On heroes in digraphs with forbidden induced forests

Alvaro Carbonero¹, Hidde Koerts¹, Benjamin Moore², and Sophie Spirkl^{*1}

¹University of Waterloo, Department of Combinatorics and Optimization, Waterloo, Canada ²Charles University, Institute of Computer Science, Prague, Czech Republic

June 9, 2023

*Emails: (ar2carbonerogonzales, hkoerts, sspirkl)@uwaterloo.ca, brmoore@iuuk.mff.cuni.cz Spirkl: We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), [funding reference number RGPIN-2020-03912]. Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG), [numéro de référence RGPIN-2020-03912]. This project was funded in part by the Government of Ontario.

Benjamin Moore is supported by project 22-17398S (Flows and cycles in graphs on surfaces) of the Czech Science Foundation.

Some of this material appeared in Carbonero's master's thesis.

Abstract

We continue a line of research which studies which hereditary families of digraphs have bounded dichromatic number. For a class of digraphs C, a hero in C is any digraph Hsuch that H-free digraphs in C have bounded dichromatic number. We show that if F is an oriented star of degree at least five, the only heroes for the class of F-free digraphs are transitive tournaments. For oriented stars F of degree exactly four, we show the only heroes in F-free digraphs are transitive tournaments, or possibly special joins of transitive tournaments. Aboulker et al. characterized the set of heroes of $\{H, K_1 + \vec{P}_2\}$ -free digraphs almost completely, and we show the same characterization for the class of $\{H, rK_1 + \vec{P}_3\}$ -free digraphs. Lastly, we show that if we forbid two "valid" orientations of brooms, then every transitive tournament is a hero for this class of digraphs.

1 Introduction

Throughout this paper, (di)graphs are finite and simple. In particular, for digraphs, between every two vertices u and v, at most one of uv and vu is present.

We will be interested in the dichromatic number of families of digraphs with forbidden induced subgraphs. Recall that a (di)graph H is an *induced subgraph* of a (di)graph G if by deleting vertices of G we obtain a (di)graph isomorphic to H. Equivalently, we say H is an induced subgraph of G if there exists a set $S \subseteq V(G)$ such that G[S] is isomorphic to H. We call S a *copy* of H in G. If G has no copy of H, then we say G is H-free. Furthermore, if Cis a set of (di)graphs, then G is C-free if for every $H \in C$, we have that G is H-free.

For a natural number k, we write [k] for the set $\{1, \ldots, k\}$. Given a digraph D, a kdicolouring of D is a function $f: V(G) \to [k]$ such that for every $i \in [k]$, the induced subdigraph $D[f^{-1}(i)]$ is acyclic; in other words, no directed cycle of D is monochromatic with respect to f. The dichromatic number $\vec{\chi}(D)$, introduced by Neumann-Lara in [10], is the minimum k such that D has a k-dicolouring. If $X \subseteq V(D)$, we also use $\vec{\chi}(X)$ to mean $\vec{\chi}(D[X])$. One may compare this definition to the chromatic number, denoted $\chi(G)$, which is the minimum k such that there is a mapping $f: V(G) \to [k]$ where for all edges e = xy, $f(x) \neq f(y)$.

We are interested in understanding for which families \mathcal{F} all \mathcal{F} -free graphs have bounded dichromatic number. To this end, let us say that \mathcal{F} is $\bar{\chi}$ -finite if there is a constant c such that all \mathcal{F} -free digraphs have dichromatic number at most c. We consider the following question, which was first systematically studied by Aboulker, Charbit, and Naserasr [2]:

Question 1.1. Which finite families \mathcal{F} of digraphs are $\vec{\chi}$ -finite?

Consider the family of tournaments: that is, the family of graphs which is obtained by orienting cliques. It is easy to see that tournaments are exactly the class of digraphs which forbid $2K_1$, that is the graph consisting of two isolated vertices. It is also well-known that there exist tournaments of arbitrarily large dichromatic number; in other words, $\{2K_1\}$ is not χ -finite. A natural question is: for which digraphs H is $\{2K_1, H\}$ a χ -finite family? This motivates the definition of heroes.

We say that H is a *hero* in \mathcal{F} -free digraphs if $\{H\} \cup \mathcal{F}$ is $\tilde{\chi}$ -finite. Heroes in tournaments are often just called heroes. A seminal paper of Berger et al. [3] completely characterizes heroes in tournaments. We need some definitions before we can fully state the theorem.

For graphs and digraphs D_1 and D_2 , we use $D_1 + D_2$ to denote the disjoint union of D_1 and D_2 , and we use rD_1 for an integer $r \ge 0$ to denote the disjoint union of r copies of D_1 . A tournament is *transitive* if it is acyclic, and a transitive tournament on k vertices is denoted as TT_k . For two digraphs D_1 and D_2 , we define $D_1 \Rightarrow D_2$ to be the digraph arising from $D_1 + D_2$ by adding all arcs d_1d_2 with $d_i \in V(D_i)$ for $i \in \{1, 2\}$. Furthermore, for digraphs D_1, D_2, D_3 , we define $\Delta(D_1, D_2, D_3)$ as the digraph arising from $D_1 + D_2 + D_3$ by adding all arcs d_id_j with $d_i \in V(D_i), d_j \in V(D_j)$ and $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$. For convenience, if D_i is a k-vertex transitive tournament, we write k for D_i in this construction.

Now we can state the characterization of heroes in tournaments:

Theorem 1.2 (Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour, and Thomassé [3]). A digraph H is a hero in tournaments if and only if one of the following holds:

•
$$H = K_1;$$

- $H = H_1 \Rightarrow H_2$ where H_1 and H_2 are heroes in tournaments; or
- $H = \Delta(1, H_1, m)$ or $H = \Delta(1, m, H_1)$ where $m \ge 1$ and H_1 is a hero in tournaments.

While this may seem like a very special case of Question 1.1, it is particularly relevant due to a theorem of Aboulker, Charbit, and Naserasr [2], who showed that if \mathcal{F} is $\vec{\chi}$ -finite, then \mathcal{F} contains both an oriented forest (a digraph whose underlying undirected graph is a forest) and a hero in tournaments. They further showed:

Theorem 1.3 (Aboulker, Charbit, and Naserasr [2]). If $\{H, F\}$ is $\vec{\chi}$ -finite where H is a hero in tournaments and F is an oriented forest, then either H is a transitive tournament, or the underlying undirected graph of F is a disjoint union of stars.

What happens in the case where H is a transitive tournament and F is an oriented forest? Chudnovsky, Scott and Seymour [4] showed that in the special case where F is an *oriented* star of degree t, that is, an orientation of $K_{1,t}$, the set $\{H, F\}$ is $\tilde{\chi}$ -finite.

Theorem 1.4 (Chudnovsky, Scott, and Seymour [4]). If F is an oriented star and H is a transitive tournament, $\{H, F\}$ is $\vec{\chi}$ -finite.

We show that if the star F has (undirected) degree at least five, then the only heroes in F-free digraphs are transitive tournaments, showing that the theorem of Chudnovsky, Scott, and Seymour cannot be strengthened for these stars.

Theorem 1.5. If F is an oriented star of degree at least 5, then $\{H, F\}$ is $\vec{\chi}$ -finite only if H is a transitive tournament.

In the case where F is an orientation of a star of degree 4, the only possible heroes are transitive tournaments and tournaments of the form $\Delta(1, m, m')$:

Theorem 1.6. If F is an oriented star of degree 4 and $\{H, F\}$ is $\vec{\chi}$ -finite, then either H is a transitive tournament or $H = \Delta(1, m, m')$ where $m, m' \ge 1$.

While the case when H is a transitive tournament is resolved by Theorem 1.4, the case when $H = \Delta(1, m, m')$ remains open.

For oriented stars of degree 2 and 3, the full picture is not yet clear. For oriented stars of degree 3, we are not aware of any results aside from Theorem 1.4. For stars of degree 2, which are isomorphic to P_3 , Aboulker et al. [1] obtain the following:

Theorem 1.7 (Aboulker, Aubian, and Charbit [1]). For every hero H in tournaments, $\{H, \vec{P}_3\}$ is $\vec{\chi}$ -finite.

Here \vec{P}_3 is the directed path on three vertices. Let us pause to introduce some convenient notation for orientations of paths. We use arrows \rightarrow and \leftarrow to denote the direction of the arcs in a path. For example, $v_1 \rightarrow v_2 \rightarrow v_3 \leftarrow v_4 \leftarrow v_5$ and $\rightarrow \rightarrow \leftarrow \leftarrow$ both denote the digraph $(\{v_1, \ldots, v_5\}, \{v_1v_2, v_2v_3, v_4v_3, v_5v_4\})$. The *directed path* \vec{P}_m on *m* vertices refers to a path on *m* vertices with orientation $\rightarrow \rightarrow \cdots \rightarrow$.

Thus returning to oriented stars, the remaining cases for oriented stars of degree 2 are when $F \in \{\rightarrow\leftarrow,\leftarrow\rightarrow\}$. These cases are the same (up to reversing all arcs), so it suffices to consider $\leftarrow\rightarrow$. Steiner gave the following partial result (where \vec{C}_3 is the cyclic triangle $\vec{C}_3 = (\{v_1, v_2, v_3\}, \{v_1v_2, v_2v_3, v_3v_1\})).$ **Theorem 1.8** (Steiner [13]). If $F = \leftrightarrow and H = \vec{C}_3 \Rightarrow TT_k$ for some integer $k \ge 1$, then $\{H, F\}$ is $\vec{\chi}$ -finite.

Moving past oriented stars, we consider another natural question. Which digraphs F have the property that all heroes in tournaments are heroes in F-free digraphs? The following two results show that this holds if $F = rK_1$, and does not hold if F contains $K_1 + \vec{P}_2$.

Theorem 1.9 (Harutyunyan, Le, Newman, and Thomassé [9]). For all $r \in \mathbb{N}$ and every hero H in tournaments, $\{rK_1, H\}$ is $\tilde{\chi}$ -finite.

Theorem 1.10 (Aboulker, Aubian, and Charbit [1]). If F contains a copy of $K_1 + \vec{P}_2$, then $\Delta(1,2,\vec{C}_3), \Delta(1,\vec{C}_3,2), \Delta(1,2,3)$, and $\Delta(1,3,2)$ are not heroes in F-free digraphs.

Complementing Theorem 1.10, Aboulker, Aubian, and Charbit [1] almost completely characterize heroes in $\{K_1 + \vec{P}_2\}$ -free digraphs:

Theorem 1.11 (Aboulker, Aubian, and Charbit [1]). The set $\{H, K_1 + \vec{P}_2\}$ is $\vec{\chi}$ -finite if:

- $H = K_1;$
- $H = H_1 \Rightarrow H_2$ where $\{H_i, K_1 + \vec{P_2}\}$ is $\vec{\chi}$ -finite for $i \in \{1, 2\}$; or
- $H = \Delta(1, 1, H_1)$ where $\{H_1, K_1 + \vec{P}_2\}$ is $\vec{\chi}$ -finite.

With this theorem, only the status of $\Delta(1,2,2)$ remains to be decided. This raises the natural question: For which forests F is it the case that $\{F,H\}$ is $\tilde{\chi}$ -finite for all H as in Theorem 1.11? In Section 4, we show:

Theorem 1.12. Let $r \in \mathbb{N}$. The set $\{H, rK_1 + \vec{P}_3\}$ is $\vec{\chi}$ -finite if:

- $H = K_1;$
- $H = H_1 \Rightarrow H_2$ where $\{H_i, rK_1 + \vec{P}_3\}$ is $\vec{\chi}$ -finite for $i \in \{1, 2\}$; or
- $H = \Delta(1, 1, H_1)$ where $\{H_1, rK_1 + \vec{P}_3\}$ is $\vec{\chi}$ -finite.

Again, using Theorem 1.10, this leaves open only the status of $\Delta(1,2,2)$.

To motivate our final result, we recall the directed Gyárfás-Sumner conjecture, posed by Aboulker, Charbit and Naserasr [2]:

Conjecture 1.13 (Aboulker, Charbit, and Naserasr [2]). If F is a directed forest and H is a transitive tournament, then $\{H, F\}$ is $\vec{\chi}$ -finite.

As noted in [2], this is a directed analog of the famous Gyárfás-Sumner conjecture.

Conjecture 1.14 (Gyárfás [8] and Sumner[14]). For every forest F and every clique K_k on k vertices, the $\{F, K_k\}$ -free graphs have bounded chromatic number.

Conjecture 1.13 is wide open. We do not even know if the conjecture holds when P is an oriented path. Recently, Cook et al. [5] showed:

Theorem 1.15 (Cook, Masařík, Pilipczuk, Reinald, and Souza [5]). For every k, the set $\{TT_k, \vec{P}_4\}$ is $\vec{\chi}$ -finite.

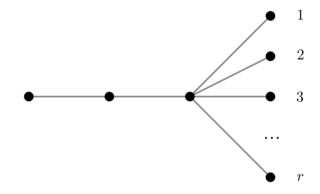


Figure 1: An illustration of B_r .

We investigate a weakening of Conjecture 1.13 by forbidding two specific oriented forests, called brooms. For an integer $r \ge 1$, let the *r*-broom, denoted by B_r , be the graph defined as follows:

$$B_r \coloneqq (\{v_1, v_2, v_3, w_1, \dots, w_r\}, \{v_1v_2, v_2v_3, v_3w_1, \dots, v_3w_r\}).$$

See Figure 1. If \mathcal{B} is an orientation of B_r and \mathcal{B}' is an orientation of B_s , then we say \mathcal{B} and \mathcal{B}' have opposing orientations if $v_2v_3 \in A(\mathcal{B})$ and $v_3v_2 \in A(\mathcal{B}')$. Furthermore, a valid orientation \mathcal{B} of B_r is an orientation such that either $\{v_3w_1, \ldots, v_3w_r\} \subseteq A(\mathcal{B})$ or $\{w_1v_3, \ldots, w_rv_3\} \subseteq A(\mathcal{B})$. We prove the following, a strengthening of an unpublished result due to Linda Cook and Seokbeom Kim (private communication):

Theorem 1.16. Let r, s, t be positive integers. If \mathcal{B} and \mathcal{B}' are valid opposing orientations of B_r and B_s respectively, then $\{\mathcal{B}, \mathcal{B}', TT_t\}$ is $\tilde{\chi}$ -finite.

We give a brief outline of how we prove our results.

