_0)$ with one point removed is conformally equivalent to the Euclidean space $\mathbf{R}^n$ so that without loss of generality, it suffices to study conformal metrics on $\mathbf{R}^n$. For our investigation we will allow singular metrics. Accordingly we call a metric $g = \chi g_0$ k-admissible if $\chi : \quad \mathcal{M} \to (0, \infty]$, χ is lower semi-continuous, $\not\equiv \infty$ and there exists a sequence of k-admissible metrics $g_m = \chi_m g_0$, $\chi_m \in C^2(\mathcal{M})$, such that $\chi_m \to \chi$ almost everywhere in $\mathcal{M}$. If g is k-admissible, then the function $v = \chi^{(n-2)/4}$ is subharmonic with respect to the operator

$$\square := -\Delta_g + \frac{n-2}{4(n-1)} R_g \tag{1.6}$$

and hence by the weak Harnack inequality [11,25], the set $\{\chi = \infty\}$ has measure zero. Our next result classifies the possible singularities of k-admissible metrics on $\mathbf{R}^n$.

Theorem B *Let g be k-admissible on $\mathbf{R}^n$ with $\frac{n}{2} < k \leq n$. Then either*

$$g(x) = \frac{C}{|x-x_0|^4} g_0(x) \tag{1.7}$$

for some point $x_0 \in \mathbf{R}^n$ and positive constant C, or the conformal factor χ is Hölder continuous with exponent $\alpha = 2 - \frac{n}{k}$, where g_0 is the standard metric on $\mathbf{R}^n$.

Remark Theorems A and B also extend to metrics where the condition $g \in [g_0]_k$ (namely $\lambda(A_g) \in \Gamma_k$) is replaced by $P_\delta(\lambda) > 0$, for some constant $\delta < 1/(n-2)$, where P_δ is the Pucci operator [11], given by,

$$P_\delta(\lambda) = \min \lambda_i + \delta \sum \lambda_i.$$

By (1.1), the Ricci curvature $\mu = (\mu_1, \ldots, \mu_n)$ is given by

$$\mu_i = (n-2)\lambda_i + \sum \lambda_j.$$

Hence the Ricci curvature $\mu_i \geq 0$ for all $1 \leq i \leq n$ if and only if $P_{\frac{1}{n-2}}(\lambda) \geq 0$. We remark that when $\lambda \in \Gamma_k$ and $k \geq \frac{n}{2}$, the Ricci curvature is nonnegative [15]. Indeed, for any $\lambda \in \Gamma_k$, let $f(x) = \frac{1}{2}\sum \lambda_k x_k^2$. By direct computation, $\Delta_p f \geq 0$ for $p \leq 2 + \frac{n(k-1)}{n-k}$ (this is the simplest case, namely $l = 1$, in Lemma 4.2 in [27]). Hence

$$\sum \lambda_i + \frac{n(k-1)}{n-k}\lambda_j \geq 0 \tag{1.8}$$

for every j. Hence $\mathrm{Ric}_g \geq 0$ if $\lambda(A_g) \in \Gamma_{n/2}$, and $\mathrm{Ric}_g > 0$ if $\lambda(A_g) \in \Gamma_k$ for $k > \frac{n}{2}$.

Theorems A and B have various interesting consequences, in particular to the existence of conformal metrics with prescribed k-curvature. This is the problem of finding a conformal metric $g \in [g_0]$ such that

$$\sigma_k(\lambda(A_g)) = f, \tag{1.9}$$

where f is a given positive smooth function on $\mathcal{M}$. When $f \equiv 1$, (1.9) is the *k-Yamabe problem*, which has been studied by many authors; see [1,22,26] for $k = 1$ and [5,6,10,14, 18,20,21,24,29] for $k \geq 2$. For $k > \frac{n}{2}$, it was recently resolved by Gursky and Viaclovski [18], by proving compactness results and characterizations of singularities for *solutions*. Our Theorems A and B above show that such results are already valid for admissible metrics, that is those in the conformal class $[g_0]$ in which solutions are sought. As well, Theorem A provides a sharp version of the solution estimates in [18].

By writing $g = v^{4/(n-2)} g_0$, we see that (1.9) is equivalent to the *conformal k-Hessian equation*

$$\sigma_k(\lambda(V)) = \varphi(x, v), \tag{1.10}$$

where

$$V = -\nabla^2 v + \frac{n}{n-2}\frac{\nabla v \otimes \nabla v}{v} - \frac{1}{n-2}\frac{|\nabla v|^2}{v} g_0 + \frac{n-2}{2} v A_{g_0}, \tag{1.11}$$

$\lambda(V)$ denotes the eigenvalues of the matrix V, and $\varphi = f v^{k\frac{n+2}{n-2}}$. When $k \geq 2$, (1.10) is a fully nonlinear partial differential equation, which is elliptic if the eigenvalues $\lambda(A_g) \in \Gamma_k$. Therefore to study problem (1.9), we always assume $[g_0]_k \neq \emptyset$. Under this assumption, the k-Yamabe problem has been solved in [24] if $2 \leq k \leq \frac{n}{2}$ and (1.9) is variational. Equation (1.9) is automatically variational when $k = 2$, but when $k \geq 3$, it is variational when the manifold is locally conformally flat. When $\frac{n}{2} < k \leq n$, the existence of solutions to (1.9) was proved in [18] for any smooth, positive function f; see also [6] for the solvability when $k = 2$ and $n = 4$, and [14,20] when the manifold is locally conformally flat. For completeness and to illustrate the application of Theorem A, we prove the following extension.

Theorem C *Let $(\mathcal{M}, g_0)$ be a compact n-manifold not conformally equivalent to the unit sphere S^n. Suppose $\frac{n}{2} < k \leq n$ and $[g_0]_k \neq \emptyset$. Then for any smooth, positive function f and any constant $p \neq k$, there exists a positive admissible solution to the equation*

$$\sigma_k(\lambda(V)) = f(x) v^p. \tag{1.12}$$

The solution is unique if $p < k$. When $p = k$, then there exists a unique constant $\theta > 0$ such that

$$\sigma_k(\lambda(V)) = \theta f(x) v^k \tag{1.13}$$

has a solution, which is unique up to a constant multiplication.

We may call the constant θ in (1.13) (with $f \equiv 1$) the *eigenvalue* of the conformal k-Hessian operator in (1.10). As a special case of Theorem C, letting $p = k\frac{n+2}{n-2}$, we obtain the existence of solutions to the k-Yamabe problem (1.9) for $\frac{n}{2} < k \le n$, as proved in [18]. As indicated above, the full strength of Theorem A is not necessary for these applications and solution estimates as in [18] would suffice. We also include some extensions of Theorem C at the end of Sect. 4.

As in [24] we will use conformal transformations of different forms,

$$g = \chi g_0 = v^{\frac{4}{n-2}} g_0 = u^{-2} g_0 = e^{-2w} g_0 \tag{1.14}$$

so that

$$u = v^{-2/(n-2)} = e^w. \tag{1.15}$$

We say u, v, or w is *conformally k-admissible*, or simply k-admissible if no confusion arises, if the metric g is k-admissible. In the smooth case, from the matrix V in (1.11), we see that u, w are k-admissible if the eigenvalues of the matrices

$$U = \left\{ u_{ij} - \frac{|Du|^2}{2u} g_0 + u A_{g_0} \right\}, \tag{1.16}$$

$$W = \left\{ w_{ij} + w_i w_j - \frac{1}{2} |Dw|^2 g_0 + A_{g_0} \right\} \tag{1.17}$$

lie in $\overline{\Gamma}_k$, the closure of Γ_k. Note that if g is the metric given by (1.7), then

$$v = \frac{C}{|x - x_0|^{n-2}} \tag{1.18}$$

is the fundamental solution of the Laplace operator.

The conformal k-Hessian equation is closely related to the k-Hessian equation

$$\sigma_k(\lambda(D^2 u)) = \varphi \quad \text{in} \quad \Omega, \tag{1.19}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain. For the k-Hessian equation (1.19), it is proved in [27] that when $\frac{n}{2} < k \le n$, a k-admissible function (relative to (1.19)) is locally Hölder continuous with Hölder exponent $\alpha = 2 - \frac{n}{k}$. The existence of solutions to (1.19) with right hand side $\varphi = f(x)|u|^p$ for some constant $p > 0$ was studied in [9] for $k \le \frac{n}{2}$ and in [8,30] for $k = n$. By the Hölder continuity one can extend the results in [8,30] to the cases $\frac{n}{2} < k \le n$. The argument in [30] uses a degree theory, which does not require a variational structure. We will employ the same degree argument to prove our Theorem C.

The proof of Theorem A can roughly be divided into two steps. In the first one we study the singularity behavior of admissible functions. Let $g_m = v_m^{\frac{4}{n-2}} g_0$ be a sequence of admissible metrics in $[g_0]_k$ for $k > \frac{n}{2}$. Then $v_m / \inf v_m$ converges in locally uniformly to an admissible function v. Moreover, if x_0 is a singular point of v, we prove that

$$v(x) = \frac{C_0 + o(1)}{|x - x_0|^{n-2}}, \tag{1.20}$$

where C_0 is a positive constant, $|x - x_0|$ denotes the geodesic distance from x to x_0 in the metric g_0. In the second step we show that if (1.20) holds at all singular points, then by Bishop's volume growth formula, as used in [18], w is smooth away from x_0, and the manifold $(\mathcal{M}, g_0)$ is conformally equivalent to the unit sphere.

