TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 10, Pages 4235–4260
S 0002-9947(01)02731-3
Article electronically published on June 6, 2001

ON HERSTEIN'S LIE MAP CONJECTURES, I

K. I. BEIDAR, M. BREŠAR, M. A. CHEBOTAR, AND W. S. MARTINDALE III

Abstract. We describe surjective Lie homomorphisms from Lie ideals of skew elements of algebras with involution onto noncentral Lie ideals (factored by their centers) of skew elements of prime algebras $\mathcal{D}$ with involution, provided that $\mathrm{char}(\mathcal{D}) \neq 2$ and $\mathcal{D}$ is not PI of low degree. This solves the last remaining open problem of Herstein on Lie isomorphisms module cases of PI rings of low degree. A more general problem on maps preserving any polynomial is also discussed.

1. Introduction

Let $\mathcal{F}$ be an associative commutative ring with 1 and $\mathcal{B}$ an associative $\mathcal{F}$-algebra. It is well–known that defining new products in $\mathcal{B}$, the *Lie product* and the *Jordan product*, by

$$[x, y] = xy - yx \quad \text{and} \quad x \circ y = xy + yx,$$

$\mathcal{B}$ becomes a Lie and a Jordan algebra, respectively. A *Lie (Jordan) subalgebra* of $\mathcal{B}$ is any $\mathcal{F}$-submodule closed under the Lie (Jordan) product. A submodule $\mathcal{T}$ of $\mathcal{B}$ satisfying a stronger condition $[\mathcal{T}, \mathcal{B}] \subseteq \mathcal{T}$ is called *a Lie ideal* of $\mathcal{B}$. Given any subset $\mathcal{R}$ of $\mathcal{B}$, we define the *center* $\mathcal{Z}(\mathcal{R})$ *of* $\mathcal{R}$ to be the set $\{r \in \mathcal{R} \mid [x, r] = 0 \text{ for all } x \in \mathcal{R}\}$. Next, we denote by $[\mathcal{R}, \mathcal{R}]$ the $\mathcal{F}$-submodule of $\mathcal{B}$ generated by $\{[x, y] \mid x, y \in \mathcal{R}\}$. Note that $\mathcal{Z} = \mathcal{Z}(\mathcal{B})$ and $[\mathcal{B}, \mathcal{B}]$ are Lie ideals of $\mathcal{B}$.

Now assume that $\mathcal{B}$ is an algebra with involution $*$. Let

$$\mathcal{S}(\mathcal{B}) = \{x \in \mathcal{B} \mid x^* = x\} \quad \text{and} \quad \mathcal{K} = \mathcal{K}(\mathcal{B}) = \{x \in \mathcal{B} \mid x^* = -x\}$$

be the set of *symmetric* and *skew* elements in $\mathcal{B}$, respectively. Clearly, $\mathcal{K}$ is a Lie subalgebra and $\mathcal{S}(\mathcal{B})$ is a Jordan subalgebra of $\mathcal{B}$. A *Lie ideal of* $\mathcal{K}$ is, of course, an $\mathcal{F}$-submodule $\mathcal{T}$ of $\mathcal{K}$ satisfying $[\mathcal{T}, \mathcal{K}] \subseteq \mathcal{T}$. For instance, $[\mathcal{K}, \mathcal{K}]$ is a Lie ideal of $\mathcal{K}$.

The study of the relationship between the associative and the Lie and Jordan structure of an associative ring $\mathcal{B}$ was initiated in the 1950's by Herstein (see, e.g., [38, 39]). Introducing elementary and clever new methods (which, incidentally, are still reflected even in the present work) he obtained, in particular, rather definitive results concerning the Lie and Jordan ideal structure in the case where $\mathcal{B}$ was a simple ring (with or without involution), thereby extending classical results of Cartan and Killing on simple finite–dimensional complex Lie algebras. Some related natural questions, however, remained unsolved, and among them, notably, all the basic questions on Lie isomorphisms. Given Lie subalgebras $\mathcal{T}$ and $\mathcal{U}$ of associative