For Theorems 1.5 and 1.6, we construct a sequence of digraphs with large dichromatic number that in the case of Theorem 1.5, have no cyclic triangle, or in the case of Theorem 1.6, only constrained cyclic triangles. We sketch the construction given for Theorem 1.5, as the construction is very similar for Theorem 1.6. Our starting point is a classical construction of a graph with no short odd cycles, and large chromatic number – the shift graph (see, for example, [12]). Shift graphs were also used by Aboulker, Aubian, and Charbit [1] in the proof of Theorem 1.10. For Theorem 1.5, we will use a 7-tuple-shift graph. To turn this into a digraph, we simply orient in the natural fashion, that is, we orient edges from $(a,b,c,d,e,f,g) \rightarrow (b,c,d,e,f,g,\bullet)$ (in other words, edges correspond to one "shift" of the sequence). This results in a graph with dichromatic number 1, so to remedy this, after four such shifts, we add a backedge (that is, adding edges of the form $(a, b, c, d, \bullet, \bullet, \bullet) \rightarrow$ $(\bullet, \bullet, \bullet, a, b, c, d)$). By a well-known theorem of Gallai-Hasse-Roy and Vitaver, this forces the graph to have large dichromatic number. Now we need to forbid oriented stars of degree at least five, and to do this we add transitive tournaments in a careful way to the neighbourhoods of vertices. Finally, one can check that the resulting graph has large dichromatic number, no cyclic triangle, and no induced oriented star of degree at least five, completing the proof.

For Theorem 1.12, we require multiple steps. First we look at the class of "k-(co)local" graphs. These are graphs that have the property that for every vertex v, the out-(in)-neighbourhood of v induces a digraph with dichromatic number at most k. We show that if we

are given a digraph F which behaves well with respect to heroes (which we call localized and colocalized) in k-local (rK_1+F) -free digraphs, then we can construct new heroes from smaller heroes by the operations given in Theorem 1.12. To prove this, we follow the ideas devised by Harutyunyan, Le, Newman, and Thomassé [9] to prove Theorem 1.9 (what they describe as characterizing superheroes), generalizing it to the setting of localized and colocalized graphs. With this in hand, to prove Theorem 1.12 it will suffice to show that \vec{P}_3 is localized, colocalized and has a property that we call cooperation. We will introduce a concept of "domination" which we show implies the localization properties, and thus prove the theorem. The final step will be to prove that \vec{P}_3 has the domination property, which will then imply the theorem immediately.

For Theorem 1.16, we follow a similar approach as Cook et al. [5] in their proof that \vec{P}_4 -free digraphs have dichromatic number bounded by a function of their clique number. For a digraph G, let $\omega(G)$ denote the clique number of the underlying undirected graph of G. Cook et al. proceed by considering what they call a *path minimizing closed tournament* and using this, they find a so called "nice set" (we defer the definition of this until later). Nice sets are a well-known concept which first appeared in [2], and if one can show they exist, it immediately implies Theorem 1.16. We will not be able to find a nice set, but by using path minimizing closed tournaments in a similar fashion to the Cook et al. proof, we will find a slightly weaker set, which we will call a k-nice-set, whose existence immediately implies Theorem 1.16. The majority of the difference in our result from the Cook et al. result is the additional complications that arise when trying to find a k-nice set rather than a nice set.

We end the introduction by outlining the structure of the paper. In Section 2 we prove Theorem 1.5 and Theorem 1.6. In Section 3 we build the critical tools which will lead to the prove of Theorem 1.12. In Section 4 we prove Theorem 1.12. In Section 5 we prove Theorem 1.16.

2 Forbidding oriented stars

In this section, we prove Theorem 1.5 and Theorem 1.6. The following definitions will be needed throughout. When considering a digraph D = (V(D), A(D)) where $uv \in A(D)$, we say u sees v, and v is seen by u. For a digraph D, when we say that $X_1 \subseteq V(D)$ is complete (resp. anticomplete) to $X_2 \subseteq V(D)$, we mean that this is the case for the underlying undirected graph of D. Additionally, X_1 is *in-complete* (resp. *out-complete*) to X_2 if every vertex in X_1 is seen by (resp. sees) every vertex in X_2 .

2.1 Heroes for oriented stars of degree at least five

In this subsection, we prove Theorem 1.5, which we restate for the reader's convenience:

Theorem 1.5. If F is an oriented star of degree at least 5, then $\{H, F\}$ is $\vec{\chi}$ -finite only if H is a transitive tournament.

To prove this theorem, as well as Theorem 1.6, we need the following family of graphs. Let n and k be integers such that n > 2k > 2. The k-tuple shift-graph with indices in [n] is the graph whose vertices are of the form (x_1, \ldots, x_k) , where $x_i \in [n]$ for every $i \in [k]$ and $x_i < x_{i+1}$ for every $i \in [k-1]$. Furthermore, two vertices (a_1, \ldots, a_k) and (b_1, \ldots, b_k) are adjacent if $a_{i+1} = b_i$ for every $i \in [k-1]$ or vice versa. In [6], Erdős proved the following. **Theorem 2.1** (Erdős [6]). For every fixed k, if G_n is the k-tuple shift-graph with indices in [n], then $\chi(G_n) \to \infty$ as $n \to \infty$.

We will also use the Gallai–Hasse–Roy–Vitaver Theorem (see [7, 11]):

Theorem 2.2 (Gallai-Hasse-Roy-Vitaver). If D has no directed path of length t as a (not necessarily induced) subgraph, and G is the underlying undirected graph of D, then $\chi(G) \leq t$.

Like Aboulker, Aubian, and Charbit [1] in the proof of Theorem 1.10, we orient the shift graph acyclically in the natural way, and add "back-edges" carefully to increase its dichromatic number. The following is the main result of this subsection:

Theorem 2.3. There exists digraphs F_1, F_2, \ldots such that:

- $\vec{\chi}(F_n) \to \infty \text{ as } n \to \infty;$
- for every $n \ge 1$ and $v \in V(F_n)$, the neighbourhood of v can be partitioned into four tournaments; and
- for every $n \ge 1$, the digraph F_n has no cyclic triangle $\Delta(1,1,1)$.

Proof. Let G_n be the 7-tuple shift-graph with indices in [n], and let D_n be the orientation of G_n where $(a_1, \ldots, a_7)(b_1, \ldots, b_7) \in A(D_n)$ if $b_i = a_{i+1}$ for every $i \in [6]$. For every $v = (a_1, \ldots, a_7) \in V(G_n)$, define $m(v) = a_4$. Let $X := A(D_n)$. That is, X is the set of edges of the form $(\bullet, b, c, d, e, f, g) \rightarrow (b, c, d, e, f, g, \bullet)$. Moreover, let D'_n be the digraph with $V(D'_n) = V(D_n)$ and $A(D'_n) = X \cup Y$ where Y is the set of arcs of the form $(a, b, c, d, \bullet, \bullet, \bullet) \rightarrow (\bullet, \bullet, \bullet, a, b, c, d)$. Note that as m is strictly increasing along arcs in X, it follows that X is acyclic. Likewise, m is strictly decreasing in Y, so Y is acyclic.

(Claim 1) For every $n \ge 1$, $\chi(G_n)/3 \le \tilde{\chi}(D'_n)$.

Proof. We will prove the claim by proving that a set of vertices that induces an acyclic set in D'_n also induces a subgraph with chromatic number at most 3 in G_n . Let Λ be a set of vertices that induces an acyclic set in D'_n . Notice that $D_n[\Lambda]$ does not have a directed path of length 3 because if such a path $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4$ exists, then $v_4v_1 \in A(D'_n)$ contradicting that Λ is an acyclic set in D'_n . Thus, by Theorem 2.2, we have $\chi(G_n[\Lambda]) \leq 3$ as desired.

Finally, let F_n be the digraph with $V(F_n) = V(D'_n)$ and $A(F_n) = X \cup Y \cup Z_1 \cup Z_2$ where we define Z_1 and Z_2 as follows. Let < be a total ordering of $V(F_n)$. Define Z_1 (resp. Z_2) as the set of edges such that $uv \in Z_1$ (resp. $uv \in Z_2$) if u < v and there exists numbers a, b, c, d (resp. d, e, f, g) such that both u and v are of the form $(a, b, c, d, \bullet, \bullet, \bullet)$ (resp. $(\bullet, \bullet, \bullet, d, e, f, g)$).

(Claim 2) If $v \in V(F_n)$, then $N_{F_n}(v)$ can be partitioned into four tournaments.

Proof. Fix $v = (a, b, c, d, e, f, g) \in V(F_n)$. The neighbours u of v such that $uv \in X$ or $vu \in X$ are of the form $(\bullet, a, b, c, d, e, f)$ and $(b, c, d, e, f, g, \bullet)$ respectively. By the definition of edges in $Z_1 \cup Z_2$, vertices of these forms each induce a tournament.

Denote by A and B the neighbours of v of the forms $(\bullet, \bullet, \bullet, a, b, c, d)$ and $(d, e, f, g, \bullet, \bullet, \bullet)$, respectively. Notice that these sets partition the neighbours of v connected to v via edges in Y. Denote by M and N the neighbours of v of the form $(a, b, c, d, \bullet, \bullet, \bullet)$ and $(\bullet, \bullet, \bullet, d, e, f, g)$, respectively. Notice that these sets partition the neighbours of v connected to v via edges in $Z_1 \cup Z_2$. By the definition of edges in $Z_1 \cup Z_2$, each of the sets A, B, M and N induces a tournament. Furthermore, M is complete to A, and B is complete to N via edges in Y. Since $A(F_n) = X \cup Y \cup (Z_1 \cup Z_2)$, these are all the neighbours of v, thus finishing the proof.

(Claim 3) F_n has no cyclic triangle.

Proof. Assume for a contradiction that there exist vertices u, v, w such that u sees v, v sees w, w sees u, and where u = (a, b, c, d, e, f, g).

We claim that no edge in the cyclic triangle is in $Z_1 \cup Z_2$. For a contradiction, assume without loss of generality that $uv \in Z_1 \cup Z_2$, so m(v) = d. Assume first that $wu \in Z_1 \cup Z_2$. Consequently, m(w) = d as well, so $vw \in Z_1 \cup Z_2$, which contradicts that the edges in $Z_1 \cup Z_2$ form an acyclic orientation. Therefore, $wu \notin Z_1 \cup Z_2$. Assume next that $wu \in X$. Consequently, m(w) = c, so $vw \in X$. Thus, $v = (\bullet, \bullet, a, b, c, d, e)$, which contradicts that m(v) = d. Therefore, $wu \notin X$. Thus, $wu \in Y$, so $w = (d, e, f, g, \bullet, \bullet, \bullet)$. But then $vw \notin X \cup (Z_1 \cup Z_2)$, so $vw \in Y$. Thus, the first index of v is g. This contradicts that m(v) = d since d < g. We conclude that no edge in the cyclic triangle is in $Z_1 \cup Z_2$.

We claim that no edge in the cyclic triangle is in X. For a contradiction, assume without loss of generality that $uv \in X$, so $v = (b, c, d, e, f, g, \bullet)$. Assume $vw \in X$. Consequently, $w = (c, d, e, f, g, \bullet, \bullet)$, so by definition $wu \notin Y$. But $wu \notin X$ since $m(u) \notin g$, which contradicts that wu is an arc and $wu \notin Z_1 \cup Z_2$. Thus, $vw \in Y$, so $w = (\bullet, \bullet, \bullet, b, c, d, e)$. But then $wu \notin X$ and $wu \notin Y$. This contradicts that wu is an arc and $wu \notin Z_1 \cup Z_2$. We conclude that no edge in the cyclic triangle is in X. But then every edge in the cyclic triangle is in Y, which contradicts that the edges in Y induce an acyclic digraph. This finishes the proof.

The second and third bullet points are proven in (Claim 2) and (Claim 3) respectively. Since F_n contains D'_n as a subgraph, it follows that $\chi(G_n)/3 \leq \bar{\chi}(F'_n)$ as well. As mentioned, the sequence $\chi(G_n) \to \infty$ as $n \to \infty$, so $\bar{\chi}(F_n) \to \infty$ as $n \to \infty$ as well. Thus, the first bullet point holds.

Proof of Theorem 1.5: Assume F is a directed star of degree at least 5. By Theorem 1.4, every transitive tournament is a hero in F-free digraphs. For the other direction, assume that H is a hero in F-free digraphs. If H is not transitive, then H contains a cyclic triangle. Thus, the cyclic triangle is a hero in F-free digraphs. This, however, contradicts Theorem 2.3 which provides a family of digraphs of arbitrarily high dichromatic number with no cyclic triangles and which is F-free (the construction is F-free because a copy of F contains a vertex whose neighbourhood has a stable set with at least 5 vertices, contradicting that the neighbourhood of every vertex can be partitioned into four tournaments). Thus, we conclude that H is transitive, which finishes the proof.

2.2 Heroes for oriented stars of degree 4

In this section, we prove Theorem 1.6, which we restate for the reader's convenience:

Theorem 1.6. If F is an oriented star of degree 4 and $\{H, F\}$ is $\vec{\chi}$ -finite, then either H is a transitive tournament or $H = \Delta(1, m, m')$ where $m, m' \ge 1$.

The proof is similar to the proof of Theorem 1.5. We start by first restricting some of the heroes in F-free digraphs when F is an oriented star of degree 4. Let the *in-triangle*, denoted by IT, be the digraph on 4 vertices a, b, c, d where d is in-complete from a, b, c and where $\{a, b, c\}$ induces the cyclic triangle. The first step towards proving Theorem 1.6 is proving the following.

Theorem 2.4. If ST is a directed star of degree at least 4, then no hero in ST-free digraphs contains the in-triangle as a subgraph.

This theorem is an immediate consequence to the following theorem.

Theorem 2.5. There exists digraphs F_1, F_2, \ldots such that:

- $\vec{\chi}(F_n) \to \infty \text{ as } n \to \infty;$
- for every $n \ge 1$ and $v \in V(F_n)$, the neighbourhood of v can be partitioned into three tournaments; and
- for every $n \ge 1$, the digraph IT is not a subgraph of F_n .

Proof. Let G_n be the 5-tuple shift-graph with indices in [n], and let D_n be the orientation of G_n where $(a_1, \ldots, a_5)(b_1, \ldots, b_5) \in A(D)$ if $b_i = a_{i+1}$ for every $i \in [4]$. For every $v = (a_1, \ldots, a_5) \in V(G_n)$, define $m(v) = a_3$. Let $X = A(D_n)$. That is, X is the set of edges of the form $(\bullet, b, c, d, e) \to (b, c, d, e, \bullet)$. Moreover, let D'_n be the digraph with $V(D'_n) = V(D_n)$ and $A(D'_n) = X \cup Y$ where Y is the set of arcs of the form $(a, b, c, \bullet, \bullet) \to (\bullet, \bullet, a, b, c)$. Note that as m is strictly increasing along arcs in X, it follows that X is acyclic. Likewise, m is strictly decreasing in Y, so Y is acyclic.