The asymptotic behavior (1.20) has been established for the blow-ups of *solutions* to the classical Yamabe problem. Also a weak form of (1.20) was proved by Gursky and Viaclovsky [18], namely if $g_m = v_m^{\frac{4}{n-2}} g_0$ is a sequence of solutions to (1.5) for $k > \frac{n}{2}$, then $v_m / \inf v_m$ converges locally uniformly away from finitely many singular points to an admissible function v, and v satisfies

$$\frac{C_1}{|x - x_0|^{n-2}} \le v(x) \le \frac{C_2}{|x - x_0|^{n-2}} \tag{1.21}$$

at any singular point x_0. From (1.21) they proved that $(\mathcal{M}, g_0)$ is conformally equivalent to the unit sphere. Basically our proof in the second step is inspired by that in [18], however our proof is somewhat simpler due to the stronger asymptotic estimate (1.20).

In this paper we also introduced a *new* technique. That is we reduce the singularity analysis of a k-admissible function to its *maximal radial function*. Given an admissible function v, we say $\tilde{v}$ is the maximal radial function of v with center at x_0 if $\tilde{v}$ is a rotationally symmetric with respect to x_0, $\tilde{v} \le v$, and $\tilde{v} \ge \varphi$ for any radial function φ satisfying $\varphi \le v$. Obviously $\tilde{v}$, as a function of $|x - x_0|$, is monotone increasing, namely $\tilde{v}(r) \ge \tilde{v}(r')$ for any $0 < r' < r$. We reduce the proof of (1.20) for v to that for $\tilde{v}$. This new technique also applies to other nonlinear elliptic equations, which will be shown in a subsequent paper [28].

This paper is arranged as follows. In Sect. 2 we first prove Theorem B for radially symmetric, k-admissible functions defined on $\mathbf{R}^n$, then extend it to general k-admissible functions by considering the corresponding minimal radial function for $w = -\frac{2}{n-2} \log v$. In Sect. 3 we prove Theorem A. By studying the corresponding minimal radial function, we show that if an admissible function w is a k-admissible function on a manifold $\mathcal{M}$, then either w is Hölder continuous, or (1.20) holds at some singular point $x_0 \in \mathcal{M}$. We then show that if (1.20) occurs, then w is smooth away from x_0 and the manifold $\mathcal{M}$ is conformally equivalent to the unit sphere. In Sect. 4 we prove Theorem C by a degree argument.

In a subsequent paper [28], we will establish the asymptotic behavior (1.20) for any blow-up functions of solutions to the k-Yamabe problem (for $1 \le k \le \frac{n}{2}$) and the problem of prescribing more general symmetric curvature functions. In particular we prove the existence and compactness of solutions to the k-Yamabe problem for $k = \frac{n}{2}$.

2 Proof of Theorem B

2.1 Radial functions

We first demonstrate our idea by considering radially symmetric functions. Let w be a radially symmetric, k-admissible function on $\mathbf{R}^n \backslash \{0\}$. For any given point $x \ne 0$, by a rotation of axes we assume $x = (0, \ldots, 0, r)$. Regard w as a function of $r = |x|$, $r \in (0, \infty)$. Then the matrix W in (1.17) is diagonal,

$$W = \operatorname{diag}\left(\frac{1}{r} w' - \frac{1}{2} w'^2, \ldots, \frac{1}{r} w' - \frac{1}{2} w'^2, w'' + \frac{1}{2} w'^2\right).$$

Denote $a = w'' + \frac{1}{2}w'^2$ and $b = \frac{1}{r}w' - \frac{1}{2}w'^2$. We have

$$\begin{aligned}\sigma_k(\lambda(W)) &= b^k C_{n-1}^k + ab^{k-1}C_{n-1}^{k-1} \\ &= C_{n-1}^{k-1}b^{k-1}\left(a + \frac{n-k}{k}b\right).\end{aligned} \tag{2.1}$$

Since $\lambda(W) \in \overline{\Gamma}_k$ and $k > \frac{n}{2}$,

$$b = \frac{w'}{r} - \frac{1}{2}w'^2 \geq 0, \tag{2.2}$$

$$a + \frac{n-k}{k}b = \left(w'' + \frac{w'}{r}\right) - (1-\theta)\left(\frac{w'}{r} - \frac{1}{2}w'^2\right) \geq 0, \tag{2.3}$$

where $\theta = \frac{n-k}{k} < 1$. It follows that

$$0 \leq w' \leq \frac{2}{r}, \tag{2.4}$$

$$w'' + \frac{w'}{r} \geq 0. \tag{2.5}$$

By sub-harmonicity of w, we have $w'(r) = 0$ for $r \in (0, r_0)$ if $w'(r_0) = 0$. Note that (2.5) can also be written as $(rw')' \geq 0$. Therefore we have

Lemma 2.1 *The function rw' is nonnegative, monotone increasing, and $rw' \leq 2$.*

It follows that w must be locally uniformly bounded from above, namely if $w(r_0) \leq 0$, then $w \leq C$ for some constant C depending only on r_0. Next we prove

Lemma 2.2 *The function w is either Hölder continuous in $\mathbf{R}^n$ with exponent $\alpha = 2 - \frac{n}{k}$, or*

$$w(r) = 2\log r + C \tag{2.6}$$

for some constant C.

Proof It suffices to prove that w is either Hölder continuous near $r = 0$ or (2.6) holds. If w is not Hölder continuous near $r = 0$, then $w' > 0$ as $v = e^{\frac{n-2}{2}w}$ is superharmonic with respect to the operator (1.6).

If $rw' \not\equiv 2$, then by Lemma 2.1, $\lim_{r\to 0} rw' = c_0 < 2$. For any $c_1 \in (c_0, 2)$,

$$w'' + \frac{w'}{r} \geq (1-\theta)\frac{w'}{r}\left(1 - \frac{1}{2}rw'\right) \geq (1-\theta)\left(1 - \frac{c_1}{2}\right)\frac{w'}{r} \tag{2.7}$$

if r is sufficiently small. Hence

$$\frac{w''}{w'} + \frac{\sigma}{r} \geq 0,$$

where $\sigma = 1 - (1-\theta)(1 - \frac{c_1}{2}) < 1$. We obtain

$$\log(w'r^\sigma)\Big|_r^{r_0} \geq 0.$$

Hence

$$w' \leq \frac{C}{r^\sigma}. \tag{2.8}$$

Hence w is bounded and continuous.

To show that w is Hölder continuous with Hölder exponent $\alpha = 2 - \frac{n}{k}$, by Lemma 2.1 it suffices to prove it at $r = 0$. Note that

$$a + \theta b = w'' + \theta \frac{w'}{r} + \frac{1-\theta}{2} w'^2 \geq 0.$$

Hence

$$\frac{w''}{w'} + \frac{\theta}{r} \geq -\frac{1-\theta}{2} w'.$$

Taking integration from r to r_0, we obtain

$$\log(w' r^\theta) \big|_r^{r_0} \geq C.$$

Hence

$$w' \leq \frac{C}{r^\theta}, \tag{2.9}$$

so that w is Hölder continuous with exponent $1 - \theta = 2 - \frac{n}{k}$. ◻

Remark 2.1 The Hölder continuity also follows from [27]. Let $u = e^w$ be as in (1.15). From the matrix U in (1.16), one sees that u is k-admissible with respect to the k-Hessian operator $\sigma_k(\lambda(D^2u))$. To show that u is Hölder continuous with exponent $\alpha = 2 - \frac{n}{k}$, one applies the comparison principle to u and $\varphi(x) = C|x-y|^{2-n/k} + u(y)$ (relative to the k-Hessian equation), where C is chosen such that $u \leq \varphi$ on $\partial B_R(y)$, ones sees that $u(x) - u(y) \leq \varphi(x)$ in $B_R(y)$.

It follows that for any constant $c > 0$, $w_c = \max(w, -c)$ is also Hölder continuous with exponent $2 - \frac{n}{k}$. In particular, if w_m converges to w a.e., then w_m converges to w uniformly in $\{w > -c\}$ for any $c > 0$.

2.2 Proof of Theorem B

Let w be a k-admissible function. For any $h \in \mathbf{R}$, denote $\Omega_h = \{w < h\}$. Since w is upper semi-continuous, Ω_h is an open set. For any given point 0, we define a function $\tilde{w}$ of one variable r by

$$\tilde{w}(r) = \inf\{h : \operatorname{dist}(0, \partial\Omega_h) > r\}. \tag{2.10}$$

We call $\tilde{w}$ the *minimal radial function* of w (with respect to 0). It is a function of $|x|$, satisfying $\tilde{w} \geq w$ and $\tilde{w} \leq \varphi$ for any radial function φ satisfying $\varphi \geq w$. Obviously $\tilde{w}$ is monotone increasing, namely $\tilde{w}(r) \geq \tilde{w}(r')$ for any $0 < r' < r$.