Received by the editors October 6, 1999 and, in revised form, June 1, 2000.
1991 *Mathematics Subject Classification.* Primary 16W10, 16W20, 16R50.
The second author was partially supported by a grant from the Ministry of Science of Slovenia.

algebras, one defines a *Lie homomorphism* of $\mathcal{T}$ into $\mathcal{U}$ to be an $\mathcal{F}$-module map $\alpha : \mathcal{T} \to \mathcal{U}$ satisfying $[x, y]^\alpha = [x^\alpha, y^\alpha]$ for all $x, y \in \mathcal{T}$ (we shall always write Lie maps as exponents). For example, an isomorphism or a negative of an anti-isomorphism of one algebra onto another is also a Lie isomorphism. One can ask whether the converse is true in some special cases. That is, does every Lie isomorphism of certain Lie subalgebras of associative algebras arise (modulo maps whose range is central) from (anti-)isomorphisms? So, in particular, this question asks whether the algebras should be (anti-)isomorphic provided that some of their Lie subalgebras are isomorphic as Lie algebras. At his 1961 AMS Hour Talk [38, pp. 528–529], Herstein conjectured that this should hold true for Lie isomorphisms of $\mathcal{B}$, $[\mathcal{B}, \mathcal{B}]$, $[\mathcal{B}, \mathcal{B}]/[\mathcal{B}, \mathcal{B}] \cap \mathcal{Z}$ and, in the case where $\mathcal{B}$ was a ring with involution, Lie isomorphisms of $\mathcal{K}$, $[\mathcal{K}, \mathcal{K}]$, and $[\mathcal{K}, \mathcal{K}]/[\mathcal{K}, \mathcal{K}] \cap \mathcal{Z}$, for the case when $\mathcal{B}$ was, neglecting some low dimensional counterexamples, an arbitrary simple ring. He also posed similar problems for another Lie type map, namely, for Lie derivations. The cases of $[\mathcal{B}, \mathcal{B}]/[\mathcal{B}, \mathcal{B}] \cap \mathcal{Z}$ and $[\mathcal{K}, \mathcal{K}]/[\mathcal{K}, \mathcal{K}] \cap \mathcal{Z}$ seem to be of special interest since these two Lie rings are, except in some very special situations, simple [38, Theorems 4 and 10] (we remark here that $\mathcal{K}$ can differ from $[\mathcal{K}, \mathcal{K}]$ even when every element of $\mathcal{Z}$ is left fixed by the involution [48]). Note that the presence of anti-isomorphisms can be avoided in results on Lie isomorphisms of skew elements—namely, if β were a negative of an anti-isomorphism of a ring $\mathcal{B}$ with involution, then the map $a \mapsto -(a^*)^\beta$ would be a homomorphism which coincides on $\mathcal{K}$ with β.

The resolution of Herstein's Lie map problems in the classical case when $\mathcal{B} = M_n(F)$, F a field, has been well–known for a long time (see, e.g., [43, Chapter 10]). In 1951 Hua [42] described Lie automorphisms of a simple Artinian ring $\mathcal{B} = M_n(D)$, D a division ring, $n \geq 3$. Later on Martindale, a student of Herstein, together with some of his students considered Herstein's problems in a series of papers [41, 50, 51, 53, 54, 55, 56, 57, 67]. Basically, the problems have been solved provided that the rings contain certain nontrivial idempotents. Also, the treatment of the problems has been extended from simple to prime rings (we mention, as a curiosity, that the utility of the concept of the extended centroid of a prime ring was realized for the first time when treating Lie maps of prime rings). Lie map problems have also been considered in operator algebras [2, 3, 4, 5, 36, 62, 63, 64] and the techniques there also rest heavily on the presence of idempotents. The question whether the results on Lie maps can be obtained in rings containing no nontrivial idempotents has been open for a long time.