(Claim 4) For every $n \ge 1$, we have $\chi(G_n)/2 \le \tilde{\chi}(D'_n)$.

Proof. We will prove the claim by proving that a set of vertices that induces an acyclic set in D'_n also induces a bipartite subgraph in G_n . Let Λ be a set of vertices that induces an acyclic set in D'_n . Notice that $D_n[\Lambda]$ does not have a directed path of length 3 because if such a path $v_1 \to v_2 \to v_3$ exists, then $v_3v_1 \in A(D'_n)$ contradicting that Λ is an acyclic set in D'_n . Thus, by Theorem 2.2, we have $\chi(G_n[\Lambda]) \leq 2$ as desired.

Finally, let F_n be the digraph with $V(F_n) = V(D'_n)$ and $A(F_n) = X \cup Y \cup Z_1 \cup Z_2$ where we define Z_1 and Z_2 as follows. Let < be a complete ordering of $V(F_n)$. Define Z_1 (resp. Z_2) as the set of edges such that $uv \in Z_1$ (resp. $uv \in Z_2$) if u < v and there exists numbers a, b, c(resp. c, d, e) such that both u and v are of the form $(a, b, c, \bullet, \bullet)$ (resp. $(\bullet, \bullet, c, d, e)$.

(Claim 5) If $v \in V(F_n)$, then $N_{F_n}(v)$ can be partitioned into three tournaments.

Proof. Fix $v = (a, b, c, d, e) \in V(F_n)$. The neighbours u of v such that $uv \in X$ or $vu \in X$ are of the form (\bullet, a, b, c, d) and (b, c, d, e, \bullet) . Vertices of the former type are complete to the vertices of the latter type by edges in Y. Thus, vertices adjacent to v via an edge in X form a clique.

Denote by A and B the neighbours of v of the forms $(\bullet, \bullet, a, b, c)$ and $(c, d, e, \bullet, \bullet)$, respectively. Notice that these sets partition the neighbours of v connected to v via edges in Y. Denote by M and N the neighbours of v of the form $(a, b, c, \bullet, \bullet)$ and $(\bullet, \bullet, c, d, e)$,

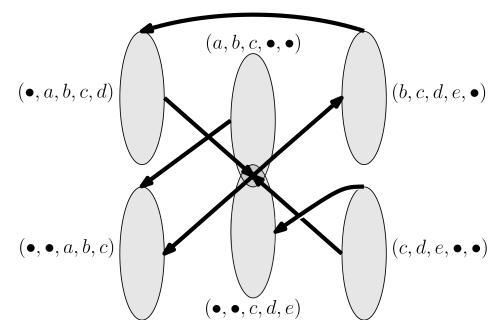


Figure 2: Illustration of the neighbourhood of the vertex (a, b, c, d, e) in F_n .

respectively. Notice that these sets partition the neighbours of v connected to v via edges in $Z_1 \cup Z_2$. By the definition of edges in $Z_1 \cup Z_2$, each of the sets A, B, M and N induce a tournament. Furthermore, M is complete to A, and B is complete to N via edges in Y. Since $A(F_n) = X \cup Y \cup (Z_1 \cup Z_2)$, these are all the neighbours of v, thus finishing the proof. Figure 2 illustrates the neighbourhood of a vertex.

(Claim 6) Every cyclic triangle in F_n has two edges in X and one edge in Y.

Proof. Let u, v, and w be vertices such that u sees v, v sees w, and w sees u. For a contradiction, assume that $uv \in Z_1 \cup Z_2$, and set v = (a, b, c, d, e). Since $uv \in Z_1 \cup Z_2$, we have m(u) = c. If $vw \in X$, then m(w) = d. Since m(u) < m(w), it follows that $wu \in Y$, so m(u) = b, a contradiction. If $vw \in Z_1 \cup Z_2$, then m(w) = c, contradicting the fact that edges in $Z_1 \cup Z_2$ induce an acyclic graph. Thus, $vw \in Y$, so m(w) = a. Since m(w) < m(u), it must be that $wu \in X$, so m(u) = b, a contradiction. We conclude that every edge in the directed triangle is not in $Z_1 \cup Z_2$. Since each of X and Y span acyclic graphs, we may assume $uv \in X$ and $vw \in Y$. Consequently, m(u) = b and m(w) = a, so m(w) < m(u). This implies that $wu \in X$, proving that directed triangles have two edges in X and one edge in Y.

The second bullet point is true by (Claim 4). Since F_n contains D'_n as a subgraph, by (Claim 5), we have $\chi(G_n)/2 \leq \bar{\chi}(F'_n)$ as well. As mentioned before, we have $\chi(G_n) \to \infty$ as $n \to \infty$. Thus, $\bar{\chi}(F_n) \to \infty$. This proves the first bullet point. We prove the third bullet point by contradiction. Assume *IT* is a subgraph of F_n . Let $u \to v \to w \to u$ be the directed cycle in F_n and x be the vertex in-complete from $\{u, v, w\}$. Without loss of generality, by (Claim 6), we may assume that $uv, vw \in X$. Set v = (a, b, c, d, e). If $ux \in X$, then m(x) = c, and since m(w) = d > m(x), it follows that $wx \in Y$. This implies that m(x) = b, a contradiction to m(x) = c. If $ux \in Y$, then m(x) < a, so m(x) < m(v). It follows that $vx \in Y$, implying

m(x) = a, a contradiction. Thus, $ux \in Z_1 \cup Z_2$, so m(x) = b. Since m(x) < m(v), it follows that $vx \in Y$, so m(x) = a, a contradiction. This shows that F_n is *IT*-free, which finishes the proof.

Proof of Theorem 1.6: Let the out-triangle, denoted by *OT*, be *IT* with arcs reversed.

(Claim 7) OT is not a hero in ST-free digraphs.

Proof. Let ST' be ST with arcs reversed. By Theorem 2.4, the $\{ST', IT\}$ -free digraphs do not have bounded dichromatic number. By reversing arcs, we get that $\{ST, OT\}$ -free digraphs do not have bounded dichromatic number.

Assume for a contradiction that there exists a hero H such that H is not acyclic and $H \neq \Delta(1, m, m')$ for integers $m, m' \geq 1$. If H is not strongly connected, then there exists nonempty tournaments $H_1 \subseteq H$ and $H_2 \subseteq H$ such that $V(H_1) \cap V(H_2) = \emptyset$, H_1 is not transitive, and either H_1 is out-complete to H_2 , or H_2 is out-complete to H_1 . Since H_1 is not transitive, it contains a directed triangle T. If T is out-complete to H_1 , then H contains a copy of IT, a contradiction. Thus, H_2 is out-complete to T, but then H contains a copy of OT, a contradiction. We conclude H is strongly connected.

Since H is strongly connected, by Theorem 1.2, it follows that $H = \Delta(1, m, H')$ or $H = \Delta(1, H', m)$ where H' is a hero in ST-free digraphs. It is then enough to prove that H' is acyclic. Suppose not. That is, assume that H' contains a directed triangle T. In either case, by the structure of strongly connected heroes, H contains a copy of IT, a contradiction. \Box

3 Localized and colocalized digraphs

In this section, we introduce the concept of localized and colocalized digraphs and how these conditions relate to Theorem 1.12. To elaborate, we need some definitions.

In a digraph D, we say the *out-neighbourhood* (resp. *in-neighbourhood*) of a set of vertices $S \subseteq V(D)$, denoted by $N^+(S)$ (resp. $N^-(S)$), is the set of vertices not in S that vertices $v \in S$ see (resp. $v \in S$ is seen by). The neighbourhood of S is $N(S) := N^+(S) \cup N^-(S)$. When $S = \{v\}$, we use $N(v), N^+(v)$, and $N^-(v)$ to denote $N(S), N^+(S)$, and $N^-(S)$ respectively. Similarly, the *non-neighbourhood* of a set S, denoted $N^0(S)$, is the set of vertices not in S or the neighbourhood of S. If $S = \{v\}$, we let $N^0(v)$ be the set of non-neighbours of v. So we have:

$$N^{+}(S) = \bigcup_{s \in S} N^{+}(s) \setminus S;$$
$$N^{-}(S) = \bigcup_{s \in S} N^{-}(s) \setminus S;$$
$$N^{0}(S) = \bigcap_{s \in S} N^{0}(s) \setminus S.$$

A digraph D is k-local if, for every $v \in V(D)$, we have $\vec{\chi}(N^+(v)) \leq k$. Furthermore, it is k-colocal if, for every $v \in V(D)$, we have $\vec{\chi}(N^-(v)) \leq k$. The concept of k-local digraphs was introduced by Harutyunyan, Le, Newman, and Thomassé [9]. A digraph F cooperates if H is a hero in F-free digraphs when one of the following three conditions hold:

- $H = K_1;$
- $H = H_1 \Rightarrow H_2$ where H_1 and H_2 are heroes in *F*-free digraphs; or
- $H = \Delta(1, 1, H_1)$ where H_1 is a hero in *F*-free digraphs.

In other words, a digraph F cooperates when their heroes can be used to construct bigger heroes by using the operations described above. Notice that Theorem 1.11 is equivalent to proving $K_1 + K_2$ cooperates, and Theorem 1.12 is equivalent to proving that $rK_1 + \vec{P}_3$ cooperates for every $r \ge 1$. Of the three conditions above, the first is true for every F. For the second, we require a result of Aboulker, Aubian and Charbit:

Theorem 3.1 (Aboulker, Aubian, and Charbit [1]). Let H_1 , H_2 and F be digraphs such that $H_1 \Rightarrow H_2$ is a hero in F-free digraphs, and H_1 and H_2 are heroes in $\{K_1 + F\}$ -free digraphs. Then $H_1 \Rightarrow H_2$ is a hero in $\{K_1 + F\}$ -free digraphs.

Theorem 3.1 shows that if the second condition above holds for F, then it also holds for $K_1 + F$. Therefore, the main goal of this section is to develop sufficient conditions for "lifting" the third condition from F to $K_1 + F$.

We require the following technical definitions. We say a digraph F is *localized* (resp. *colocalized*) if for every $r \ge 1$, the following two:

- $\Delta(1,1,H)$ is a hero in $\{(r-1)K_1 + F\}$ -free digraphs; and
- *H* is a hero in $\{rK_1 + F\}$ -free digraphs

imply that, for every fixed $k \ge 1$, $\Delta(1, 1, H)$ is a hero in k-local (resp. colocal) $\{rK_1 + F\}$ -free digraphs. These definitions are meant to describe the properties a digraph F needs to have for the proof strategy of Theorem 1.9 to apply to F, where in Theorem 1.9, $F = K_1$.

The first step to proving Theorem 1.12 is proving the following.

Theorem 3.2. Let F be a localized and colocalized digraph. If F cooperates, then $rK_1 + F$ cooperates for every $r \ge 0$.

Aboulker, Aubian, and Charbit [1], in Question 5.4, ask the following. If H is a hero in $\{K_1 + F\}$ -free digraphs and $\Delta(1, 1, H)$ is a hero in F-free digraphs, does it follow that $\Delta(1, 1, H)$ is a hero in $\{K_1 + F\}$ -free digraphs? Theorem 3.2 proves that the question has an affirmative answer if F is localized and colocalized.

The following lemma simplifies using Theorem 3.2 for certain cases.

Lemma 3.3. Let F' be a digraph isomorphic to F with every arc reversed. If F is localized, then F' is colocalized.

Proof. Assume that $\Delta(1, 1, H)$ is a hero in $\{(r-1)K_1 + F'\}$ -free digraphs, and assume that H is a hero in $\{rK_1 + F'\}$ -free digraphs. Let H' denote the digraph obtained from H by reversing all arcs. It follows that $\Delta(1, 1, H')$ is a hero in $\{(r-1)K_1 + F\}$ -free digraphs, and that H' is a hero in $\{rK_1 + F\}$ -free digraphs.

Fix $k \ge 1$. We want to prove that $\Delta(1, 1, H)$ is a hero in k-colocal $\{rK_1 + F'\}$ -free digraphs. Let D be a k-colocal $\{rK_1 + F', \Delta(1, 1, H)\}$ -free digraph. Let D' be D with every arc reversed, and notice that D' is k-local and $\{rK_1 + F, \Delta(1, 1, H')\}$ -free. Since F is localized, $\Delta(1, 1, H')$ is a hero in k-local $\{rK_1 + F\}$ -free digraphs. Thus, there exists an integer c (depending only on r, k, H, F) such that $\vec{\chi}(D') \leq c$ and hence $\vec{\chi}(D) \leq c$. Therefore, $\Delta(1, 1, H)$ is a hero in $\{rK_1 + F'\}$ -free digraphs. This finishes the proof.

In this section, we prove Theorem 3.2. The following lemma reduces the task of proving Theorem 3.2 to proving the case where r = 1.

Lemma 3.4. If F is localized, then $K_1 + F$ is localized. Similarly, if F is colocalized, then $K_1 + F$ is colocalized.

Proof. This is immediate from the definition of (co)localized graphs.

We will use the proof strategy devised by Harutyunyan, Le, Newman, and Thomassé [9] to prove that if H is a hero in tournaments, then $\Delta(1, m, H)$, where $m \ge 1$, is a hero in rK_1 -free digraphs, for $r \ge 2$.

Their strategy relies on the analysis of bag chains. A β -bag is a subset B of V(D) such that $\vec{\chi}(B) = \beta$, and a (c, β) -bag-chain is a sequence of β -bags B_1, \ldots, B_t such that for every $1 \le i \le t$ and $v \in B_i$, we have:

- $\vec{\chi}(N^+(v) \cap B_{i-1}) \leq c$, and
- $\vec{\chi}(N^-(v) \cap B_{i+1}) \leq c.$

The *length* of the (c, β) -bag-chain is t.

As a brief outline for the upcoming proof, for a $\{\Delta(1,1,H), rK_1 + F\}$ -free digraph D, we want to prove the following:

- 1. For some choice of an integer c, and for every $\beta \in \mathbb{N}$, the absence of a (c, β) -bag-chain of length 8 implies that the digraph has bounded dichromatic number.
- 2. For some choice of c and β' , (c, β') -bag-chains have a bounded dichromatic number.
- 3. For some choice of c and β' , if there is a (c, β') -bag-chain, then vertices not in a maximal (c, β') -bag-chain have bounded dichromatic number as well.

We will need the following Lemma:

Lemma 3.5 (Aboulker, Aubian, and Charbit [1]). Let D be a digraph and let (X_1, \ldots, X_n) be a partition of V(D). Suppose that k is an integer such that:

- for every $1 \le i \le n$, we have $\vec{\chi}(X_i) \le k$, and
- for every $1 \le i < j \le n$, if there is an arc uv with $u \in X_j$ and $v \in X_i$, then $\vec{\chi}(X_{i+1} \cup \cdots X_j) \le k$.