Let $x_h \in \partial\Omega_h$ such that $|x_h| = r_h := \operatorname{dist}(0, \partial\Omega_h)$. Assume that $\partial\Omega_h$ and w are smooth at x_h. Rotate the axes such that $x_h = (0, \ldots, 0, r_h)$. Then the x_n-axis is the outer normal of $\partial\Omega_h$ at x_h. Hence

$$\begin{aligned} \tilde{w}(r_h) &= w(x_h), \\ \tilde{w}(r_h + t) &\geq w(x_h + te_n) \end{aligned} \tag{2.11}$$

for t near 0, where $e_n = (0, \ldots, 0, 1)$. We obtain

$$\begin{aligned} \tilde{w}'(r_h) &= w_n(x_h) = |Dw|(x_h), \\ \tilde{w}''(r_h) &\geq w_{nn}(x_h) \end{aligned} \tag{2.12}$$

provided $\tilde{w}$ is twice differentiable point at r_h.

Let $\kappa_1, \ldots, \kappa_{n-1}$ be the principal curvatures of $\partial\Omega_h$ at x_h. Then, after a rotation of the axes $(x_1, \ldots, x_{n-1})$,

$$w_{ij} = |Dw|\kappa_i\delta_{ij} \quad i, j \le n-1. \tag{2.13}$$

By our choice of x_h, we have

$$\kappa_i \le \frac{1}{r}, \tag{2.14}$$

where $r = r_h$. Hence the matrix

$$(w_{ij})_{i,j=1}^{n-1} \le \frac{1}{r}|Dw|I. \tag{2.15}$$

At x_h, the matrix W is given by

$$W = \left\{w_{ij} + w_i w_j - \frac{1}{2}|Dw|^2 I\right\}$$

$$= \begin{pmatrix} w_{11} - \frac{1}{2}|Dw|^2, & 0, & \cdots, & w_{1n} \\ 0, & w_{22} - \frac{1}{2}|Dw|^2, & \cdots, & w_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ w_{1n}, & w_{2n}, & \cdots, & w_{nn} + \frac{1}{2}|Dw|^2 \end{pmatrix}$$

Let

$$\hat{W} = \text{diag}\left(w_{11} - \frac{1}{2}|Dw|^2, w_{22} - \frac{1}{2}|Dw|^2, \ldots, w_{nn} + \frac{1}{2}|Dw|^2\right) \tag{2.16}$$

be a diagonal matrix. By Lemma 2.3 below, the eigenvalues $\lambda(\hat{W}) \in \overline{\Gamma}_k$.

Lemma 2.3 *Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix with eigenvalues $\lambda(A) \in \Gamma_k$. Then $\lambda(\hat{A})) \in \Gamma_k$, where $\hat{A} = \begin{pmatrix} A', & 0 \\ 0, & a_{nn} \end{pmatrix}$, $A' = (a_{ij})_{1\le i,j\le n-1}$.*

In particular if $\lambda(A) \in \Gamma_k$, then the vector $(a_{11}, \ldots, a_{nn}) \in \Gamma_k$.

Proof It suffices to prove $\sigma_j(\lambda(\hat{A})) \ge 0$ for all $j = 1, \ldots, k$. Let us prove it for $j = k$. Recalling that $\sigma_k(\lambda(A))$ is the sum of all principal $k \times k$ minors, we have

$$\sigma_k(\lambda(A)) = \sigma_k(\lambda(\hat{A})) - \sum_{i<n} \sigma_{k-2}(\lambda(A_{|in}))a_{in}^2, \tag{2.17}$$

where $A_{|ij}$ denotes the matrix obtained by canceling the ith row and jth column of A. Since $\lambda(A) \in \overline{\Gamma}_k$, we have

$$\sigma_{k-2}(\lambda(A_{|in})) = \frac{\partial^2\sigma_k(\lambda(A))}{\partial a_{ii}\partial a_{nn}} \ge 0. \tag{2.18}$$

Hence $\sigma_k(\lambda(\hat{A})) \ge \sigma_k(\lambda(A)) \ge 0$. □

From (2.15),

$$\hat{W} \le \text{diag}\left(\frac{1}{r}\tilde{w}' - \frac{1}{2}(\tilde{w}')^2, \ldots, \frac{1}{r}\tilde{w}' - \frac{1}{2}(\tilde{w}')^2, \tilde{w}'' + \frac{1}{2}(\tilde{w}')^2\right). \tag{2.19}$$

Therefore as in Sect. 2.1, we see that $\tilde{w}$ satisfies

$$\frac{\tilde{w}'}{r} - \frac{1}{2}(\tilde{w}')^2 \geq 0 \tag{2.20}$$

$$\left(\tilde{w}'' + \frac{\tilde{w}'}{r}\right) - (1-\theta)\left(\frac{\tilde{w}'}{r} - \frac{1}{2}(\tilde{w}')\right)^2 \geq 0 \tag{2.21}$$

if $\tilde{w}$ is twice differentiable at r.

To proceed further we make a remark.

Remark 2.2 Inequalities (2.20) and (2.21) are exactly (2.2) and (2.3). In the present case, the function $\tilde{w}$ may not be smooth. But by definition, w can be approximated by smooth k-admissible functions. Hence $\tilde{w}$ can be approximated by piecewise smooth functions, which satisfy $\lim_{r\to r_0^-} \tilde{w}'(r) \leq \lim_{r\to r_0^+} \tilde{w}'(r)$ by (2.11).

For a piecewise smooth function w with nonsmooth points $r_1 > r_2 \cdots > r_j > \cdots$ at which $\lim_{r\to r_j^-} \tilde{w}'(r) < \lim_{r\to r_j^+} \tilde{w}'(r)$, we can mollify w at the points r_j to get a smooth function w^* which satisfies $\lim_{r\to 0} w^*(r) = \lim_{r\to 0} w(r)$, and also satisfies (2.20) and (2.21). The proof of Lemma 2.2 implies that if $\lim_{r\to 0} w^*(r) = -\infty$, then $r(w^*)' \equiv 2$; if $\lim_{r\to 0} w^*(r) > -\infty$, then w^* is Holder continuous with its Holder norm independent of the mollification.

We can now prove Theorem B easily. First we consider the case when w is unbounded from below.

Lemma 2.4 *Let w be a k-admissible function which is locally unbounded from below, then there exists a point $x_0 \in \mathbf{R}^n$ and a constant C such that*

$$w(x) \equiv 2\log|x - x_0| + C. \tag{2.22}$$

Proof If w is locally unbounded from below, the singular set $S = \bigcap_{\{c<0\}}\{w < c\}$ is not empty. Choose a point $0 \in S$. By (2.20) and (2.21), and from the argument in Sect. 2.1, we must have $\tilde{w}(r) = 2\log r + C$ for some constant C.

Let $\hat{w} = 2\log|x| + C$. Then

$$\sigma_1(\lambda(W_{\hat{w}})) = 0,$$
$$\sigma_1(\lambda(W_w)) \geq \sigma_k^{1/k}(\lambda(W_w)) \geq 0,$$

where $W_{\hat{w}}$ is the matrix corresponding to $\hat{w}$, given in (1.17). Note that $\sigma_1(\lambda(W))$ is indeed the Laplace operator in $\mathbf{R}^n$ (in terms of v, by the relation (1.15)). By our definition of $\tilde{w}$, we have $\hat{w} \geq w$. Since $\tilde{w} = 2\log r + C$, we see that $w - \hat{w}$ attains its local maximum 0 at some interior point. By the maximum principle for the Laplace equation, we conclude that $w \equiv \hat{w}$. □

Next we consider the case when w is bounded from below.

Lemma 2.5 *Let w be a k-admissible function w. Suppose w is bounded from below. Then w is Hölder continuous with exponent $\alpha = 2 - \frac{n}{k}$.*

Proof For any given point x_0, we may take x_0 as the origin and let $\tilde{w}$ be the minimal radial function of w with respect to x_0, as defined in (2.10). Then to prove that w is Hölder continuous at x_0 with exponent $\alpha = 2 - \frac{n}{k}$, it suffices to show that $\tilde{w}$ is Hölder continuous with exponent α. But by (2.20), (2.21), the Hölder continuity of $\tilde{w}$ readily follows from the argument in Sect. 2.1, see (2.9). □

The Hölder continuity also follows from Remark 2.1 above.

Note that the function $w = 2\log|x|$ is k-admissible. By truncating at $w = -K$ (for large K), we see that the set of Hölder continuous k-admissible functions is not compact.

2.3 Applications

First we remark that, by the above proof, Theorem B also holds for k-admissible functions defined on a domain. Here we restate the theorem for the function $v = e^{-\frac{n-2}{2}w}$. Note that by Lemma 2.1, w is locally uniformly bounded, and so v is locally strictly positive when $k > \frac{n}{2}$.