It seems appropriate at this point to say a few words about the analogous Jordan map project. Generally speaking, the Jordan case is usually easier than the Lie case; in Herstein's words [38, p. 518]: "... in general, in considering such questions as we shall, be they about the appropriate ideal structure, homomorphisms, derivatives, the Jordan situation is much easier to study than the corresponding Lie one. One reason for this is that in the Lie case the center of $\mathcal{R}$ constantly gets in our way, so much so, that many questions, completely answered for the Jordan case are virtually untouched in their Lie analogs." The definition of a Jordan homomorphism is analogous to that of a Lie homomorphism. Jordan homomorphisms of an (associative) ring onto a prime (associative) ring were characterized already by Herstein [37]. In 1967, Martindale described Jordan homomorphisms of a Jordan ring of symmetric elements in a ring with involution, containing some nontrivial orthogonal idempotents [52] (see also an extension of Jacobson [44]). Though the

Jordan case is supposed to be easier, the proofs of the results replacing the condition on idempotents by more intrinsic conditions [58, 60] are quite involved and are based on Zelmanov's path–breaking work on Jordan algebras [71].

The first idempotent free result on Lie maps was obtained in 1993 by Brešar [27]. Under some mild technical assumptions (which were removed somewhat later [22, 33]), he described the form of a Lie isomorphism between arbitrary prime rings. This was also the first paper based on applications of the theory of functional identities. The main idea of the proof can be easily described. Every Lie isomorphism α of an associative ring $\mathcal{B}$ clearly satisfies $[x^\alpha, (x^2)^\alpha] = 0$ for every $x \in \mathcal{B}$. This identity can be viewed, at least from the present point of view, as a rather simple functional identity (see Section 2). Under suitable assumptions, one can obtain all possible solutions of functional identities. Hence one obtains the form of $(x^2)^\alpha$. Knowing how α acts on squares, and hence on the Jordan product, and at the same time knowing, by the very definition of the Lie map, how α acts on the Lie product, it is then easy to describe the action of α on the initial associative product. The same idea also works in the semiprime case [6]. In 1994, Beidar, Martindale and Mikhalev described Lie isomorphisms of $\mathcal{K} = \mathcal{K}(\mathcal{B})$, where $\mathcal{B}$ was a prime ring with $\mathrm{char}(\mathcal{B}) \neq 2, 3$ and with involution of the first kind [17] (see also [18, Chapter 9]). This proof is considerably more difficult, but its main idea is essentially the same: every Lie isomorphism α of $\mathcal{K}$ gives rise to a functional identity $[x^\alpha, (x^3)^\alpha] = 0$ for all $x \in \mathcal{K}$ (here, the fact that the cube of a skew element is skew again was used). The case of $\mathrm{char}(\mathcal{B}) = 3$ was investigated by Chebotar [35] who also obtained a considerably shorter proof of the main result of [17]. A consideration of Lie isomorphisms of Lie ideals of $\mathcal{B}$ and $\mathcal{K}$ (in particular, $[\mathcal{B}, \mathcal{B}]$ and $[\mathcal{K}, \mathcal{K}]$) is more entangled since there is no such obvious way of how to arrive at appropriate functional identities (in particular, these Lie rings are not closed under any powers). Nevertheless, functional identities can be produced in these cases as well.

Though in the Jordan case surjective homomorphisms were described some time ago, in the case of Lie maps only the isomorphisms have been investigated. The first results on surjective Lie maps were obtained recently by Beidar and Chebotar [12] who described Lie homomorphisms of Lie ideals of algebras onto noncentral Lie ideals of prime algebras (factored by their centers). As a crucial tool in this investigation, in [10, 11] the useful concept of a d-free subset of a ring was introduced which, in particular, allows for a unified approach to a variety of mapping problems involving different subsets of rings. Very roughly speaking, a subset of a ring is d-free if every appropriate functional identity in less than d variables has only the "standard" (i.e., "obvious") solution. In [10, 11, 12] many basic results were proved for d-free subsets of rings. These results can be applied to rings, Lie ideals of rings, skew elements of rings with involution, etc. as soon as the d-freeness of these subsets has been established. Since the d-freeness of noncentral Lie ideals of the skew elements of a prime ring with involution was proved in [9], we shall apply a number of these basic results in the present paper, enabling us in particular to prove our key result, Theorem 3.5. In a subsequent paper and part II of the present paper, we shall characterize Lie derivations by reducing the problem to Lie isomorphisms onto a d-free set, which further illustrates the usefulness of the d-free concept. We remark that one can usually establish the d-freeness of appropriate subsets of the ring in question unless the ring is PI of low degree. Therefore it will turn out that in order to obtain complete solutions of Herstein's Lie isomorphism problems classical structure theory (for the PI case) seem to be required along with the d-free methods (for the non–PI case).