Then $\vec{\chi}(D) \leq 2k$.

We need a generalization of Lemma 3.8 in [1], which in turn is an adaptation of 4.4 in [3]. Our proof differs from theirs only slightly.

Lemma 3.6. Assume that there exists an integer m such that:

- $\{\Delta(1,1,H),F\}$ -free digraphs D have $\vec{\chi}(D) \leq m$; and
- $\{H, K_1 + F\}$ -free digraphs D have $\vec{\chi}(D) \leq m$.

If D is a $\{\Delta(1,1,H), K_1 + F\}$ -free digraph with a partition (X_1, \ldots, X_n) of V(D), and m' an integer such that:

- for every $1 \le i \le n$, we have $\vec{\chi}(X_i) \le m'$;
- for every $1 \le i \le n$ and for every $v \in X_i$, we have $\vec{\chi}(N^+(v) \cap (X_1 \cup \cdots \cup X_{i-1})) \le m'$; and
- for every $1 \le i \le n$ and for every $v \in X_i$, we have $\vec{\chi}(N^-(v) \cap (X_{i+1} \cup \cdots \cup X_n)) \le m'$;

then $\vec{\chi}(D) \le 6(m+m') + 2$.

Proof. We start with the following claim.

(Claim 8) $\vec{\chi}(N^0(v)) \le m$ for every $v \in D$.

Proof. Since D is $K_1 + F$ -free, it follows that $N^0(v)$ is F-free. Furthermore, since D is $\Delta(1, 1, H)$ -free, we have by the definition of m that $\vec{\chi}(N^0(v)) \leq m$.

Set k' = 2(m + m') + m + 1. It suffices to show that the partition (X_1, \ldots, X_n) satisfies the hypothesis of Lemma 3.5 with k = k' + m'. Let uv be an edge such that $u \in X_j$, $v \in X_i$, and i < j, and set $X = X_{i+1} \cup \cdots \cup X_{j-1}$. For a contradiction, assume that $\vec{\chi}(X) > k'$. Let $A = (N^-(v) \cup N^0(v)) \cap X$. By the hypothesis and by (Claim 8), the dichromatic number of Ais at most m+m'. Similarly, the set $B = (N^+(u) \cup N^0(u)) \cap X$ has dichromatic number at most m+m'. Thus, the set $X' = X \setminus (A \cup B)$ has $\vec{\chi}(X') > k' - 2(m + m') > m$. Consequently, there exists a copy X'' of H in X'. But then, by the definitions of A and B, it follows that $\{u, v\} \cup X''$ induces a copy of $\Delta(1, 1, H)$, a contradiction. Thus, $\vec{\chi}(X) \le k'$, so $\vec{\chi}(X \cup X_j) \le k' + m'$, as desired.

We dedicate the rest of the section to proving Theorem 3.2.

Proof of Theorem 3.2. By Lemma 3.4, it is enough to prove the result for r = 1. That is, we want to prove that $K_1 + F$ cooperates. Evidently, $H = K_1$ is a hero in every class of graphs. Assume then that H_1 and H_2 are heroes in $\{K_1 + F\}$ -free digraphs. Consequently, they are heroes in F-free digraphs, and since F cooperates, it follows that $H_1 \Rightarrow H_2$ is a hero in F-free digraphs. Thus, by Theorem 3.1, it follows that $H_1 \Rightarrow H_2$ is a hero in $\{K_1 + F\}$ -free digraphs.

It remains to show that $\Delta(1, 1, H)$ is a hero in $\{K_1 + F\}$ -free digraphs whenever H is a hero in $\{K_1 + F\}$ -free digraphs. However, since we exclude $\Delta(1, 1, H)$ instead of $\Delta(1, k, H)$, in some places we are able to simiplify the proofs.

Let us assume that H is a hero in $\{K_1 + F\}$ -free digraphs. Let c be an integer such that $\{K_1 + F, H\}$ -free digraphs D have $\vec{\chi}(D) \leq c$. Since H is a hero in $K_1 + F$ -free digraphs, H is a hero in F-free digraphs as well, and since F cooperates, it follows that $\Delta(1, 1, H)$ is a hero in F-free digraphs. Let b' be such that $\{F, \Delta(1, 1, H)\}$ -free digraphs D have $\vec{\chi}(D) \leq b'$. Since F is localized, set $f_1(r, k, H)$ as the function such that $\{\Delta(1, 1, H), rK_1 + F\}$ k-local digraphs D have $\vec{\chi}(D) \leq f_1(r, k, H)$ whenever $\Delta(1, 1, H)$ is a hero in $\{(r-1)K_1 + F\}$ -free

digraphs, and H is a hero in $\{rK_1 + F\}$ -free digraphs. Let $f_2(r, k, H)$ be the equivalent but from the fact that F is colocalized. Now let $f(r, k, H) = \max\{f_1(r, k, H), f_2(r, k, H)\}$. Set

$$\hat{f}(\beta) \coloneqq 2f(1, 2f(1, 2f(1, \beta, H) + 1, H) + 1, H).$$

Finally, set $\beta' = 2|V(H)|(c+b') + b' + 1$. We will show that $\{\Delta(1,1,H), K_1 + F\}$ -free digraphs D have $\vec{\chi}(D) \leq b$ where

$$b = 6(\max\{b', c\} + \beta') + 3\hat{f}(\beta') + 2.$$

Assume that D is a $\{\Delta(1,1,H), K_1 + F\}$ -free digraph. Henceforth, we will use the terms β -bags and β -bag-chains to refer to (c,β) -bags and (c,β) -bag-chains. To achieve the first objective, we start by proving that the absence of a β -bag-chain of length 2 bounds the dichromatic number. Call a vertex $v \beta$ -red if $\vec{\chi}(N^+(v)) \leq \beta$, and β -blue if $\vec{\chi}(N^-(v)) \leq \beta$. The following two claims are the equivalent of Lemma 4.11 in [9], although our proof is significantly simpler as we deal with $\Delta(1,1,H)$ instead of $\Delta(1,m,H)$ for some m.

(Claim 9) For every $\beta \in \mathbb{N}$, if D does not have a β -bag-chain of length 2, then $\vec{\chi}(D) \leq 2f(1,\beta,H)$.

Proof. Set R, B, and U as the sets of β -red, β -blue and uncoloured vertices respectively. We start by proving that U is empty. For the sake of a contradiction, assume that $u \in U$. Set $B_1 = N^-(u)$ and $B_2 = N^+(u)$. We claim that B_1, B_2 is a β -bag-chain. Let $v \in B_1$. If $\vec{\chi}(N^-(v) \cap B_2) > c$, then there exists a copy X of H in B_2 . But then $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$ in D, a contradiction. A symmetric argument proves that if $v \in B_2$, then $\vec{\chi}(N^+(v) \cap B_1) \leq c$. That is, B_1, B_2 is β -bag-chain of length 2, a contradiction. Thus, U is empty. Notice that D[R] is d-local, so $\vec{\chi}(R) \leq f(1, d, H)$. Similarly, D[B] is β -colocal, so $\vec{\chi}(B) \leq f(1, \beta, H)$, and hence $\vec{\chi}(D) \leq 2f(1, \beta, H)$, as claimed.

(Claim 10) For every $\beta \in \mathbb{N}$, if D does not have a β -bag-chain of length 8, then $\vec{\chi}(D) \leq \hat{f}(\beta)$.

Proof. We proceed by contrapositive. Assume that $\vec{\chi}(D) > \hat{f}$. By (Claim 9), there exists a $(2f(1, 2f(1, \beta, H) + 1, H) + 1)$ -bag-chain of length 2, say A_1, A_2 . By definition of a bag and by (Claim 9), it follows that A_1 contains a $(2f(1, \beta, H) + 1)$ -bag-chain of length 2 consisting of bags A_1^1, A_1^2 . Similarly, A_2 contains the $(2f(1, \beta, H) + 1)$ -bag-chain A_2^1, A_2^2 . Finally, using the same reasoning, we can split each of these bags into the β -bag-chain B_1, \ldots, B_8 where B_1, B_2 is the β -bag-chain of A_1^1 , where B_3, B_4 is the β -bag-chains of A_1^2 , and so on. But then B_1, \ldots, B_8 is a β -bag-chain of length 8, finishing the proof.

With the first objective achieved, we now prove the second objective. From now on, we assume B_1, \ldots, B_t is a β' -bag-chain in D with t maximum, where $\beta' = 2|V(H)|(c+b')+b'+1$. For convenience, define $B_{i,j}$, where $i \leq j$, as the union of the bags B_i, \ldots, B_j .

(Claim 11) $\vec{\chi}(N^0(v)) \leq b'$ for every $v \in V(D)$.

Proof. This is a consequence of the fact that the set of non-neighbours of v is $\{F, \Delta(1, 1, H)\}$ -free. Thus, the result holds by the definition of b'.

The following is the equivalent of Claim 4.3 in [9], although we are able to prove a stronger statement.

(Claim 12) For every $i \ge 1$, $v \in B_i$, and s > 1,

- $N^+(v) \cap B_{i-s} = \emptyset$, and
- $N^{-}(v) \cap B_{i+s} = \emptyset$.

Proof. For a contradiction, let s > 1 be the smallest integer such that there exist vertices u and v such that $u \in N^+(v) \cap B_{i-s}$ or $u \in N^-(v) \cap B_{i+s}$. We deal first with the former.

Suppose first that s = 2. Let $A = (N^-(u) \cup N^+(v)) \cap B_{i-1}$, and $B = (N^0(u) \cup N^0(v)) \cap B_{i-1}$. By the definition of a β -bag-chain and (Claim 11), $\vec{\chi}(A) \leq 2c$ and $\vec{\chi}(B) \leq 2b'$. Thus, $\vec{\chi}(B_{i-1} \setminus (A \cup B)) \geq \beta' - 2c - 2b' > c$. By the definition of c, there exists a copy X of H in $B_{i-1} \setminus (A \cup B)$. But by the definition of A and B, this implies that $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$, a contradiction.

Suppose then that s > 2. The proof for this case is very similar. Let $A = (N^{-}(u) \cup N^{+}(v)) \cap B_{i-1})$ and $B = (N^{0}(u) \cup N^{0}(v)) \cap B_{i-1}$. By the minimality of s, and since s > 1, we have $A = N^{+}(v) \cap B_{i-1}$. By (Claim 11), it follows that $\vec{\chi}(B) \leq 2b'$. Thus, $\vec{\chi}(B_{i-1} \setminus (A \cup B)) \geq \beta' - 2b' - c > c$. By the definition of c, there exists a copy X of H in $B_{i-1} \setminus (A \cup B)$. But then, by the definition of A and B, this implies that $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$, a contradiction.

The proof for the case where $u \in N^-(v) \cap B_{i+s}$ is analogous with arcs reversed.

The following is the equivalent of Claim 4.4 and Claim 4.5 in [9].

(Claim 13) For every i and $v \in B_i$,

- $\vec{\chi}(N^+(v) \cap B_{1,i-1}) \leq c, and$
- $\vec{\chi}(N^-(v) \cap B_{i+1,t}) \leq c.$

Proof. The result is immediate from (Claim 12) and the definition of β' -bag-chains.

We can now prove our second objective:

(Claim 14) $\vec{\chi}(B_{1,t}) \le 6(\max\{b',c\} + \beta') + 2.$

Proof. Applying Lemma 3.6 with $m = \max\{b', c\}$, and $m' = \beta'$, where the hypothesis holds by (Claim 13), it follows that $\vec{\chi}(B_{1,t}) \leq 6(\max\{b', c\} + \beta') + 2$.

For our final objective, we will partition the vertices of $V(D) \\ \\ > B_{1,t}$ in such a way that they behave similarly to a bag chain as well. We partition $V(D) \\ > B_{1,t}$ into sets Z_i we call zones such that $v \\ \in Z_i$ if i is the largest index such that $\chi(N^-(v) \cap B_i) > c$, and $v \\ \in Z_0$ if no such i exists. Furthermore, for convenience, set $Z_{i,j} := Z_i \cup \cdots \cup Z_j$ for $i \\ \le j$. We proceed to prove claims that will allow us to bound $\chi(Z_{0,t})$ by using Lemma 3.6. To this end, in (Claim 15–Claim 17), we will show that zones interact with the bag chain and each other in limited ways.

(Claim 15) For every *i* and every $v \in Z_i$,

- $\vec{\chi}(N^-(v) \cap B_{i+r}) \leq c$ for $r \geq 1$, and
- $N^+(v) \cap B_{i-r} = \emptyset$ for $r \ge 2$.

Proof. The first bullet point is true by the definition of Z_i . We prove the second. For a contradiction, assume that there exists a vertex u such that $u \in N^+(v) \cap B_{i-r}$. We claim that $\tilde{\chi}(N^-(v) \cap B_{i-1}) \leq b' + 2c$. For a contradiction, assume this is not the case. Set

$$A := (N^0(u) \cup N^-(u)) \cap (N^-(v) \cap B_{i-1}).$$

By (Claim 11) and (Claim 13), $\vec{\chi}(A) \leq b' + c$, so $\vec{\chi}((N^-(v) \cap B_{i-1}) \setminus A) > c$. Thus, there exists a copy X of H in $(N^-(v) \cap B_{i-1}) \setminus A$. But then $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$, a contradiction.

Thus, $\vec{\chi}(N^-(v) \cap B_{i-1}) \leq b' + 2c$. Since $\vec{\chi}(N^0(v) \cap B_{i-1}) \leq b'$ by (Claim 11), and since B_{i-1} is a β' -bag, it follows that $\vec{\chi}(N^+(v) \cap B_{i-1}) \geq |V(H)|(b'+c) + 1$. By the definition of a zone, there exists a copy X' of H in $N^-(v) \cap B_i$. Set

$$A' := \bigcup_{x \in X'} (N^0(x) \cup N^+(x)) \cap (N^+(v) \cap B_{i-1}).$$

By (Claim 11) and (Claim 13), it follows that $\vec{\chi}(A') \leq |V(H)|(b'+c)$. Thus, $\vec{\chi}((N^+(v) \cap B_{i-1}) \setminus A') > 0$, so there exists a vertex u' in $(N^+(v) \cap B_{i-1}) \setminus A'$. This, however, implies that $\{u', v, X'\}$ induces a copy of $\Delta(1, 1, H)$, a contradiction.

(Claim 16) For every $i \ge 0$, $v \in B_i$, and $r \ge 2$, we have $N^+(v) \cap Z_{i-r} = \emptyset$.