Theorem B′ *Let Ω be a domain in $\mathbf{R}^n$. Let v be a k-admissible function in Ω with $\frac{n}{2} < k \le n$. If v is unbounded from above near some point $x_0 \in \Omega$, then*

$$v(x) = C|x - x_0|^{2-n}. \tag{2.23}$$

Otherwise v is locally Hölder continuous in Ω with exponent $\alpha = 2 - \frac{n}{k}$.

It was proved in [20] that if v is a k-admissible function, so is the function v_ψ in a punctured small ball $B_r(0)\backslash\{0\}$, where

$$v_\psi = |J_\psi|^{\frac{n-2}{2n}} v \circ \psi \tag{2.24}$$

$\psi(x) = \frac{x}{|x|^2}$, and J_ψ is the Jacobian of the mapping ψ. From Theorem B′ we have

Corollary 2.5 *Let v be a k-admissible function defined in $\mathbf{R}^n\backslash B_1(0)$ with $\frac{n}{2} < k \le n$. Then either $v \equiv constant$ or $|x|^{n-2}v(x)$ converges to a positive constant as $x \to \infty$.*

Proof We cannot apply Theorem B′ directly, as the function v_ψ has a singular point at 0. Denote $w = \frac{-2}{n-2}\log v_\psi$. If $w(x) \to -\infty$ as $x \to 0$, the argument in Sect. 2.2 implies that $w = 2\log|x| + C$. Changing back to v, we obtain $v \equiv$ constant. Otherwise it suffices to show that w is continuous at 0.

Let $w(0) = \overline{\lim}_{x\to 0} w(x)$ so that w is upper semi-continuous. If $a := \underline{\lim}_{x\to 0} w(x) < w(0)$, by subtracting a constant we assume $w(0) = 0$. Let $x_m \to 0$ such that $w(x_m) = -\frac{a}{2}$. Let $\tilde{w} = \tilde{w}_{x_m}$ be the minimal radial function of w with respect to x_m, as defined in (2.10). We claim that when m is sufficiently large, the point x_h in (2.11) at $h = 0$ cannot be the origin. In other words, there is a point y_m with $w(y_m) = 0$ such that $|y_m - x_m| < |x_m|$. Indeed, if $x_h = 0$, by the Hölder continuity of w in $B_{r_m}(x_m)$, where $r_m = |x_m|$ (Remark 2.1), we see that $w(x) \le -\frac{a}{2}$ in $B_{\delta r_m}(x_m)$, for some $\delta > 0$ independent of m. Namely $v > e^{\frac{(n-2)a}{4}} > 1$ in $B_{\delta r_m}(x_m)$, where $v = v_\psi$. Recall that $\underline{\lim}_{x\to 0} v(x) \ge 1$, by the mean value inequality for super-harmonic function, we have

$$v(0) \ge \frac{1}{|B_{2r_m}|(0)} \int_{B_{2r_m}(0)} v(x)dx > 0.$$

We reach a contradiction.

When the singular point $x = 0$ is not the extreme point x_h in (2.11), the argument in Sect. 2.2 applies to $\tilde{w} = \tilde{w}_{x_m}$ in $B_1(0)$. In particular $\tilde{w}$ is uniformly Hölder continuous. Hence if $w(0) = 0$ and $w(x_m) \le -1$, we have $|x_m| \ge c_0 > 0$ for some c_0 independent of m. This is again a contradiction. Hence w is continuous at 0, and so $|x|^{n-2}v(x)$ converges to a positive constant as $x \to \infty$. □

Theorem B also implies the non-existence of solutions to the Dirichlet problem in general. Let Ω be a bounded domain in $\mathbf{R}^n$ which is not a ball. Then if $k > \frac{n}{2}$, there is no solution to the Dirichlet problem

$$\begin{aligned} \sigma_k(\lambda(V)) &= f \quad \text{in} \quad \Omega, \\ v &= c \quad \text{on} \quad \partial\Omega \end{aligned} \tag{2.25}$$

in general, where c is any positive constant, and f is a positive smooth function. Indeed, let $\{f_m\}$ be a sequence of smooth, positive functions which converges to zero locally uniformly in $\Omega\backslash\{x_0\}$ such that $\sup v_m \to \infty$, where v_m is the corresponding solution (if the solution v_m does not exist, we are through). Then v_m must converge to the function $v = C|x - x_0|^{2-n}$ by Theorem B. Hence Ω must be a ball centered at x_0.

For the existence of solutions to the Dirichlet problem, it was proved recently by Guan [12] that for any smooth, bounded domain with smooth boundary data, the Dirichlet problem is classically solvable if there exists a smooth sub-solution with the same boundary trace.

3 Proof of Theorem A

3.1 Hölder continuity

We start with a Hölder continuity property of k-admissible functions.

Lemma 3.1 *Let $(\mathcal{M}, g_0)$ be a compact manifold. Suppose $g = u^{-2}g_0 \in [g_0]_k$ and $k > \frac{n}{2}$. Then u is Hölder continuous with exponent $\alpha = 2 - \frac{n}{k}$,*

$$\frac{u(x) - u(y)}{|x - y|^\alpha} \leq C \int_{\mathcal{M}} u, \tag{3.1}$$

where C is independent of u.

Proof By approximation it suffices to prove (3.1) for smooth functions. For any given point $0 \in \mathcal{M}$, there exists a conformal normal metric [2,4,16], still denoted by g_0, such that in the normal coordinates at 0,

$$\det(g_0)_{ij} \equiv 1 \quad \text{near} \quad 0. \tag{3.2}$$

Let

$$u_0(x) = |x|^{2-\frac{n}{k}}, \tag{3.3}$$

where $|x|$ denotes the geodesic distance from 0. Note that under condition (3.2), the Laplacian Δ on $\mathcal{M}$ is equal to the Euclidean Laplacian when applying to functions of $r = |x|$ alone [19,23]. Hence

$$\Delta_{g_0} u_0 = \frac{n(k-1)(2k-n)}{k^2} r^{-\frac{n}{k}}. \tag{3.4}$$

Denote by

$$P[u] = \min \lambda_i + \delta \sum_i \lambda_i, \quad \left(\delta = \frac{n-k}{n(k-1)}\right) \tag{3.5}$$

the Pucci minimal operator [11], where $(\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of the Hessian matrix $(\nabla_{ij} u_0)$. Obviously we have

$$\min \lambda_i \le \partial_r^2 u_0 = -\frac{(2k-n)(n-k)}{k^2} r^{-\frac{n}{k}}. \tag{3.6}$$

Therefore u_0 satisfies

$$P[u_0] \le 0 \quad \text{in } B_{0,r} \backslash \{0\}.$$

where $B_{y,r}$ denotes the geodesic ball with center y and radius r.

On the other hand, since $\lambda(U) \in \overline{\Gamma}_k$, where U is given in (1.16), we have $\lambda(u_{ij} + uA_{g_0}) \in \overline{\Gamma}_k \subset \overline{\Gamma}_1$. Namely $\Delta u + \mathrm{tr}(A_{g_0})u \ge 0$. By the weak Harnack inequality [11] it follows

$$\sup u \le C \int_{\mathcal{M}} u, \tag{3.7}$$

where C is a constant independent of u. Therefore to prove (3.1) we may assume that $\int_{\mathcal{M}} u = 1$ and u is uniformly bounded.

Let $u_a = u + a|x|^2$. Then $\nabla^2 u_a > \nabla^2 u + aI$ near 0, where I is the unit matrix. Since $\lambda(\nabla^2 u + uA_{g_0}) \in \overline{\Gamma}_k$, we have $\lambda(\nabla^2 u_a) \in \Gamma_k$ when a is suitably large. Taking $l = 1$ in the proof of Lemma 4.2 in [27], one has

$$\lambda_i + \frac{n-k}{n(k-1)} \sum_i \lambda_i \ge 0, \tag{3.8}$$

namely $P[u_a] \ge 0$ near 0. Hence by applying the comparison principle to the functions u_a and u_0 with respect to the operator P, we conclude the Hölder continuity (3.1). □

Remark 3.1 The estimate (3.1) (with exponent $\alpha < 2 - \frac{n}{k}, k > \frac{n}{2}$) also follows from gradient estimates from our reduction to p-Laplacian subsolution in [27]. Since $\lambda(U) \in \Gamma_k$, we have $\lambda(D^2 u + uA_{g_0}) \in \Gamma_k$. By (3.8) it follows that

$$\Delta_p u := \nabla_i (|\nabla u|^{p-2} \nabla_i u) \ge -Cu|\nabla u|^{p-2} \tag{3.9}$$

for $p - 2 = \frac{n(k-1)}{n-k}$ and some constant C. From our argument in [27], we obtain $\int_{\mathcal{M}} |\nabla u|^q \le C$ for any $q < nk/(n-k)$, whence by the Sobolev inequality, we infer (3.1) for $\alpha < 2 - \frac{n}{k}$; (see also [17]).

By the relation $u = e^w$, we have the following

Corollary 3.2 *Let w be a k-admissible function. Suppose $w \le 0$. Then for any $K > 0$, there exists $C = C_K > 0$, independent of w, such that when $w(y) > -K$,*

$$\frac{w(x) - w(y)}{|x-y|^\alpha} \le C, \tag{3.10}$$

with the same α as above.