As indicated earlier there has been over the years a long series of papers settling Herstein's Lie map conjectures in a variety of special situations. The main goal of the present paper is to give solutions to all Lie isomorphism problems in rings with involution modulo cases when rings satisfy PI of low degree. In particular, we settle the most difficult of his conjectures, the one involving Lie isomorphisms between Lie rings of the form $[\mathcal{K},\mathcal{K}]/[\mathcal{K},\mathcal{K}] \cap \mathcal{Z}$, $\mathcal{K}$ the skew elements of a prime ring with involution. In Part III of the present paper, using structure theory of PI rings, we remove these restrictions on degree of PI in both Parts I and II of the present paper and in [12], and thereby solve Herstein's problem in full generality.

We mention here several recent papers [19, 20, 21, 23, 24, 46, 61, 65, 69, 70] which are connected in some way with our present work.

The paper is organized as follows. In Section 2 we briefly survey basic concepts and some results on functional identities, thereby providing a necessary tool for attacking the Lie map problem. The theory of functional identities has been developed recently and much of this development has been motivated by the Lie map problems; we refer to [26, 28, 29, 31, 49] for some initial results and to [7, 8, 9, 10, 11, 16, 30, 34] for the more advanced results. In Section 2 we shall basically present only some extractions from the recent papers [9, 10, 11]. The main reason for including Section 2 in the paper is to make the paper readable and as self-contained as possible.

The body of the paper is Section 3 which treats Lie maps with d-free range. As a matter of fact, for reasons that become clear in Section 4 and for most of Section 3 we consider not only Lie maps but maps α satisfying a somewhat more general condition $[x,y]^\alpha = \lambda[x^\alpha, y^\alpha]$, where λ is a nonzero central element. In our most general result on Lie maps, Theorem 3.5, there is no restriction on the nature of the rings involved, the only essential restriction being that the range of the Lie homomorphism is d-free, specifically $d = 9$. The d-freeness condition seems to be the "proper condition" when considering such problems, and the class of rings satisfying this condition certainly also includes various nonprime rings. Nevertheless, because of historic reasons we are primarily interested in prime (and simple) rings (with involution). As an application of the d-free approach we shall then obtain several results which settle Herstein's Lie isomorphism conjectures for these rings. We are presently going to state these results, the proofs of which are given in Section 3. But first we have to set some notation in place.

In what follows, $\mathcal{F}$ is a commutative ring with 1. Given a nonempty subset $\mathcal{T}$ of an $\mathcal{F}$-algebra $\mathcal{A}$, we denote by $\langle \mathcal{T} \rangle$ the subalgebra of $\mathcal{A}$ generated by $\mathcal{T}$. Next, by $\mathcal{B}$ we denote a prime $\mathcal{F}$-algebra with maximal right (left) ring of quotients $\mathcal{Q}_{mr} = \mathcal{Q}_{mr}(\mathcal{B})$ (respectively $\mathcal{Q}_{ml} = \mathcal{Q}_{ml}(\mathcal{B})$) and Martindale extended centroid $\mathcal{C} = \mathcal{C}(\mathcal{B})$ (see [18, Chapter 2]). We let $\mathcal{Q}$ be either $\mathcal{Q}_{mr}$ or $\mathcal{Q}_{ml}$. It is well-known that both $\mathcal{Q}$ and $\mathcal{C}$ are $\mathcal{F}$-algebras and $\mathcal{B}$ is a subalgebra of $\mathcal{Q}$. We recall that an involution $*$ on $\mathcal{B}$ is said to be of the first kind if $*$ acts as the identity on $\mathcal{C}$; otherwise, $*$ is of the second kind.