Proof. For a contradiction, assume that there exists a vertex u such that $u \in N^+(v) \cap Z_{i-r}$. Now let

$$A = (N^{0}(u) \cup N^{-}(u)) \cap B_{i-1},$$

and let

$$B = (N^+(v) \cup N^0(v)) \cap B_{i-1}$$

By the definition of zones and by (Claim 11), $\vec{\chi}(A) \leq b' + c$, and by the definition of a β -bagchain and (Claim 11), $\vec{\chi}(B) \leq b' + c$. Thus, $\vec{\chi}(B_{i-1} \smallsetminus (A \cup B)) \geq \beta' - (b' + c) - (b' + c) > c$. Consequently, there exists a copy X of H in $B_{i-1} \smallsetminus (A \cup B)$. But by the definition of A and $B, \{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$, a contradiction.

(Claim 17) For every $i, v \in B_i$, and $r \ge 3$, we have $N^-(v) \cap Z_{i+r} = \emptyset$.

Proof. For a contradiction, assume that there exists a vertex u such that $u \in N^-(v) \cap Z_{i+r}$. Now let

$$A \coloneqq (N^0(u) \cup N^+(u)) \cap B_{i+1}$$

and let

$$B \coloneqq (N^0(v) \cup N^-(v)) \cap B_{i+1}$$

By (Claim 11) and (Claim 16), $\vec{\chi}(A) \leq b'$. Furthermore, by (Claim 11) and the definition of bags, $\vec{\chi}(B) \leq b' + c$. Thus, $\vec{\chi}(B_{i+1} \setminus (A \cup B)) \geq \beta' - b' - (b' + c) > c$. Consequently, there exists a copy X of H in $B_{i+1} \setminus (A \cup B)$. But by the definition of A and B, it follows that $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$, a contradiction.

Finally, we are ready to bound $\vec{\chi}(Z_i)$. The following is the equivalent of Claim 4.10 in [9].

(Claim 18) For every $i, \vec{\chi}(Z_i) \leq \hat{f}(\beta')$.

Proof. By (Claim 10), it is enough to prove that zones do not have a β' -bag-chain of length 8. We will do this by using the maximality of t. Assume for a contradiction that Y_1, \ldots, Y_8 is a β' -bag-chain of length 8 in Z_i . By (Claim 15), (Claim 16) and (Claim 17), $B_1, \ldots, B_{i-3}, Y_1, \ldots, Y_8, B_{i+3}, \ldots, B_t$ is a longer β' -bag-chain than B_1, \ldots, B_t which contradicts the maximality of t.

To finish the proof, it remains to show we can partition $Z_{0,t}$ such that we are able to colour each part. The following is the equivalent of Claim 4.9 in [9].

(Claim 19) For every i and $v \in Z_i$,

- $N^+(v) \cap Z_{0,i-3} = \emptyset$, and
- $N^-(v) \cap Z_{i+3,t} = \emptyset$.

Proof. Let us prove the first bullet point. Suppose for a contradiction that there exists a vertex u such that $u \in N^+(v) \cap Z_{0,i-3}$. Now let

$$A \coloneqq (N^0(u) \cup N^-(u)) \cap B_{i-2},$$

and

$$B := (N^0(v) \cup N^+(v)) \cap B_{i-2}.$$

By (Claim 11) and the definition of zones, $\vec{\chi}(A) \leq b' + c$. Similarly, $\vec{\chi}(B) \leq b'$ by (Claim 11) and (Claim 15). Since B_{i-2} is a β' -bag, we have $\vec{\chi}(B_{i-2} \setminus (A \cup B)) > \beta' - (b' + c) - b' > c$. By the definition of c, there exists a copy X of H in $B_{i-2} \setminus (A \cup B)$. But then, by the definitions of A and B, it follows that $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$, a contradiction. A similar argument, using the established claims, gives the second bullet point.

We are ready to prove that $\vec{\chi}(Z_{0,t})$ is bounded.

(Claim 20) $\vec{\chi}(Z_{0,t}) \leq 3\hat{f}(\beta').$

Proof. Let $Z_i = \bigcup_{j \cong i \mod 3} Z_j$. By (Claim 19), every strongly connected component in Z_i is contained in a zone Z_j . Thus, by (Claim 18), $\vec{\chi}(Z_i) \leq \hat{f}(\beta')$. Since Z_1, Z_2, Z_3 is a partition of $Z_{0,t}$, it follows that $\vec{\chi}(Z_{0,t}) \leq 3\hat{f}(\beta')$ as claimed.

We are ready to finish the proof. Since $V(D) = B_{1,t} \cup Z_{0,t}$, and by (Claim 14) and (Claim 20), we have:

$$\vec{\chi}(D) \le \vec{\chi}(B_{1,t}) + \vec{\chi}(Z_{0,t}) \le 6(\max\{b',c\} + \beta') + 2 + 3f(\beta)$$

as claimed.

4 Forbidding $rK_1 + \vec{P}_3$

In this section, we prove Theorem 1.12, which we restate for the reader's convenience.

Theorem 1.12. Let $r \in \mathbb{N}$. The set $\{H, rK_1 + \vec{P}_3\}$ is $\vec{\chi}$ -finite if:

•
$$H = K_1;$$

- $H = H_1 \Rightarrow H_2$ where $\{H_i, rK_1 + \vec{P}_3\}$ is $\vec{\chi}$ -finite for $i \in \{1, 2\}$; or
- $H = \Delta(1, 1, H_1)$ where $\{H_1, rK_1 + \vec{P}_3\}$ is $\vec{\chi}$ -finite.

Equivalently, we will prove that for every $r \ge 1$, the digraph $rK_1 + \vec{P}_3$ cooperates. We will use Theorem 3.2 to do this. Thus, we need to prove that \vec{P}_3 cooperates, and that \vec{P}_3 is localized and colocalized. The fact that \vec{P}_3 cooperates is a consequence of Theorem 1.7. Notice that by Lemma 3.3, we only need to show that \vec{P}_3 is localized.

To prove that \vec{P}_3 is localized, we use domination. We say a set of vertices S_1 dominates a set of vertices S_2 , or equivalently S_1 is a dominating set for S_2 , if every vertex in $S_2 \\ S_1$ is seen by a vertex in S_1 . A digraph F dominates if, for every $r \ge 1$, the following two:

- $\Delta(1,1,H)$ is a hero in $\{(r-1)K_1 + F\}$ -free digraphs;
- *H* is a hero in $\{rK_1 + F\}$ -free digraphs;

imply that there exists a function g(r, k, H) such that for every $\{\Delta(1, 1, H), rK_1 + F\}$ -free k-local digraph D, either $\vec{\chi}(D) \leq g(r, k, H)$, or F-free acyclic induced subsets S of V(D) have a dominating set in D of size at most g(r, k, H). While this definition is rather technical, it allows us to formulate a proof in such a way that parts of it are more general than the case of \vec{P}_3 .

We want to prove that if F dominates, then F is localized. The concept that a digraph F dominates, as well as how this implies that F is localized, is meant to generalize the proof strategy devised by Harutyunyan, Le, Newman, and Thomassé [9] to prove that k-local rK_1 -free digraphs, where $r \ge 2$, have bounded dichromatic number.

To prove that digraphs that dominate are localized, we use a concept introduced in [9]. A family of digraphs \mathcal{C} is *tamed* if, for every m, there exists integers M and l such that if $D \in \mathcal{C}$ has $\vec{\chi}(D) \geq M$, then there exists a subset $X \subseteq V(D)$ such that $|X| \leq l$ and $\vec{\chi}(X) \geq m$. The following proof is a slight generalization of the proof of Claim 2.4 in [9].

Lemma 4.1. If F dominates and the following two hold:

- $\Delta(1,1,H)$ is a hero in $\{(r-1)K_1 + F\}$ -free digraphs, and
- *H* is a hero in $\{rK_1 + F\}$ -free digraphs,

then, for every $k \ge 1$, the family of $\{\Delta(1,1,H), rK_1 + F\}$ -free k-local digraphs is tamed.

Proof. We proceed by induction on m (from the definition of tamed). The case when m = 1 is immediate. Assume the statement holds for m. Let M and l be the corresponding integers. Let c be an integer such that $\{rK_1 + F, H\}$ -free digraphs D have $\vec{\chi}(D) \leq c$, and let b be an integer such that $\{(r-1)K_1 + F, \Delta(1, 1, H)\}$ -free digraphs D have $\vec{\chi}(D) \leq b$. Since F

dominates, let g(r, k, H) be the associated function. Furthermore, let p = M + bl + kl + 1, and let d = m((g(r, k, H) + r)p + 1) + 1. Note that, by the pigeonhole principle, d is the smallest number such that if a set S of size d is m-coloured, then there exists a monochromatic subset of size at least (g(r, k, H) + r)p + 2. We claim that the statement holds for m + 1 when $M' = \max\{g(r, k, H) + 1, kd, M + d(b + k + 1)\}$ and $l' = d + l + l\binom{d}{(g(r, k, H) + r)p + 2}$.

Assume that D is a $\{\Delta(1,1,H), rK_1 + F\}$ -free k-local digraph, and assume $\vec{\chi}(D) \ge M'$. We start with the following claim.

(Claim 21) $\vec{\chi}(N^0(v)) \leq b$ for every $v \in V(D)$.

Proof. Since D is $\{rK_1 + F, \Delta(1, 1, H)\}$ -free, it follows that $N^0(v)$ is $\{(r-1)K_1 + F, \Delta(1, 1, H)\}$ -free, so the claim follows by definition of b.

Since $\vec{\chi}(D) \ge M'$, we have $\vec{\chi}(D) > g(r, k, H)$. Let *B* be a minimum dominating set for *D*. Since *D* is *k*-local, it follows that $\vec{\chi}(D) \le |B|k$, so $|B| \ge M'/k \ge d$. Pick $W \subseteq B$ such that |W| = d. By the choice of *M'* and the size of *B*, we know this subset exists. Notice that $\vec{\chi}(\bigcup_{w \in W} N^0(w)) \le bd$ by (Claim 21), and $\vec{\chi}(\bigcup_{w \in W} N^+(w)) \le kd$ since *D* is *k*-local. Since $\vec{\chi}(D \setminus W) \ge M' - d$, it follows that the set \mathcal{A} of vertices out-complete to *W* has dichromatic number at least $M' - d - bd - kd \ge M$. By the definition of *M*, there exists a set *A* out-complete to *W* of size at most *l* and dichromatic number at least *m*.

We will define a set A_S for every subset S of W of size (g(r,k,H)+r)p+2 as follows. Let S be such a set, and let $Y = \bigcup_{s \in S} N^+(s)$. For a contradiction, assume that $\vec{\chi}(Y) \leq p$. Let Y_1, \ldots, Y_p be a partition of Y into p acyclic sets. For each set Y_i , pick a vertex y_i^1 with no in-neighbours. Having picked vertex y_i^j for some $1 \leq j \leq r-1$, pick another vertex y_i^{j+1} in $Y_i \setminus \bigcup_{k \leq j} N^+[y_i^k]$ (unless this set is empty) with no in-neighbours in $Y_i \setminus \bigcup_{1 \leq k \leq j} N^+[y_i^k]$. Then, for every i, the vertices y_i^1, \ldots, y_i^r form a stable set, and so the set $Y_i' = Y \setminus \bigcup_{1 \leq k \leq r} N^+[y_i^k]$ is acyclic and F-free. Since $\vec{\chi}(D) > g(r,k,H)$, there exists a dominating set Z_i for Y_i' of size at most g(r,k,H), so the set $Z_i' = Z_i \cup \{y_i^1, \ldots, y_i^r\}$ is a dominating set for Y_i of size at most g(r,k,H) + r.

Thus, the set $Z = Z'_1 \cup \cdots \cup Z'_p$ is a dominating set for Y of size at most (g(r, k, H) + r)p. Adding a vertex z from A, we get a dominating set for $N^+[S]$ of size at most (g(r, k, H) + r)p + 1. Then $(B \setminus S) \cup Z \cup \{z\}$ is a dominating set for D of size at most |B| - 1, contradicting that B is a smallest dominating set. Thus, $\vec{\chi}(Y) > p$.

Because $|A| \leq l$, by (Claim 21), and by the fact that D is k-local, we have

$$\vec{\chi}(N^0(A) \cap Y) \le bl,$$

$$\vec{\chi}(N^+(A) \cap Y) \le kl.$$

Thus, the set A' of vertices of Y out-complete to A has dichromatic number at least p-bl-kl > M, which implies by the inductive hypothesis that A' contains a set A_S with $\bar{\chi}(A_S) \ge m$ and $|A_S| \le l$. This is how we define A_S for every subset S of W where |S| = (g(r,k,H) + r)p + 2. Figure 3 illustrates this process.

Finally, take

$$V \coloneqq W \cup A \cup \bigcup A_S.$$

where the union happens over all subsets S of W of size exactly (g(r,k,H)+r)p+2. This set has size at most $d+l+l\binom{d}{g(r,k,H)p+2} = l'$. By the definition of d, every m-colouring f

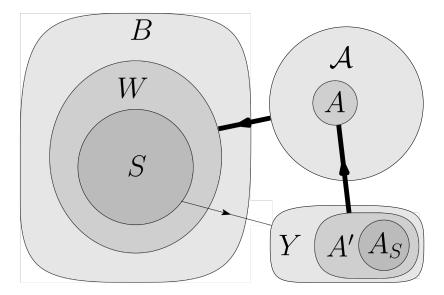


Figure 3: Illustration of the proof of Lemma 4.1.

of V contains a monochromatic set $S \subseteq W$ of size g(r,k,H)p + 2. Let $f(S) = \{\gamma\}$. Since $\vec{\chi}(A), \vec{\chi}(A_S) \ge m$, it follows that there exists $a \in A$ and $a' \in A_S$ with $f(a) = f(a') = \gamma$. Now let $s \in S$ be an in-neighbour of a' (which exists since $A_S \subseteq N^+(S)$). It follows that $\{a, a', s\}$ is a cyclic triangle monochromatic under f. Since f was an arbitrary m-colouring, this argument applies to every m-colouring of V. We conclude that $\vec{\chi}(V) \ge m + 1$, and so V is the desired set for m + 1, finishing the inductive argument.

The following is analogous to the proof of Theorem 2.3 in [9].

Lemma 4.2. If F dominates, then F is localized.

Proof. Assume that

- $\Delta(1,1,H)$ is a hero in $\{(r-1)K_1 + F\}$ -free digraphs, and
- *H* is a hero in $\{rK_1 + F\}$ -free digraphs.