From (3.10), we see that if $w(x) \le -K - 1$, then $|x - y| \ge C_{K+1}^{1/\alpha}$. Also note that in Corollary 3.2, if we assume that $w \le 0$ in $B_{y,r}$, then (3.10) holds for $x, y \in B_{y,r/2}$ for some C depending on r.

3.2 Singularity behaviour of k-admissible functions

Suppose w is a k-admissible function. At any given point $0 \in \mathcal{M}$, we choose a conformal normal coordinate near 0 such that (3.2) holds. In the conformal metric, the Ricci curvature vanishes at 0 [19,23]. Hence

$$|A_{g_0}(x)| \le Cr \quad \text{near } 0, \tag{3.11}$$

where r is the geodesic distance from 0. Let $\tilde{w}$ be the minimal radial function of w with respect to 0, as defined in (2.10). Then the argument thereafter is still valid, except that (2.14) should be replaced by $\kappa_i \le \frac{1}{r} + C$. Also note that in the conformal normal coordinate, at the point $(0, \dots, 0, r)$ in the exponential map, the metric $g_0 = I + J$, where I is the unit matrix, $|J| = O(r)$ and its nth-row and nth-column vanishes. Hence from (2.19), we have

$$(\tilde{b}, \dots, \tilde{b}, \quad \tilde{a}) \in \overline{\Gamma}_k, \tag{3.12}$$

where

$$\tilde{b} = \left(\frac{1}{r} + C\right)\tilde{w}' - \frac{1 - Cr}{2}(\tilde{w}')^2 + Cr,$$
$$\tilde{a} = \tilde{w}'' + \frac{1}{2}(\tilde{w}')^2 + Cr.$$

Similarly to the argument in Sect. 2.1, we have $\tilde{b} \ge 0$ and $\tilde{a} + \frac{n-k}{k}\tilde{b} \ge 0$. Note that $\tilde{b} \ge 0$ is equivalent to that $\frac{1}{2}(\tilde{w}')^2 \le \left(\frac{1}{r} + C\right)\tilde{w}' + Cr$. Hence

$$\tilde{a} + \frac{n-k}{k}\tilde{b} \ge \left[\tilde{w}'' + \left(\frac{1}{r} + C\right)\tilde{w}' + Cr\right] - (1 - \theta)\left[\left(\frac{1}{r} + C\right)\tilde{w}' - \frac{1}{2}(\tilde{w}')^2 + Cr\right] \ge 0,$$

where $\theta = \frac{n-k}{k}$ is as in (2.3). It follows that, when $r > 0$ is small,

$$\tilde{w}' \le \frac{2}{r} + \frac{Cr}{\tilde{w}'} + C, \tag{3.13}$$

$$\tilde{w}'' + \left(\frac{1}{r} + C\right)\tilde{w}' + Cr \ge 0. \tag{3.14}$$

From (3.13),

$$\tilde{w}' \le \frac{2}{r} + C$$

for a different C. It follows that $\tilde{w}(r) > -\infty$ for any $r > 0$, namely the set $\{w = -\infty\}$ cannot an open set containing the origin 0. But this property also follows from the weak Harnack inequality for the operator (1.6), as noted there.

From (3.14), we therefore obtain

$$(r\tilde{w}')' + C \ge 0.$$

It follows that $r\tilde{w}' + Cr$ is increasing. By the compactness of $\mathcal{M}$, a k-admissible function w must be bounded from above.

If $r\tilde{w}' < 2$ near $r = 0$, then similarly to (2.7) (2.8), $\tilde{w}$ is bounded and Hölder continuous. If $r\tilde{w}' \to 2$ as $r \to 0$, then $r\tilde{w}' + Cr \ge 2$, namely $\tilde{w}' \ge \frac{2}{r} - C$. Hence we obtain

$$\frac{2}{r} + C \ge \tilde{w}' \ge \frac{2}{r} - C. \tag{3.15}$$

We obtain

$$\tilde{w}(r) = 2\log r + C' + O(r). \tag{3.16}$$

By subtracting a constant we assume that $C' = 0$.

Lemma 3.3 *If $\tilde{w}$ satisfies (3.16), then near* 0,

$$w(x) = 2\log|x| + o(1). \tag{3.17}$$

Proof We prove (3.17) by a blow-up argument. In a normal coordinate system at 0, let $y = c_m x$ and $w_m(y) = w(x) + 2\log c_m$, where c_m is any sequence converging to infinity. Let $\tilde{w}_m$ be the radial function corresponding to w_m. Then by (3.16),

$$\tilde{w}_m(r) = 2\log r + O(c_m^{-1}). \tag{3.18}$$

Hence $\tilde{w}_m \to 2\log r$.

For any fixed $r_0 > 0$ small, let $w_m(y_m) = \tilde{w}_m(r_0)$ $(|y_m| = r_0)$. We may assume that $y_m \to y_0$. By the Hölder continuity (Corollary 3.2), we may also assume that in a neighborhood of y_0, w_m converges uniformly to w_∞. Then w_∞ is a k-admissible function defined on $\mathbf{R}^n$. The comparison principle argument of Lemma 2.4 implies that $w_\infty \equiv 2\log r$ in a neighborhood of y_0. The Hölder continuity in Corollary 3.2 implies that if $w_\infty = 2\log r$ at some point, w_∞ is well-defined nearby. The comparison principle then implies that $w_\infty \equiv 2\log r$ near the point. Hence $\tilde{w}_m$ converges uniformly to $w_\infty = 2\log r$ on $|x| = r_0$, and converges locally uniformly to $w_\infty = 2\log r$ in $\mathbf{R}^n\backslash\{0\}$. Hence (3.17) is proved. □

Note that (3.17) is equivalent to (1.20), by the relation $v = e^{-\frac{n-2}{2}w}$.

Remark 3.2 From the proof of Lemma 3.3, we see that w has only isolated singularities. For if there is a sequence of singular points $x_m \in \mathcal{M}$ which converges to a point 0, we may choose $c_m = |x_m|^{-1}$ in the above argument. Then the limit function w_∞ has at least two singular points 0 and $x^* = \lim x_m/|x_m|$. To see that x^* is a singular point of the limit function w_∞, we notice that the constant C' is uniformly bounded from above if w is negative in a neighbourhood of 0, which in turn implies that $\lim_{x\to x^*} w_\infty(x^*) = -\infty$. But the above argument shows that $w_\infty = 2\log r$. This is a contradiction.

Similarly one can show that the set of all isolated singular points has no limit points, for if there is a sequence of singular points $y_m \to 0$, we may for each $m > 1$ choose a conformal normal coordinate with origin at y_m and consider the sequence $x_m = y_{m+1} - y_m$. The above argument also leads to a contradiction.

Next we show that w has at most one singular point.

Lemma 3.4 *Let w be a k-admissible function. Then the singularity set*

$$S_w = \bigcap_{h<0}\{x \in \mathcal{M} \mid w(x) < h\} \tag{3.19}$$

contains at most one point.

Proof We adapt the corresponding argument in [18]. If S_w is not empty, it consists of finitely many isolated points. Let $g = e^{-2w}g_0$. By Lemma 3.3, $(\mathcal{M}\backslash S_w, g)$ is a complete manifold with finitely many ends. Now fixing a point $y \notin S_w$, we consider the ratio

$$Q(r) = \frac{\mathrm{Vol}(B_{y,r})}{r^n}, \tag{3.20}$$

where $B_{y,r} = B_{y,r}[g]$ is the geodesic ball of $(\mathcal{M}, g)$. By definition, as well as the local Hölder continuity (Lemma 3.1), there is a sequence of smooth k-admissible functions w_m which converges to w locally uniformly. It is easy to verify that for any fixed y and r, $\text{Vol}(B_{y,r}[g_m]) \to \text{Vol}(B_{y,r}[g])$ as $m \to \infty$, where $g_m = e^{-2w_m} g_0$. Note that when $k \geq n/2$ the Ricci curvature of $(\mathcal{M}, g_m)$ is positive, as shown in (1.8). Hence by the Bishop Theorem, the ratio $Q_m(r) = \text{Vol}(B_{y,r}[g_m])/r^n$ is decreasing for all m. Sending $m \to \infty$, we see that Q is non-increasing in r. Hence

$$Q(0) \leq \lim_{r \to 0} Q(r) \leq \frac{1}{n}\omega_n, \tag{3.21}$$

where ω_n is the area of the unit sphere S^{n-1}.