Let c be an integer such that $\{rK_1 + F, H\}$ -free digraphs D have $\vec{\chi}(D) \leq c$. Furthermore, let b be an integer such that $\{(r-1)K_1 + F, \Delta(1, 1, H)\}$ -free digraphs D have $\vec{\chi}(D) \leq b$. Fix an integer $k \geq 1$. By Lemma 4.1, $\{rK_1 + F, \Delta(1, 1, H)\}$ -free k-local digraphs are tamed. Let M and l be the corresponding integers following the definition of tameness when m = k + b + 1.

Let D be a $\{rK_1 + F, \Delta(1, 1, H)\}$ -free k-local digraph. To prove that F is localized, it is enough to show that $\vec{\chi}(D) \leq \max\{M, lk\}$. Assume that $\vec{\chi}(D) > M$. By definition, there exists a set $X \subseteq D$ such that $|X| \leq l$ and $\vec{\chi}(X) \geq m$. We claim that X is a dominating set of D. Assume for a contradiction that there exists a vertex v not in $\bigcup_{x \in X} N^+(x)$. Consequently, $X \subseteq N^0(v) \cup N^+(v)$. By the definition of k and b, it follows that $\vec{\chi}(X) \leq k+b$, a contradiction. Thus, X dominates D. But D is k-local, so $\vec{\chi}(D) \leq kl$ thus finishing the proof.

Now that we have proven that digraphs F that dominate are localized, it only remains to show that \vec{P}_3 dominates.

Lemma 4.3. The digraph \vec{P}_3 dominates.

Proof. Suppose that D is a k-local $\{\Delta(1,1,H), rK_1 + \vec{P}_3\}$ -free digraph. Suppose that

- $\Delta(1,1,H)$ is a hero in $\{(r-1)K_1 + \vec{P}_3\}$ -free digraphs; and
- *H* is a hero in $\{rK_1 + \vec{P}_3\}$ -free digraphs.

We set some constants:

- Let b be an integer such that $\{\Delta(1, 1, H), (r-1)K_1 + \vec{P_3}\}$ -free digraphs have dichromatic number at most b.
- Let c be an integer such that $\{H, rK_1 + P_3\}$ -free digraphs have dichromatic number at most c.

Let $g(r, k, H) = \max\{4r + 5, b + 1 + k + 2c + (|V(H)| + 1)(kr + b), 3r|V(H)|\}$. We will show that either *D* has dichromatic number at most g(r, k, H), or for every acyclic \vec{P}_3 -free set *S*, there is a dominating set in *D* for *S* of size at most g(r, k, H). Suppose for a contradiction that neither of these outcomes holds.

Let S be an acyclic \vec{P}_3 -free set. By possibly adding vertices to S, we assume that S is a vertex-maximal acyclic \vec{P}_3 -free set. As S is maximal, all vertices in $V(D) \\ \\S$ have a neighbour in S. We start by noting that acyclic \vec{P}_3 -free digraphs with small independence number can be dominated with few vertices.

(Claim 22) Suppose that X is an acyclic \vec{P}_3 -free digraph with independence number q. Then there is a dominating set B of X contained inside X of at most q vertices.

Proof. Let $B \subseteq V(X)$ be a minimal dominating set for X, and suppose that B contains at least q + 1 vertices. As the independence number of X is at most q, there is an arc uv in X[B]. As $B \setminus \{v\}$ is not a dominating set, there is a vertex w such that $vw \in A(X)$ but $uw \notin A(X)$. If $wu \notin A(X)$, then X contains an induced \vec{P}_3 , a contradiction. So $wu \in A(X)$. But then $\{u, v, w\}$ induces a cyclic triangle, contradicting that X is acyclic.

(Claim 23) If Y is an induced copy of \vec{P}_3 in $D \setminus S$, then a vertex in Y has an in-neighbour in S.

Proof. Suppose not. Then, no vertex in Y has an in-neighbour in S. Let $S' = S \cap N^+(Y)$. As no vertex in Y has an in-neighbour in S, the set of vertices $v \in S \setminus S'$ are common nonneighbours of all of the vertices in Y. Thus $S \setminus S'$ has independence number at most r-1, as otherwise D contains $rK_1 + \vec{P}_3$. Thus there is a dominating set B for $S \setminus S'$ of size at most r-1 by (Claim 22). But then $Y \cup B$ is a dominating set for S of size at most r+2, a contradiction.

(Claim 24) If X is an induced subgraph of $D \setminus S$ and $N^-(X) \cap S = \emptyset$, then $\vec{\chi}(X) \leq b$.

Proof. By (Claim 23), we conclude that X is \vec{P}_3 -free. Then, by the definition of b, it follows that $\vec{\chi}(X) \leq b$.

Let $S_1 \subseteq S$ be the set of vertices in S with no in-neighbour in S.

(Claim 25) The set S_1 is a stable set, and all vertices in $S - S_1$ have an in-neighbour in S_1 .

Proof. This is immediate from the fact that \vec{P}_3 -free acyclic digraphs are directed comparability graphs. We give a self-contained proof for completeness.

The fact that S_1 is a stable set follows directly from the definition of S_1 . Now let $v \in S \setminus S_1$. Let P be a maximal directed (not necessarily induced) path of the form $x_1 \to x_2 \to \cdots \to x_t \to v$ in D[S]. We claim that $x_1 \in S_1$. If not, then x_1 has an in-neighbour $x_0 \in S$. Since x_0 cannot be added to P to make a longer path, it follows that $x_0 \in \{x_2, \ldots, x_t, v\}$. But then D[S] has a directed cycle, a contradiction. So $x_1 \in S_1$. Now let Q be a shortest directed path from x_1 to v. Then Q is an induced path (since D[S] is acyclic); but since S is \vec{P}_3 -free, it follows that Q has at most one edge; in other words, $x_1v \in A(D)$. Since v was chosen arbitrarily, the claim follows.

We observe that S_1 is a dominating set for S, and thus $|S_1| \ge g(r, k, H) + 1$ by our assumptions.

Let $Q \subseteq V(D) \setminus S$ be the set of vertices such that for each vertex $v \in Q$, we (Claim 26) have that the in-neighbours of v in S can be dominated by a set B of at most r+1 vertices where $B \subseteq S$. Then $\vec{\chi}(Q) \leq b$.

Proof. Suppose not. Then *Q* contains an induced copy *P* of \vec{P}_3 . Partition *S* into $(N^-(P) \cap S)$, $(N^0(P) \cap S)$, and $(N^+(P) \cap S)$ (choosing arbitrarily if a vertex is both an in- and outneighbour of some vertex in *P*). Note this is a partition of *S*: if a vertex is neither an in-neighbour or out-neighbour of a vertex in *P*, then it is a non-neighbour of all of the vertices of *P*. Then the digraph induced by $N^0(P) \cap S$ has independence number at most r-1, as otherwise *D* contains $rK_1 + \vec{P}_3$, a contradiction. Thus by (Claim 22), there exists a dominating set for $N^0(P) \cap S$ of size at most r-1. By the assumption, $N^-(P) \cap S$ can be dominated by at most 3(r+1) vertices. Lastly, $N^+(S)$ is dominated by *P*, and thus *S* can be dominated by at most $3(r+1) + r + 2 = 4r + 5 \le g(r,k,H)$ vertices, a contradiction.

Let Q be the set of vertices defined as in (Claim 26). Let $T = D \setminus (S \cup Q)$. As $\vec{\chi}(Q) \leq b$, and S is acyclic, it follows that $\vec{\chi}(T) \geq \vec{\chi}(D) - b - 1$.

(Claim 27) If there exists a vertex $v \in V(T)$ such that v has an out-neighbour in S_1 , then v has at most r-1 non-neighbours in S_1 .

Proof. Let v be a vertex in T and suppose that v has at least r non-neighbours in S_1 . Let u be an out-neighbour of v in S_1 . Let X be any set of r non-neighbours of v in S_1 . The set $S \cap N^-(v)$ cannot be dominated by $X \cup \{u\}$, as the in-neighbours of v in S cannot be dominated by r + 1 vertices by the definition of Q, and thus there is at least one in-neighbour of v in S, say w, such that w is not in $N^+(X \cup \{u\})$. Since $N^-(S_1) \cap S = \emptyset$, it follows that w is not adjacent to any vertex in $X \cup \{u\}$. Then $\{w, v, u\} \cup X$ induces an $rK_1 + \vec{P}_3$, a contradiction.

(Claim 28) If v is in T, and v has an out-neighbour in S_1 , then the non-neighbours of v can be dominated with at most $\max\{r+1, 2r-1\} \leq 2r$ vertices inside S.

Proof. Let u be an out-neighbour of v in S_1 . First suppose that v has an in-neighbour in S_1 , say w. Then $\{u, v, w\}$ induces a \vec{P}_3 . Let $Y = S \cap N^0(\{u, v, w\})$. Then, since $\{u, v, w\}$ induces a copy of \vec{P}_3 , we have that Y has independence number at most r, and thus it follows from (Claim 22) that Y has a dominating set X of size at most r - 1. Consequently, the

non-neighbours of v in S can be dominated by $X \cup \{u, w\}$, which is at most r - 1 + 2 = r + 1 vertices.

Therefore, we may assume that v only has out-neighbours and non-neighbours in S_1 . By (Claim 27), v has at most r - 1 non-neighbours in S_1 . Let X be this set. Let $Y \subseteq S_1 \setminus X$ be minimal with respect to inclusion such that $(S \setminus S_1) \cap N^0(v) \subseteq N^+(X \cup Y)$. This set exists as $S \setminus S_1 \subseteq N^+(S_1)$ by (Claim 25). Then, if $|Y| \leq r$, the claim holds as $X \cup Y$ is the desired set; so we may assume that $|Y| \geq r + 1$. It follows that Y contains r + 1 distinct vertices, say y_1, \ldots, y_{r+1} . For each $i \in \{1, \ldots, r+1\}$, the set $X \cup (Y \setminus \{y_i\})$ does not dominate $(S \setminus S_1) \cap N^0(v)$, and so there is a vertex $y'_i \in (S \setminus S_1) \cap N^0(v)$ such that $N^-(y'_i) \cap (X \cup Y) = \{y_i\}$. But now $\{v, y_1, y'_1, y'_2, \ldots, y'_{r+1}\}$ induces $rK_1 + P_3$, a contradiction.

From now on, let X be the set of vertices in $V(D) \setminus S$ with no out-neighbour in S_1 .

(Claim 29) Either $|N^0(X) \cap S_1| \le r - 1$, or $\vec{\chi}(X) \le b$.

Proof. If not, then $|N^0(X) \cap S_1| \ge r$, and thus X does not induce a \vec{P}_3 , as otherwise D would contain a copy of $rK_1 + \vec{P}_3$. But then by the definition of b, it follows that $\vec{\chi}(X) \le b$, a contradiction.

Since D is k-local, (Claim 29) implies that $\vec{\chi}(X) \leq \max\{b, rk\}$ (because if $|N^0(X) \cap S_1| \leq r-1$, then choosing r vertices in S_1 yields a dominating set for X). In addition, every vertex v in $T \setminus X$ has an out-neighbour in S_1 , and thus, by (Claim 28), we have $|S_1 \cap N^0(v)| \leq r-1$. (Claim 30) Let R be the set of vertices in $T \setminus X$ that have an in-neighbour in S_1 . Then $\vec{\chi}(R) \leq (r-1)|V(H)|k + (k+c) + (b+k)|V(H)|$.

Proof. Suppose not. By removing one vertex at a time from S_1 , we create a subset S' of S_1 such that

$$\vec{\chi}(R) - (k+c) \le \vec{\chi}(N^+(S') \cap R) < \vec{\chi}(R) - c$$

(which is possible as D is k-local and $\vec{\chi}(R) > c$). Let $Z = R \setminus N^+(S')$. Then $\vec{\chi}(Z) > c$, and it follows that there exists a copy X' of H in Z.

Let S'' be the set of vertices $s \in S'$ such that s is a neighbour of every vertex in X', and note that from the definition of Z, we have that s is an out-neighbour of every vertex in X'in this case. It follows that X' is out-complete to S''. As every vertex in $X' \subseteq T \setminus X$ has at most r-1 non-neighbours in S', we have that $|S''| \ge |S'| - (r-1)|V(H)|$ (and thus implying S'' is non-empty). Let $Y = N^+(S'') \cap R$. Then, as D is k-local and from the choice of S', it follows that

$$\vec{\chi}(Y) \ge \vec{\chi}(N^+(S') \cap R) - (r-1)|V(H)|k \ge \vec{\chi}(R) - (r-1)|V(H)|k - (k+c) > b|V(H)| + k|V(H)|.$$

Let $A = \bigcup_{x \in X'} N^0(x) \cap Y$ and $B = \bigcup_{x \in X'} N^+(x) \cap Y$. As $N^0(x)$ is $\{(r-1)K_1 + \vec{P}_3\}$ -free for every $x \in D$, we have that $\vec{\chi}(A) \leq b|V(H)|$; and $\vec{\chi}(B) \leq k|V(H)|$ as D is k-local. Thus $\vec{\chi}(Y \setminus (A \cup B)) \geq \vec{\chi}(Y) - b|V(H)| - k|V(H)| \geq 1$ and therefore $Y' = Y \setminus (A \cup B)$ is not empty. Let $y \in Y'$ and $s \in S''$ be an in-neighbour of y in S'', which exists by the definition of Y. By definition, s is in-complete from X', and X' is in-complete from y. Thus the set $\{s, y\} \cup X'$ induces a $\Delta(1, 1, H)$, a contradiction. Putting this all together, as D has large dichromatic number, by (Claim 30) and since $\vec{\chi}(T) \geq \vec{\chi}(D) - b - 1$, it follows that the set of vertices $U = T \setminus (X \cup R)$ (where R is defined as (Claim 30)) with only out-neighbours and non-neighbours in S_1 has dichromatic number at least

$$\vec{\chi}(U) \ge \vec{\chi}(D) - b - 1 - \max\{b, rk\} - k - c - |V(H)|(kr + b) > c.$$

As D[U] has dichromatic number more than c and is $\{rK_1 + \vec{P}_3\}$ -free, it contains a copy X' of H. Since $U \subseteq T \setminus X$, and from the definition of X, it follows that each vertex of X' has an out-neighbour in S_1 , and therefore, by (Claim 27), at most r-1 non-neighbours in S_1 . Let Y' be the set of vertices in S_1 with a non-neighbour in X'. Then $|Y'| \leq (r-1)|V(H)|$. Moreover, by (Claim 28), there is a set X'' of at most 2r|V(H)| vertices in S such that X'' dominates the set of all vertices in S with a non-neighbour in X'.

If $Z' = X' \cup X'' \cup Y'$ is a dominating set for S, then it has size at most $3r|V(H)| \le g(r, k, H)$, a contradiction. Therefore, there is a vertex s such that:

- $s \in S$ is not an out-neighbour and not a non-neighbour of any vertex in X', so s is out-complete to X'; in particular, $s \notin S_1$; and
- $s \in S \setminus N^+(Y')$, and so, since S_1 is a dominating set for S, it follows that s has an in-neighbour s' in $S_1 \setminus Y'$. As $s' \notin Y'$, it follows that X' is out-complete to s'.

But now s, s' and X' form a copy of $\Delta(1, 1, H)$, a contradiction.

Proof of Theorem 1.12. By Theorem 3.2, it suffices to show that \vec{P}_3 cooperates, is localized, and is colocalized. It follows from Theorem 1.7 that \vec{P}_3 cooperates. By Lemma 4.3, \vec{P}_3 dominates, so by Lemma 4.2 \vec{P}_3 is localized. By Lemma 3.3, \vec{P}_3 is colocalized as well, thus finishing the proof.

5 Forbidding brooms

In this section, we prove Theorem 1.16, which we restate for the reader's convenience.

Theorem 1.16. Let r, s, t be positive integers. If \mathcal{B} and \mathcal{B}' are valid opposing orientations of B_r and B_s respectively, then $\{\mathcal{B}, \mathcal{B}', TT_t\}$ is $\vec{\chi}$ -finite.

As mentioned in the introduction, we follow the technique designed by Cook, Masařík, Pilipczuk, Reinald, and Souza [5] to prove that if P is an orientation of P_4 , then P-free digraphs are $\vec{\chi}$ -bounded. We will need a lemma about so called k-nice sets. A set $S \neq \emptyset$ is k-nice if there exists a partition S_1, S_2 of S such that every vertex in S_1 (resp. S_2) has at most k in-neighbours (resp. k out-neighbours) in $V(D) \smallsetminus S$. Recall that a hereditary class of digraphs C is a class of digraphs such that if $G \in C$, all induced subdigraphs are in C.

Lemma 5.1. Let $k \ge 0$, and let C be a hereditary class of digraphs. If there exists an integer c such that every $D \in C$ has a k-nice set S with $\vec{\chi}(S) \le c$, then $\vec{\chi}(D) \le 2c(k+1)$ for every $D \in C$.

Proof. Fix C. We proceed by induction on |V(D)|. The statement holds if |V(D)| = 1. Assume the statement holds for digraphs with fewer than |V(D)| vertices. By the assumption D has

a k-nice set S with $\vec{\chi}(S) \leq c$. Let S_1 and S_2 be the partitioning of S as in the definition of a k-nice set.

By induction, the digraph induced by $V(D) \setminus S$ has a 2c(k+1)-dicolouring. Let f_0 : $(V(D) \setminus S) \rightarrow \{1, \ldots, k+1\} \times \{1, \ldots, 2c\}$ be such a (2c(k+1))-dicolouring. Furthermore, let f_1 be a *c*-dicolouring of S_1 using colours in $\{1, \ldots, c\}$, and let f_2 be a *c*-dicolouring of S_2 using colours in $\{c+1, \ldots, 2c\}$.

We define a function $m: S \to \{1, \ldots, k+1\}$ as follows. Let $u \in S$. If $u \in S_1$, then u has at most k in-neighbours in $V(D) \smallsetminus S$. Thus, $|f_0(N^-(u) \cap (V(D) \smallsetminus S))| \le k$. Consequently, there exists a number m(u) such that no colour in $f_0(N^-(u) \cap (V(D) \smallsetminus S))$ has m(u) as its first coordinate. We define m(v) when $v \in S_2$ similarly, where we use its out-neighbourhood in $V(D) \smallsetminus S$ instead. Using these, we can define the following colouring.

$$f(v) = \begin{cases} f_0(v) & \text{if } v \notin S; \\ (m(v), f_1(v)) & \text{if } v \in S_1; \\ (m(v), f_2(v)) & \text{if } v \in S_2. \end{cases}$$

We claim that f is a (2c(k+1))-dicolouring of D. The first index of the coordinate has k+1 values, and the second index at most 2c. Thus, this indeed uses at most 2c(k+1) colours. For a contradiction, assume that C is a directed monochromatic cycle in D. Since f_0, f_1 and f_2 are dicolourings, C is not contained in neither of the sets S_1, S_2 and $V(D) \\S$. Since f_1 and f_2 use colours that do not overlap, it follows that V(C) does not intersect both S_1 and S_2 , so V(C) is not contained completely in S. By the same reason, if V(C) intersects both S and $V(D) \\S$, then V(C) intersects only one of S_1 and S_2 .

Thus, either V(C) intersects with $V(D) \\ S$ and S_1 , or V(C) intersects with $V(D) \\ S$ and S_2 . We will only show the first situation leads to a contradiction - the second follows similarly. Assume V(C) intersects with S_1 . Thus, there is an edge e = uv in C such that $u \\ \in V(D) \\ S$ and $v \\ \in S_1$. But then, by the definition of m(v), the first coordinate of f(u) is not equal to the first coordinate of f(v), contradicting that C is monochromatic. \Box

Before we can prove Theorem 1.16 we need to introduce some more tools developed in [5]. For a not strongly connected tournament K, let K_1, \ldots, K_k be the partition of V(K) into its strongly connected components. Let K^* be the tournament that results from contracting each of these parts into a single vertex each. It follows that digraph K^* has vertices u^* and v^* such that $N_{K^*}^-(u) \cap V(K^*) = \emptyset$ and $N_{K^*}^+(v) \cap V(K^*) = \emptyset$. If u is in the component that got contracted to the vertex u^* , then we call u a source vertex. If v is in the component that got contracted to the vertex v^* , then we call v a sink vertex.

We say C is a path-minimizing closed tournament (PMCT) if either V(C) = K, where K is a strongly connected tournament with $\omega(D) = |K|$, or $V(C) = K \cup V(P)$ where K is a tournament that is not strongly connected, $\omega(D) = |K|$, and P is a directed path from a sink vertex to a source vertex of K. Furthermore, K is picked such that $|V(C)| = |V(K) \cup V(P)|$ is minimized. Notice that if D has a strongly connected tournament on $\omega(D)$ vertices, then every PMCT is a tournament. Otherwise, if C is a PMCT, then C is not a tournament, and K is picked such that |V(P)| is as small as possible.

Eventually, we need to go into four different cases. For that, we will illustrate the different cases that we will have. There are 8 types of orientations to consider that we separate into four types. These are illustrated on Figure 4a, Figure 4b, Figure 5a, and Figure 5b. Since \mathcal{B}

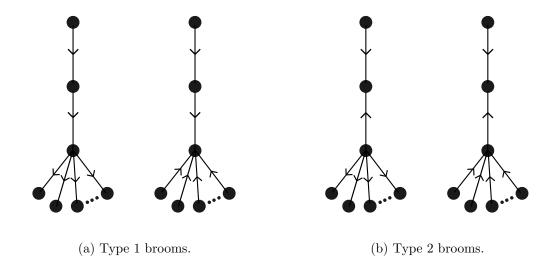


Figure 4: Type 1 and type 2 brooms.

and \mathcal{B}' are of opposing orientation, we may assume that \mathcal{B} is of type 1 or type 3, and that \mathcal{B} is of type 2 or type 4, giving four cases.

Proof of Theorem 1.16: Let \mathcal{C} be the set of $\{\mathcal{B}, \mathcal{B}'\}$ -free digraphs. To prove that \mathcal{C} is $\vec{\chi}$ -bounded, we proceed by induction on $\omega(D)$. The result is immediate if $\omega(D) = 1$. For a digraph D, assume that the statement holds for every $\omega < \omega(D)$. That is, assume that there exists a number γ such that if $\omega(D') < \omega(D)$ and D' is $\{\mathcal{B}, \mathcal{B}\}$ -free, then $\vec{\chi}(D') \leq \gamma$. Finally, let $k = \max\{R(r, \omega(D)), R(s, \omega(D))\}$ where R is the graph Ramsey number. We want to prove that

$$\vec{\chi}(D) \le 2(\omega(D)(\gamma+1) + \gamma(6k+25) + 2)(k+1).$$

We may assume that D is strongly connected as the strongly connected components of a digraph can be coloured independently. Let C be a PMCT, which exists since D is strongly connected. Let X be the set of vertices $v \notin C$ such that v has an in-neighbour and an out-neighbour in $C, Z = N(V(C)) \setminus X$, and $Y = N(X) \setminus N[V(C)]$.

(Claim 31) $\begin{array}{l} If \ S \ is \ a \ set \ of \ vertices \ in \ D \ such \ that \ |S| \ge k, \ then \ S \ contains \ a \ stable \ set \ of \ size \ at \ least \ \max\{r,s\}. \end{array}$

Proof. The proof is immediate from the definition of the graph Ramsey number.

The following is the analog of the proof of Lemma 3.1 from [5].

(Claim 32) $N[C \cup X]$ is a k-nice set.

Proof. We want to prove that if $v \in N[C \cup X]$, then either v has at most k in-neighbours in $V(D) \setminus N[C \cup X]$, or v has at most k out-neighbours in $V(D) \setminus N[C \cup X]$. For this purpose, notice that if $v \in C \cup X$, then the result follows immediately.

For a contradiction, assume that there exists a vertex $v \in N(C \cup X)$ such that v has at least k in-neighbours and out-neighbours not in $N[C \cup X]$. Let $S^- := N^-(v) \setminus N[C \cup X]$ and $S^+ := N^+(v) \setminus N[C \cup X]$. Either $v \in Y$ or $v \in Z$. If $v \in Y$, then by the definition of Y, there

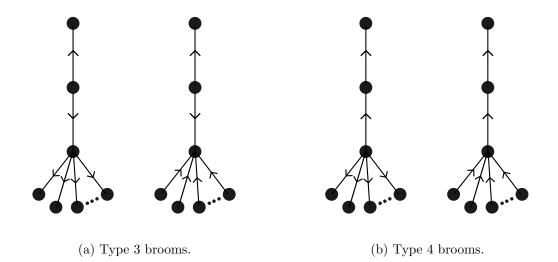


Figure 5: Type 3 and type 4 brooms.

exists $x \in X$ such that x is a neighbour of v. Since $x \in X$, there exists vertices $c_1, c_2 \in C$ such that $c_1x, xc_2 \in A(D)$. Note that as $v \in Y$, v is non-adjacent to c_1, c_2 . Furthermore, notice that $\{x, c_1, c_2\}$ is anticomplete to $S^- \cup S^+$. Since \mathcal{B} and \mathcal{B}' have opposing orientations, both cases $xv \in A(D)$ and $vx \in A(D)$ each imply that there exists a copy of \mathcal{B} or \mathcal{B}' in $\{c_1, c_2, x, v\} \cup S^- \cup S^+$. Since D is $\{\mathcal{B}, \mathcal{B}'\}$ -free, we conclude $v \notin Y$.

It follows that $v \in \mathbb{Z}$. Since $v \notin X$, v has either only in-neighbours or only out-neighbours in C. Furthermore, since C contains a clique of maximal size, $N^0(v) \cap C$ is nonempty. Thus, since C is strongly connected, there is an arc from $N^0(v) \cap C$ to $N(v) \cap C$, and an arc from $N(v) \cap C$ to $N^0(v) \cap C$. Let these arcs be x_1y_1 and y_2x_2 , respectively. Note that $\{x_1, x_2, y_1, y_2\}$ are anti-complete to $S^+ \cup S^-$. As before, both cases where v has only in-neighbours in C or out-neighbours in C imply that the set $\{x_1, x_2, y_1, y_2, v\} \cup S^- \cup S^+$ contain a copy of \mathcal{B} or \mathcal{B}' . Since D is $\{\mathcal{B}, \mathcal{B}'\}$ -free, both lead to contradictions. We conclude $N[C \cup X]$ is a k-nice set.

By using Lemma 5.1, it is enough to bound $\vec{\chi}(N[C \cup X])$. If C is a strongly connected tournament, then we consider P to be the empty path. As noted by Cook, Masařík, Pilipczuk, Reinald, and Souza [5],

$$\vec{\chi}(N[C \cup X]) \le \vec{\chi}(N[K]) + \vec{\chi}(P) + \vec{\chi}(N(P) \setminus N[K]) + \vec{\chi}(Y).$$

For an illustration of $N[C \cup X]$, see Figure 6. Thus, we want to bound each of these. By the minimality of |V(P)| and by Observation 4.1 in [5], we have $\vec{\chi}(P) \leq 2$. Furthermore, since $\vec{\chi}(N(v)) \leq \gamma$ for every $v \in V(D)$ by the definition of γ , we have $\vec{\chi}(N[K]) \leq \omega(D) + \omega(D)\gamma = \omega(D)(\gamma + 1)$. We proceed to bound $\vec{\chi}(Y)$. The following is the analog of Lemma 4.3 and Corollary 4.4 in [5]. However, we use k-nice sets to get brooms rather than paths.

(Claim 33) $\vec{\chi}(Y) \leq 2\gamma(k+1).$

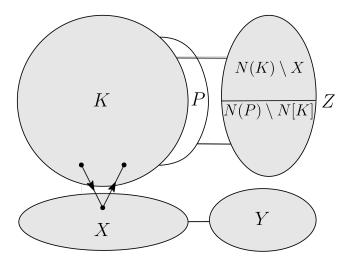


Figure 6: An illustration of $N[C \cup X]$.

Proof. We proceed by proving that every non-empty induced subgraph Y' of Y has a k-nice set S such that $\vec{\chi}(S) \leq \gamma$, which finishes the proof by Lemma 5.1. The statement is true for $Y' = \emptyset$, so we may assume Y' is not empty.

By the definition of Y, there exists a vertex $x \in X$ such that $N(x) \cap Y' \neq \emptyset$. By the definition of X, there exists vertices $c_1, c_2 \in C$ such that $c_1x, xc_2 \in A(D)$. Set $S = N(x) \cap Y'$. By the definition of γ , we get $\vec{\chi}(S) \leq \gamma$. It suffices to prove that S is a k-nice set. For a contradiction, assume that there exists a vertex $s \in S$ which has at least k in-neighbours and out-neighbours in $Y' \smallsetminus N(x)$. Let $S^- := N^-(s) \cap (Y' \smallsetminus N(x))$ and $S^+ := N^+(s) \cap (Y' \smallsetminus N(X))$. If $xs \in A(D)$, then $\{c_1, x, s\} \cup S^+$ induces a subgraph that contains a copy of \mathcal{B}' by Claim (Claim 31). A similar argument works if $sx \in A(D)$. This proves that S is a k-nice set, thus finishing the proof.