On the other hand, denote $A_{r_1,r_2} = B_{0,r_2}[g_0] - B_{0,r_1}[g_0]$, where $r_2 > r_1 > 0$ are sufficiently small. We identify A_{r_1,r_2} with the Euclidean annulus $A^e_{r_1,r_2} = \{x \in \mathbf{R}^n \mid r_1 < |x| < r_2\}$ by the exponential map. By the asymptotic behavior (3.17), the volume of A_{r_1,r_2} in the metric $g = e^{-2w} g_0$ is a lower order perturbation of that in the metric $g' = e^{-2w'} g_0$, where $w' = 2\log|x|$. But in our normal coordinates at 0, by (3.2) the volume of A_{r_1,r_2} in g' is the same as that of $A^e_{r_1,r_2}$ with the metric $g'_e = e^{-2w'} g_e$, where g_e is the standard Euclidean metric. Hence $\text{Vol}_{g'} A_{r_1,r_2} = \frac{1}{n}\omega_n(r_1^{-n} - r_2^{-n})$. Therefore as $r \to \infty$, each end of the metric g will contribute to the ratio $Q(r)$ a factor $\frac{1}{n}\omega_n$. Therefore we obtain

$$\lim_{r \to \infty} Q(r) = \frac{m}{n}\omega_n, \tag{3.22}$$

where m is the number of singular points of w. From (3.21) and (3.22) we see that if S_w is not empty, then m must be equal to 1, namely S_w is a single point. □

3.3 Smoothness of k-admissible functions

In this subsection we prove the following smoothness result. The proof is again inspired by the corresponding proof in [18].

Lemma 3.5 *Let w be a k-admissible function w with a singular point* 0. *Then w is C^∞ smooth away from* 0.

Proof First we prove

$$\sigma_k(\lambda(A_g)) \equiv 0 \quad \text{in } \mathcal{M}\backslash\{0\}, \tag{3.23}$$

where $g = e^{-2w} g_0$. It suffices to prove that for any given point $x_0 \neq 0$ and a sufficiently small $r > 0$ ($r < \frac{1}{4}|x_0|$), (3.23) holds in $B_{x_0,r} = B_{x_0,r}[g_0]$.

By definition, there exists a sequence w_m of smooth k-admissible functions which converges to w in $B_{x_0,2r}$ uniformly. Let φ_m be the admissible solution of the Dirichlet problem [12]

$$\begin{aligned} \sigma_k(\lambda(A_{g_{\varphi_m}})) &= \varepsilon_m \quad && \text{in } B_{x_0,r}, \\ \varphi_m &= w_m \quad && \text{on } \partial B_{x_0,r}, \end{aligned} \tag{3.24}$$

where $g_{\varphi_m} = e^{-2\varphi_m} g_0$, and ε_m is a small positive constant ($\varepsilon_m \to 0$ as $m \to \infty$) such that $\sigma_k(\lambda(A_{g_{w_m}})) > \varepsilon_m$ ($g_{w_m} = e^{-2w_m} g_0$). By the comparison principle we have $\varphi_m \geq w_m$ in $B_{x_0,r}$. Let $\hat{w}_m = w_m$ in $\mathcal{M} - B_{x_0,r}$ and $\hat{w}_m = \varphi_m$ in $B_{x_0,r}$. Then $\hat{w}_m$ is k-admissible (see Corollary 3.8 below). Let $\hat{w} = \lim_{m\to\infty} \hat{w}_m$. Then $\hat{w}$ is a k-admissible function with singularity point 0. Define the metric $\hat{g} = e^{-2\hat{w}} g_0$ and the ratio $\hat{Q}(r) = \frac{\mathrm{Vol}(B_{y,r}[\hat{g}])}{r^n}$. Then from the proof of Lemma 3.4, we also have $\hat{Q} \equiv \frac{1}{n}\omega_n$.

To prove (3.23) it suffices to show that $\hat{w} \equiv w$. Noting that $\hat{w} = w$ in $\mathcal{M} - B_{x_0,r}$ and $\hat{w} \ge w$ in $B_{x_0,r}$, we have $B_{y,r}[\hat{g}] \supset B_{y,r}[g]$ for any $r > 0$ and $y \ne 0$. If there exists a point $y \in B_{x_0,r}$ such that $\hat{w} > w$ at y, then there exists a positive constant $\delta > 0$ such that for any $r > 1$,

$$B_{y,r}[\hat{g}] \supset B_{y,r+\delta}[g].$$

But this is impossible as both the ratios $Q(r)$ and $\hat{Q}(r)$ are constant.

By the interior second order derivative estimate [13], we see that w is $C^{1,1}$ smooth. Next we prove that w is C^∞ smooth away from 0. By the regularity of linear elliptic equations [11], it suffices to prove that $v = e^{-\frac{n-2}{2}w} \in C^{1,1}$ is a strong solution to the uniformly elliptic equation

$$-\Delta_{g_0} v + \frac{n-2}{4(n-1)} R_{g_0} v = 0 \quad \text{in } \mathcal{M}\backslash\{0\}, \tag{3.25}$$

where R_{g_0} is the scalar curvature of $(\mathcal{M}, g_0)$; namely the scalar curvature of $g = e^{-2w} g_0$ vanishes identically.

Equation (3.25) may be verified as in Sect. 7.6 in [18]. Here we provide a proof for completeness. Since $w \in C^{1,1}$, it is twice differentiable almost everywhere. Suppose at a point 0, w is twice differentiable and the scalar curvature $R > 0$. Then with respect to normal coordinates of g at 0, we have the expansion

$$\det g_{ij} = 1 - \frac{1}{3} R_{ij} x_i x_j + o(|x|^2), \tag{3.26}$$

see (5.2) in [19]. Hence

$$\begin{aligned} \mathrm{Vol}(B_{0,r}[g]) &= \int_{B_{0,r}} \sqrt{\det g_{ij}} \\ &= \int_{B_{0,r}} \left[1 - \frac{1}{6} R_{ij} x_i x_j + o\left(|x|^2\right)\right] \\ &= \frac{1}{n}\omega_n r^n \left[1 - \frac{R}{6(n+2)} r^2 + o(r^2)\right], \end{aligned} \tag{3.27}$$

where R_{ij} and R are respectively the Ricci curvature and the scalar curvature in g. This is a contradiction when $R > 0$ at 0, as the ratio Q is a constant. Hence the scalar curvature of g vanishes almost everywhere. □

3.4 End of proof of Theorem A

From Sects. 3.2 and 3.3, we see that if $(\mathcal{M}, g_0)$ is a compact manifold and there exists a k-admissible function w with singularity at some point 0, then w has the asymptotic formula (3.17) and w is smooth away from 0. The manifold $\mathcal{M}\backslash\{0\}$ equipped with the metric $g = e^{-2w} g_0$ is a complete manifold with nonnegative Ricci curvature, and satisfies furthermore the volume growth formula $Q(r) \equiv 1$. Hence $(\mathcal{M}\backslash\{0\}, g)$ is isometric to the Euclidean space [7]. Hence $(\mathcal{M}, g_0)$ is conformally equivalent to the unit sphere S^n. See also [18].

To finish the proof of Theorem A, it suffices to prove

Lemma 3.6 *Let $(\mathcal{M}, g_0)$ be a compact manifold. If $(\mathcal{M}, g_0)$ is not conformally equivalent to the unit sphere S^n, then there exists $K > 0$ such that if w is a k-admissible function,*

$$\sup_{\mathcal{M}} w - \inf_{\mathcal{M}} w \le K, \tag{3.28}$$

$$|w(x) - w(y)| \le K|x-y|^{2-\frac{n}{k}}. \tag{3.29}$$

Proof If (3.28) is not true, there exists a sequence of k-admissible functions w_m such that $\sup_{\mathcal{M}} w_m = 0$ (by subtracting a constant if necessary) and $\inf_{\mathcal{M}} w_m \to -\infty$. Suppose that $w_m(0) \to -\infty$. By Lemma 3.1, $\{e^{w_m}\}$ are uniformly Holder continuous. Hence we may assume that e^{w_m} converges locally uniformly to e^w in $\mathcal{M}$. The Hölder continuity also implies that $e^w(0) = 0$, namely $\lim_{x\to 0} w(x) = -\infty$. By a weak Harnack inequality for (1.6), the set $\{w = -\infty\}$ has measure zero and so w is k-admissible. Applying Lemma 3.3 to the limit function w we conclude that w has the asymptotic behavior (3.17) and that 0 is an isolated singularity of w (see Remark 3.2). Therefore by Lemmas 3.4, 0 is the unique singular point of w. By Lemma 3.5, w is C^∞-smooth away from 0. Hence as above, we see that $(\mathcal{M}, g_0)$ is conformally equivalent to the unit sphere S^n, which is ruled out by our assumption. Hence (3.28) holds.

The Hölder continuity (3.29) follows from Lemma 3.1. □

3.5 Remarks on the set $[g_0]_k$

In this section we prove some properties of k-admissible functions.

Lemma 3.7 *If w_1, w_2 are smooth and k-admissible, then $w = \max(w_1, w_2)$ is k-admissible.*

Proof It is convenient to consider the function $u = e^w$. By approximation we suppose u_1 and u_2 are smooth and k-admissible functions such that the eigenvalues $\lambda(U)$ lie strictly in the open convex cone Γ_k, where U is the matrix (1.16) with $u = u_1$ and u_2. Hence when $r > 0$ is sufficiently small, the eigenvalues of the matrix

$$U_r = \left\{ u_{ij} - \frac{|\nabla u|^2}{2u_{x,r}} + uA_{g_0} \right\} \tag{3.30}$$

lie in Γ_k for $u = u_1$ and u_2, where $u_{x_0,r} = \inf_{B_{x_0,r}} u$.