It remains to bound $\vec{\chi}(N(P) \setminus N[K])$. How we do it mimics the technique used in Section 5 of [5]. If C is a strongly connected tournament, then P is the empty path, so $\vec{\chi}(N(P)) = 0$. Assume then that K is not strongly connected. If P has at most four vertices, then $\vec{\chi}(N(P)) \setminus N[K] \leq 4\gamma$. Assume then that P has more than four vertices. Let P' be the path P with the first, the second first, the last, and the second-to-last vertices deleted. Let Q be the set of these four vertices. We want to bound the dichromatic number of $N(P') \setminus (N[K] \cup N(Q))$.

Let v_1, \ldots, v_n denote the vertices of P' labeled such that $v_i v_{i+1} \in A(D)$ for every $i \in \{1, \ldots, m-1\}$. When talking about vertices v in N(P'), we use first (in/out-)neighbour of v to refer to the vertex v_i that is a (in/out-)neighbour of v such that i is minimized. Similarly, the last (in/out-)neighbour of v refers to the vertex v_i that is a (in/out-)neighbour of v such that i is maximized.

The rest of the proof is by cases. Notice that since \mathcal{B} and \mathcal{B}' have opposing consistent orientations, we may assume without loss of generality that \mathcal{B} is of type 1 or type 3, and \mathcal{B}' is of type 2 or type 4.

Let A^- (resp A^+) be the set of vertices in $N(P') \setminus (N[K] \cup N(Q))$ such that their first neighbour is an in-neighbour (resp out-neighbour). Furthermore, let B^+ (resp. B^-) be the set of vertices in $N(P') \setminus (N[K] \cup N(Q))$ such that their last neighbour is an out-neighbour (resp. in-neighbour). The following claim will make the proofs of each case less repetitive.

(Claim 34) The following are true.

- If \mathcal{B} is of type 1, then $\vec{\chi}(A^-) \leq 2\gamma(k+1)$.
- If \mathcal{B}' is of type 2, then $\vec{\chi}(A^+) \leq 2\gamma(k+1)$.
- If \mathcal{B} is of type 3, then $\vec{\chi}(B^-) \leq 2\gamma(k+1)$.
- If \mathcal{B}' is of type 4, then $\vec{\chi}(B^+) \leq 2\gamma(k+1)$.

Proof. Let us prove the first bullet point. We will use Lemma 5.1 to bound $\vec{\chi}(A^{-})$. Let i be the smallest integer such that $N^+(v_i) \cap A^- \neq \emptyset$. By the definition of γ , we have $\vec{\chi}(N^+(v_i) \cap A^-) \leq \gamma$. We claim $N^+(v_i) \cap A^-$ is a k-nice set in $D[A^-]$. Let $v \in N^+(v_i) \cap A^-$ be such that v has at least k in-neighbours and at least k out-neighbours not in $A^- \setminus N^+(v_i)$. Let $S^+ := N^+(v) \cap$ $(A^- \setminus N^+(v_i))$ be the set of out-neighbours of v in A^- , and let $S^- := N^-(v) \cap (A^- \setminus N^+(v_i))$ be the set of out-neighbours of v in $A^- \setminus N^+(v_i)$.

Since vertices in A^- have v_i as their first neighbour, v_{i-1} , where we pick v_{i-1} as the second vertex of P if i = 1, is anticomplete to $S^+ \cup S^-$. Furthermore, since vertices in $S^+ \cup S^-$ are not in $N^+(v_i)$, and $S^+ \cup S^- \subseteq A^-$, we have that v_i is anticomplete to $S^- \cup S^+$.

This, however, implies that $\{v_{i-1}, v_i, v\} \cup S^+ \cup S^-$ contains a copy of \mathcal{B} . We conclude that every vertex in $N^+(v_i) \cap A^-$ contains at most k out-neighbours in $A^- \setminus N^+(v_i)$. This finishes the proof that $N^+(v_i) \cap A^-$ is a k-nice set in $D[A^-]$, and so the proof that $\vec{\chi}(A^-) \leq 2\gamma(k+1)$.

Similar arguments prove the remaining bullet points.

In the following sections, we will prove that, for every case, $\vec{\chi}(N(P') \setminus (N[K] \cup N(Q))) \leq$ $\gamma(4k+19)$. This will finish the proof since then,

$$\vec{\chi}(N[K \cup X]) \le \vec{\chi}(N[K]) + \vec{\chi}(P) + \vec{\chi}(N(P) \setminus N[k]) + \vec{\chi}(Y) \le \omega(D)(\gamma + 1) + 2 + 4\gamma + \gamma(4k + 19) + \gamma(2k + 2) = \omega(D)(\gamma + 1) + \gamma(6k + 25) + 2,$$

where the 4γ in the second line came from N(Q), the neighbourhood of the vertices in P that are not in P'. By Lemma 5.1,

$$\vec{\chi}(D) \le 2(\omega(D)(\gamma+1) + \gamma(6k+25) + 2)(k+1),$$

as claimed.

Brooms of type 1 and type 2 5.1

Assume that \mathcal{B} is a broom of type 1, and that \mathcal{B}' is a broom of type 2. By (Claim 34), $\vec{\chi}(A^- \cup A^+) \leq 4\gamma(k+1)$. However, A^-, A^+ partitions $N(P') \smallsetminus (N[K] \cup N(Q))$, so $\vec{\chi}(N(P') \searrow N(Q))$ $(N[K] \setminus N(Q)) \leq \gamma(4k+4) \leq \gamma(4k+19)$, as claimed.

5.2 Brooms of type 3 and type 4

Assume that \mathcal{B} is a broom of type 3, and that \mathcal{B}' is a broom of type 4. By (Claim 34), $\vec{\chi}(B^- \cup B^+) \leq 4\gamma(k+1)$. However, B^-, B^+ partitions $N(P') \smallsetminus (N[K] \smallsetminus N(Q))$, so $\vec{\chi}(N(P') \smallsetminus (N[K] \searrow N(Q))) \leq \gamma(4k+4) \leq \gamma(4k+19)$, as claimed.

5.3 Brooms of type 2 and type 3

Assume that \mathcal{B} is a broom of type 3, and that \mathcal{B}' is a broom of type 2. By (Claim 34), $\vec{\chi}(A^+ \cup B^-) \leq 4\gamma(k+1)$. Let $C' = N(P') \setminus (N[K] \cup N(Q) \cup A^+ \cup B^-)$, and for every $i \in \{1, \ldots, m\}$, let L_i be the set of vertices v in C' such that $v_i v \in A(D)$ and i is minimized.

(Claim 35) For every $i \ge 1$ and $j \ge i+3$, if $v \in L_i$, then $N^+(v) \cap L_j = \emptyset$.

Proof. Assume for a contradiction that there exists a vertex $u \in N^+(v) \cap L_j$. Since $u \notin A^+$, v_j is the first neighbour of v in P. Since $u \notin B^-$, u has an out-neighbour v_l in P with l > j. But then we can shorten the path P by replacing vertices v_{i+1}, \ldots, v_{l-1} with v and u. This contradicts C is a PMCT.

Set

$$C_i = \{c \in L_j : i \text{ is congruent to } j \text{ modulo } 3\}.$$

That is, for example, C_1 is the union of L_1, L_4, L_7 , and so on. Furthermore, by (Claim 35), every strongly connected subdigraph D' of C_i is contained in a set L_i . Since L_i is contained in the neighbourhood of v_i , it follows that $\vec{\chi}(L_i) \leq \gamma$. Thus, $\vec{\chi}(C_i) \leq \gamma$.

We can now finish the proof. Since C_1, C_2, C_3 is a partition of C', it follows that $\vec{\chi}(C') \leq 3\gamma$. Thus,

$$\vec{\chi}(N(P') \setminus N[K]) \leq \vec{\chi}(A^- \cup B^-) + \vec{\chi}(C')$$
$$\leq 4\gamma(k+1) + 3\gamma$$
$$\leq \gamma(4k+7)$$
$$\leq \gamma(4k+19).$$

as claimed.

5.4 Brooms of type 1 and type 4

Assume that \mathcal{B} is a broom of type 1, and that \mathcal{B}' is a broom of type 4. By (Claim 34), $\vec{\chi}(A^- \cup B^+) \leq 4\gamma(k+1)$. Let $C' = N(P') \setminus (N[K] \cup N(Q) \cup A^- \cup B^+)$. C' does not contain a strongly connected tournament on $\omega(D)$ vertices by the minimality of |V(P)|. For every $i \in \{1, \ldots, m\}$, let L_i be the set of vertices v in C' such that $vv_i \in A(D)$ and i is minimized. Notice that since $v \notin A^- \cup B^+$, it follows that v has both an in-neighbour and out-neighbour in P', and so L_1, \ldots, L_m partitions C'. Finally, let C_1, \ldots, C_5 be such that:

 $C_i = \{c \in L_j : i \text{ is congruent to } j \text{ modulo } 5\}.$

That is, for example, C_1 is the union of L_1, L_6, L_{11} , and so on. We will bound $\vec{\chi}(C_i)$ by partitioning each of C_1, \ldots, C_5 into three sets each with a clique number strictly smaller than

 $\omega(D)$. This will imply that $\vec{\chi}(C') \leq 15\gamma$. The following claim will allow us to make such a partition.

(Claim 36) Let $1 \le i \le 5$, and let $v \in C_i$. If K_1 and K_2 are tournaments in C_i each of size $\omega(D)$, then v is not both a sink vertex of K_1 and a source vertex of K_2 .

Proof. For a contradiction, assume the claim does not hold. That is, suppose there exists a vertex v and tournaments K_1 and K_2 each of size $\omega(D)$ such that v is a source vertex of K_1 and a sink vertex of K_2 . Let u be a sink vertex in K_1 and w be a source vertex in K_2 . Note that this implies that $wv, vu \in A(D)$. Let v_i and v_j be the first out-neighbour and in-neighbour of v respectively. Furthermore, let v_x be the first in-neighbour of w, and let v_y be the last out-neighbour of u. Since $v \notin A^- \cup B^+$, we have i < j. Furthermore, if $i \leq x$, then K_1 and $v_i \to \cdots \to v_x$ contradict the minimality of C as a PMCT. Thus, x < i. Using similar logic, we also have j < y.

By the definition of C_i , we have $x \cong j \mod 5$, and since x < j, we have that $|x - j| \ge 5$. Consequently, the path P'' which is P' with vertices v_x, \ldots, v_y replaced by v_x, w, v, u, v_y , is strictly smaller. This contradicts the minimality of C, thus finishing the proof.

(Claim 37) For every $1 \le i \le 5$, we have $\vec{\chi}(C_i) \le 3\gamma$.

Proof. Let X_i (resp. Y_i) be the set of vertices $v \in N(P') \setminus (N[K] \cup N(Q))$ such that there exists a tournament K' with $|K'| = \omega(D)$ in C_i where v is a sink vertex (resp. source vertex) of K', and let $Z_i = C_i \setminus (X_i \cup Y_i)$. If $\omega(X_i) = \omega(D)$, then there exists a tournament K' in X_i with a source vertex v. But since $v \in X_i$, then v is a sink vertex of another tournament, this contradicts (Claim 36). Thus, $\omega(X_i) < \omega(D)$. By similar logic, $\omega(Y_i) < \omega(D)$. As for Z_i , each $\omega(D)$ -vertex tournament in Z_i is strongly connected by the choice of X_i and Y_i . But since $P \neq \emptyset$, this contradicts that C is a PMCT. Thus, $\omega(Z_i) < \omega(D)$. We conclude $\tilde{\chi}(C_i) \leq 3\gamma$.

We can now finish the proof. Since C_1, \ldots, C_5 is a partition of C', it follows that $\vec{\chi}(C') \leq 15\gamma$. Thus,

$$\vec{\chi}(N(P') \smallsetminus (N[K] \cup N(Q))) \leq \vec{\chi}(A^- \cup B^+) + \vec{\chi}(C')$$
$$\leq 4\gamma(k+1) + 15\gamma$$
$$\leq \gamma(4k+19).$$

as claimed, which finishes the proof of this case, and so the proof of Theorem 1.16. \Box

References

- [1] Pierre Aboulker, Guillaume Aubian, and Pierre Charbit. Heroes in oriented complete multipartite graphs. *arXiv preprint arXiv:2202.13306*, 2022.
- [2] Pierre Aboulker, Pierre Charbit, and Reza Naserasr. Extension of Gyárfás-Sumner conjecture to digraphs. *Electronic Journal of Combinatorics*, 2021.
- [3] Eli Berger, Krzysztof Choromanski, Maria Chudnovsky, Jacob Fox, Martin Loebl, Alex Scott, Paul Seymour, and Stephan Thomassé. Tournaments and colouring. *Journal of Combinatorial Theory, Series B*, 103(1):1–20, 2013.

- [4] Maria Chudnovsky, Alex Scott, and Paul Seymour. Induced subgraphs of graphs with large chromatic number. XI. Orientations. *European Journal of Combinatorics*, 76:53–61, 2019.
- [5] Linda Cook, Tomáš Masařík, Marcin Pilipczuk, Amadeus Reinald, and Uéverton S Souza. Proving a directed analogue of the Gyárfás-Sumner conjecture for orientations of P₄. arXiv preprint arXiv:2209.06171, 2022.
- [6] Paul Erdős and András Hajnal. On chromatic number of graphs and set-systems. Acta Math. Acad. Sci. Hungar, 17(61-99):1, 1966.
- [7] Tibor Gallai. On directed paths and circuits. Theory of Graphs, pages 115–118, 1968.
- [8] András Gyárfás. Problems from the world surrounding perfect graphs. Number 177. MTA Számítástechnikai és Automatizálási Kutató Intézet, 1985.
- [9] Ararat Harutyunyan, Tien-Nam Le, Alantha Newman, and Stéphan Thomassé. Coloring dense digraphs. *Combinatorica*, 39:1021–1053, 2019.
- [10] Victor Neumann-Lara. The dichromatic number of a digraph. Journal of Combinatorial Theory, Series B, 33(3):265–270, 1982.
- [11] Bernard Roy. Nombre chromatique et plus longs chemins d'un graphe. Revue française d'informatique et de recherche opérationnelle, 1(5):129–132, 1967.
- [12] Alex Scott and Paul Seymour. A survey of χ -boundedness. Journal of Graph Theory, 95(3):473–504, 2020.
- [13] Raphael Steiner. On coloring digraphs with forbidden induced subgraphs. Journal of Graph Theory, 103(2):323–339, 2023.
- [14] David P Sumner. Subtrees of a graph and chromatic number. The Theory and Applications of Graphs, (G. Chartrand, ed.), John Wiley & Sons, New York, 557:576, 1981.