Let $u = \max(u_1, u_2)$. Since u_1, u_2 are smooth functions, u is twice differentiable almost everywhere. Let $\rho \in C_0^\infty(\mathbf{R}^n)$ be a mollifier. In particular we choose ρ to be a radial, smooth, nonnegative function, supported in the unit ball $B_{0,1}$, with $\int_{B_{0,1}} \rho = 1$. Let

$$u_{[\varepsilon]}(x) = \int\limits_{B_{x,\varepsilon}} \varepsilon^{-n} \rho\left(\frac{|x-y|}{\varepsilon}\right) u(y)\sqrt{\det(g_0)_{ij}}dy \tag{3.31}$$

be the mollification of u, where $B_{x,\varepsilon}$ is the geodesic ball. For each point x, using normal coordinates and the exponential map, we have, by (3.26),

$$\begin{aligned} u_{[\varepsilon]}(x) &= \int\limits_{B_{0,1}} \rho(y)u(x-\varepsilon y)\sqrt{\det(g_0)_{ij}}\, dy \\ &= \int\limits_{B_{0,1}} \rho(y)u(x-\varepsilon y)\left(1 - \frac{\varepsilon^2}{6}R_{ij}(x)y_i y_j + O(\varepsilon^3)\right) dy, \end{aligned} \tag{3.32}$$

where $B_{0,1}$ is the Euclidean space. If g_0 is a flat metric, we have

$$\nabla u_{[\varepsilon]} = \int_{B_{0,1}} \rho(y)\nabla u(x-\varepsilon y)dy, \tag{3.33}$$

$$\nabla^2 u_{[\varepsilon]} \geq \int_{B_{0,1}} \rho(y)\nabla^2 u(x-\varepsilon y)dy, \tag{3.34}$$

$$\begin{aligned} |\nabla u_{[\varepsilon]}|^2 &= \left[\int_{B_{0,1}} \rho(y)\nabla u(x-\varepsilon y)dy\right]^2 \\ &\leq \int_{B_{0,1}} \rho(y)|\nabla u(x-\varepsilon y)|^2 dy. \end{aligned} \tag{3.35}$$

Hence $u_{[\varepsilon]}$ is k-admissible by (3.30). If g_0 is not flat, by (3.32), an extra term of magnitude $O(\varepsilon^2)$ arises. Letting $\varepsilon > 0$ be sufficiently small and noting that the eigenvalues of U (with respect to u_1 and u_2) lie strictly in the open set Γ_k, we conclude again that $u_{[\varepsilon]}$ is k-admissible. □

Corollary 3.8 *Suppose φ is a smooth k-admissible function on $\mathcal{M}$ with $\sigma_k(\lambda(A_{g_\varphi})) > f$, where $g_\varphi = e^{-2\varphi}g_0 \in [g_0]_k$ and f is a smooth, positive function. Let w be the admissible solution of*

$$\begin{aligned} \sigma_k(\lambda(W)) &= f \quad \text{in } \Omega, \\ w &= \varphi \quad \text{on } \partial\Omega, \end{aligned} \tag{3.36}$$

where W is given in (1.17), and Ω is a smooth domain on $\mathcal{M}$. Extend w to $\mathcal{M}$ by letting $w = \varphi$ on $\mathcal{M} - \Omega$. Then w is k-admissible.

It was proved in [12] that (3.36) admits a solution w, smooth up to the boundary. By the comparison principle we have $w > \varphi$ in Ω and $\partial_\nu(\varphi - w) > 0$ on $\partial\Omega$, where ν is the unit outward normal. Hence we can extend w to a neighbourhood of Ω such that it is k-admissible. Hence Corollary 3.8 follows from Lemma 3.7.

Corollary 3.9 *Consider the Dirichlet problem (3.36). Suppose the set of sub-solutions W_{sub} is not empty. Let*

$$w(x) = \sup\{\varphi(x) \mid \ \varphi \in W_{sub}\}. \tag{3.37}$$

If w is bounded from above, then it is a solution to (3.36).

By the interior a priori estimates [13], the proof is standard. Note that in Corollary 3.9, we allow Ω to be the whole manifold $\mathcal{M}$.

4 Proof of Theorem C

We divide the proof into three cases, according to $p < k$, $p = k$, and $p > k$.

Case 1 $p < k$. By (1.15), we can write Eq. (1.12) as

$$\sigma_k(\lambda(W)) = fe^{aw}, \tag{4.1}$$

where f is a constant multiple of that in (1.12),

$$a = \frac{1}{2}(n-2)(k-p). \tag{4.2}$$

For any given k-admissible function w, the functions $w+c$ and $w-c$ are respectively a super and a sub solution of (4.1) provided the constant c is sufficiently large. By the a priori estimates in [13] and the comparison principle, the solution of (4.1) is uniformly bounded. When $a>0$, the linearized equation of (4.1) is invertible. Hence by the continuity method, there is a unique smooth solution to (4.1).

Case 2 $p=k$. We prove that for any positive smooth function f, there is a unique constant $\theta>0$ such that the equation

$$\sigma_k(\lambda(W)) = \theta f \tag{4.3}$$

has a solution. For $a>0$ small, let w_a be the solution of (4.1). Let $c_a=\inf w_a$. We write (4.1) in the form

$$\sigma_k(\lambda(W_a)) = (f e^{ac_a}) e^{a(w_a-c_a)}, \tag{4.4}$$

where W_a is the matrix (1.17) relative to w_a. Assume $g_0\in[g_0]_k$ so that $\lambda(A_{g_0})\in\Gamma_k$. Then at the maximum point of w_a,

$$\sigma_k(\lambda(A_{g_0})) \ge \sigma_k(\lambda(W_a)) \ge f e^{ac_a}.$$

At the minimum point of w_a,

$$\sigma_k(\lambda(A_{g_0})) \le \sigma_k(\lambda(W_a)) = f e^{ac_a}.$$

Hence e^{ac_a} is strictly positive and uniformly bounded as $a\to 0$. By the a priori estimates [13], where the estimates depend only on $\inf(w_a-c_a)$, we see that w_a-c_a is uniformly bounded from above and sub-converges to a solution w_0 of (4.3) with $\theta=\lim_{a\to 0} e^{ac_a}$. By the maximum principle it is easy to see that if w' is another solution, then necessarily $w'=w_0+const$; and furthermore (4.3) has no (k-admissible) solution for different θ.

Case 3 $p>k$. In this case we adopt the degree argument from [30], see the proof of Theorem 5.1 there. Alternatively we can also use the degree argument in Sect. 3 of [30]. We will study the auxiliary problem

$$\sigma_k(\lambda(V)) = t(\delta_t + f v^p), \tag{4.6}$$

where $t\ge 0$ is a parameter and δ_t is a positive constant depending on t, $\delta_t=\delta_0\le 1$ when $t\le 1$ and $\delta_t=1$ when $t>2$, and δ_t is smooth and monotone increasing when $1\le t\le 2$.

Claim 1 For any $t_0>0$, the solution of (4.6) is uniformly bounded when $t\ge t_0$. Indeed, if there exists a sequence of solutions (t_j, v_j) of (4.6) such that $t_j\ge t_0$ and $\sup v_j\to\infty$, we have $m_j=\inf v_j\to\infty$ by (1.5). The function $v_j'=v_j/m_j$ satisfies

$$\begin{aligned}\sigma_k(\lambda(V')) &\ge t_j f m_j^{p-k}(v_j')^p)\\ &\ge t_j f m_j^{p-k}\to\infty,\end{aligned} \tag{4.7}$$

where V' is the matrix (1.11) relative to v'. From (4.7) and the comparison principle we have $\sup v_j'\to\infty$. Hence $\inf v_j'\to\infty$ by (1.5), which contradicts to the definition of v_j'.

Define the mapping T_t so that for any $v_1 \in C^2(\mathcal{M})$, $T_t(v_1)$ is the solution of

$$\sigma_k(\lambda(V)) = t(\delta_t + f v_1^p). \tag{4.8}$$

Then a solution of (4.6) is a fixed point of T_t.

Claim 2 There is a solution of (4.6) when $t > 0$ is small. Indeed, for any smooth, positive function φ^*, denote $\Phi = \{\varphi \in C^2(\mathcal{M}) \mid 0 < \varphi < \varphi^*, \|\varphi\|_{C^2} < R\}$ for some large constant $R > 0$. Then when $\delta_0, t > 0$ are small, $T(\Phi)$ is strictly contained in Φ. Hence the degree $\deg(I - T_t, \Phi, 0)$ is well defined for $t > 0$ small. Extend T_t to $t = 0$ by letting $T_t(v) = 0$ for all v, so that T_t is also continuous at $t = 0$. Hence

$$\deg(I - T_t, \Phi, 0) = \deg(I - T_0, \Phi, 0) = 1. \tag{4.9}$$

Hence T_t has a fixed point in Φ for $t > 0$ small.

Note that when δ_0 is sufficiently small, letting φ^* be a small positive constant, we see that (4.6) has a solution for $t \le 1$.

Claim 3 Let $t^* = \sup\{t \mid (4.6)$ admits a solution$\}$. Then t^* is finite. Indeed, if $t^* = \infty$, there is a sequence $t_j \to \infty$ such that (4.6) has a solution v_j. We have obviously $m_j = \inf v_j \to \infty$, which is a contradiction with Claim 1.

Claim 4 Equation (4.6) has a solution at $t = t^*$. Indeed, let $t_j \nearrow t^*$ and v_j be the corresponding solution of (4.6). By claim 1, v_j is uniformly bounded. Hence v_j sub-converges to a solution v^* of (4.6) with $t = t^*$.

Now we choose $\varphi^* = v^*$ and define Φ as above. For any $v_1 \in \Phi$, let v be the solution of (4.8). Since for any $t \in (0, t^*)$, v^* is a super-solution of (4.6). We have $0 < v < v^*$ by the maximum principle. Hence by (4.9), $\deg(I - T_t, \Phi, 0) = 1$ for $t \in [0, t^*)$.

On the other hand, for any given $t_0 > 0$, since the solution of (4.6) is uniformly bounded for $t \ge t_0$, the degree $\deg(I - T_t, B_R, 0)$ is well defined for $t \in (t_0, t^* + 1]$ for sufficiently large R, where $B_R = \{v \in C^2(\mathcal{M}) \mid 0 < v < R, \|v\|_{C^2} < R\}$. But when $t > t*$, (4.6) has no solution. Hence $\deg(I - T_t, B_R, 0) = 0$. Hence for any $t \ge t_0$, (4.6) has a solution $v \notin \Phi$ with degree -1.

Let $v = v_{\delta_0} \notin \Phi$ be a solution of (4.6) at $t = 1$. We have $\sup v > \inf v^* > 0$. Let $\delta_0 \to 0$. Since the solution is uniformly bounded, it converges to a solution of (1.12). This completes the proof. □

From the above argument, we have the following extensions.

Corollary 4.1 *Let $(\mathcal{M}, g_0)$ be a compact n-manifold not conformally equivalent to the unit sphere S^n. Suppose $\frac{n}{2} < k \le n$ and $[g_0]_k \ne \emptyset$. Suppose there exists a constant $c_0 > 0$ such that*

$$\varphi(x, t) \ge c_0, \tag{4.10}$$

$$\lim_{t\to\infty} t^{-k}\varphi(x, t) = \infty. \tag{4.11}$$

Then there exists a constant $t^* > 0$ such that the equation

$$\sigma_k(\lambda(V)) = t\varphi(x, v) \tag{4.12}$$

has at least two solutions for $0 < t < t^*$, one solution at $t = t^*$, and no solution for $t > t^*$.

Corollary 4.2 *Let* $(\mathcal{M}, g_0)$ *be as in Corollary 4.1,* $\frac{n}{2} < k \leq n$*. Suppose* $\varphi > 0$,

$$\lim_{t\to 0} t^{-k}\varphi(x,t) = 0, \tag{4.13}$$

and (4.11) *holds. Then there exists a solution to* (1.10).

In the above theorems, we can also allow that the right hand side depends on the gradient ∇v. Furthermore, (4.11) and (4.13) can be relaxed to

$$\lim_{t\to \infty} t^{-k}\varphi(x,t) > \theta, \tag{4.14}$$

$$\lim_{t\to 0} t^{-k}\varphi(x,t) < \theta, \tag{4.15}$$

where θ is the eigenvalue of (1.13) (with $f \equiv 1$). See [30] for the Monge-Ampère equation.

We remark that when $1 \leq k \leq \frac{n}{2}$, Theorem C holds for $p < k\frac{n+2}{n-2}$. Indeed, when $p \leq k$, the proof of the Cases 1 and 2 above also applies to the cases $1 \leq k \leq \frac{n}{2}$. When $k < p < k\frac{n+2}{n-2}$, by a blow-up argument and the Liouville theorem [20], it is known that the set of solutions to (4.6) is uniformly bounded. Hence by the above degree argument, one also obtains the existence of solutions.

Corollary 4.3 *Let* $(\mathcal{M}, g_0)$ *be a compact n-manifold with* $[g_0]_k \neq \emptyset$, $1 \leq k \leq n$*. Then for any smooth, positive function* f *and any constant* $p \neq k$, $p < k\frac{n+2}{n-2}$, *there exists a positive solution to the* (1.12)*. The solution is unique if* $p < k$*. When* $p = k$*, there exists a unique constant* $\theta > 0$ *such that* (1.13) *has a solution. The solution is unique up to a constant multiplication.*

Note that in Corollary 4.3 we allow that $(\mathcal{M}, g_0)$ is the unit sphere.

References

1. Aubin, T.: Equations différentielles non linéaires et problm̀e de Yamabe concernant la courbure scalaire. J. Math. Pures Appl. **55**, 269–296 (1976)
2. Aubin, T.: Some nonlinear problems in Riemannian geometry. Springer, Heidelberg (1998)
3. Caffarelli, L.A., Nirenberg, L., Spruck, J.: Dirichlet problem for nonlinear second order elliptic equations III. Functions of the eigenvalues of the Hessian. Acta Math. **155**, 261–301 (1985)
4. Cao, J.-G.: The existence of generalized isothermal coordinates for higher-dimensional Riemannian manifolds. Trans. Am. Math. Soc. **324**, 901–920 (1991)
5. Chang, A., Gursky, M., Yang, P.: An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature. Ann. Math. **155**(2), 709–787 (2002)
6. Chang, A., Gursky, M., Yang, P.: An a priori estimate for a fully nonlinear equation on four-manifolds. J. Anal. Math. **87**, 151–186 (2002)
7. Chavel, I.: Riemannian geometry—a modern introduction. Cambridge Univ. Press, London (1993)
8. Chou, K.S. (Tso, K.): On a real Monge-Ampère functional. Invent. Math. **101** 425–448 (1990)
9. Chou, K.S., Wang, X.J.: A variational theory of the Hessian equation. Comm. Pure Appl. Math. **54**, 1029–1064 (2001)
10. Ge, Y., Wang, G.: On a fully nonlinear Yamabe problem. Ann. Sci. École Norm. Sup. **39**(4), 569–598 (2006)
11. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Springer, Heidelberg (1983)
12. Guan, B.: Conformal metrics with prescribed curvature functions on manifolds with boundary. Am. J. Math. **129**, 915–942 (2007)
13. Guan, P., Wang, G.: Local estimates for a class of fully nonlinear equations arising from conformal geometry. Int. Math. Res. Not. (26), 1413–1432 (2003)
14. Guan, P., Wang, G.: A fully nonlinear conformal flow on locally conformally flat manifolds. J. Reine Angew. Math. **557**, 219–238 (2003)

15. Guan, P., Viaclovsky, J., Wang, G.: Some properties of the Schouten tensor and applications to conformal geometry. Trans. Am. Math. Soc. **355**, 925–933 (2003)
16. Günther, M.: Conformal normal coordinates. Ann. Global Anal. Geom. **11**, 173–184 (1993)
17. Gursky, M., Viaclovsky, J.: Convexity and singularities of curvature equations in conformal geometry, Int. Math. Res. Not., Art. ID 96890, pp. 43 (2006)
18. Gursky, M., Viaclovsky, J.: Prescribing symmetric functions of the eigenvalues of the Ricci tensor. Ann. Math. **166**(2), 475–531 (2007)
19. Lee, J.M., Parker, T.H.: The Yamabe problem. Bull. Am. Math. Soc. **17**, 37–91 (1987)
20. Li, A., Li, Y.Y.: On some conformally invariant fully nonlinear equations. Comm. Pure Appl. Math. **56**, 1416–1464 (2003)
21. Li, A., Li, Y.Y.: On some conformally invariant fully nonlinear equations II, Liouville, Harnack, and Yamabe. Acta Math. **195**, 117–154 (2005)
22. Schoen, R.: Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differ. Geom. **20**, 479–495 (1984)
23. Schoen, R., Yau, S.T.: Lectures on Differential geometry. International Press (1994)
24. Sheng, W.M., Trudinger, N.S., Wang, X.-J.: The Yamabe problem for higher order curvatures. J. Differ. Geom. **77**, 515–553 (2007)
25. Trudinger, N.S.: On Harnack type inequalities and their application to quasilinear elliptic equations. Comm. Pure Appl. Math. **20**, 721–747 (1967)
26. Trudinger, N.S.: Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. Ann. Scuola Norm. Sup. Pisa **22**(3), 265–274 (1968)
27. Trudinger, N.S., Wang, X.J.: Hessian measures II. Ann. Math. **150**(2), 579–604 (1999)
28. Trudinger, N.S., Wang, X.-J.: The intermediate case of the higher order Yamabe problem, (in press)
29. Viaclovsky, J.: Conformal geometry, contact geometry, and the calculus of variations. Duke Math. J. **101**(2), 283–316 (2000)
30. Wang, X.-J.: Existence of multiple solutions to the equations of Monge-Ampère type. J. Differ. Eqs. **100**, 95–118 (1992)