Let $x \in \mathcal{Q}$. By $\deg(x)$ we shall mean the degree of x over $\mathcal{C}$ (if x is algebraic over $\mathcal{C}$) or ∞ (if x is not algebraic over $\mathcal{C}$). Given a nonempty subset $\mathcal{R} \subseteq \mathcal{Q}$, we set

$$\deg(\mathcal{R}) = \sup\{\deg(x) \mid x \in \mathcal{R}\}.$$

If $\mathcal{B}$ is prime and $\deg(\mathcal{B}) = n < \infty$, then it follows from results of the theory of rings with polynomial identities [66, 68] that the ring $\mathcal{B}$ is isomorphic to a subring $\mathcal{D}$ of the ring of $n\times n$ matrices over $\overline{\mathcal{C}}$, the algebraic closure of $\mathcal{C}$, such that $\mathcal{D}\overline{\mathcal{C}} = M_n(\overline{\mathcal{C}})$. This is also equivalent to the condition that $\mathcal{BC}$ is of dimension n^2 over $\mathcal{C}$, as well as to the condition that $\mathcal{B}$ satisfies the standard polynomial identity of degree $2n$.

The next theorem must be regarded as the principal result of this paper since it constitutes the breakthrough by which all of Herstein's Lie isomorphism problems for simple and prime rings with involution can be solved (see [38, p. 529, problem 5]).

Theorem 1.1. *Let $\mathcal{A}$ be an $\mathcal{F}$-algebra with involution, let $\mathcal{L}$ be the skew elements of $\mathcal{A}$, and let $\mathcal{S}$ be a Lie ideal of $\mathcal{L}$. Let $\mathcal{B}$ be a prime $\mathcal{F}$-algebra with involution, let $\mathcal{K}$ be the skew elements of $\mathcal{B}$, let $\mathcal{R}$ be a noncentral Lie ideal of $\mathcal{K}$, and set $\overline{\mathcal{R}} = \mathcal{R}/\mathcal{R}\cap\mathcal{C}$. Further, let $\alpha : \mathcal{S} \to \overline{\mathcal{R}}$ be a surjective Lie homomorphism. Suppose that deg$(\mathcal{B}) > 20$ and $\mathrm{char}(\mathcal{B}) \neq 2$. Then there exists an algebra homomorphism $\psi : \langle\mathcal{S}\rangle \to \langle\mathcal{R}\rangle\mathcal{C} + \mathcal{C}$ such that $\overline{x^{\psi}} = x^{\alpha}$ for all $x \in \mathcal{S}$. Moreover, if the involution of $\mathcal{B}$ is of the first kind, then $\langle\mathcal{S}\rangle^{\psi} = \langle\mathcal{R}\rangle$.*

In essence Theorem 1.1 reduces Lie isomorphism problems for prime rings with involution to the case where the ring $\mathcal{B}$ is PI of "low" degree. This latter situation is handled separately in [25, 56]. Combining Theorem 1.1 with results from [25, 56] in Part III of the present paper we shall obtain the the ultimate result that all of Herstein's Lie isomorphism conjectures for prime rings with involution are finally solved in full generality. Special cases of these conjectures, where, among other assumptions, the existence of nontrivial idempotents was required, were considered by Martindale [56] and Rosen [67], as well as, quite recently, by Ayupov and Azamov [3, 4] who treated Lie isomorphisms of $[\mathcal{K}(\mathcal{B}), \mathcal{K}(\mathcal{B})]$, $\mathcal{B}$ being a real factor (and hence a prime real von Neumann algebra). As a corollary to Theorem 1.1 and [17, 25, 35] we have the following result which solves Herstein's Lie isomorphism conjectures for simple rings.

Theorem 1.2. *Let $\mathcal{A}$ be a simple $\mathcal{F}$-algebra with involution and with extended centroid $\mathcal{T}$, let $\mathcal{L}$ be the skew elements of $\mathcal{A}$, let $\mathcal{S} = [\mathcal{L}, \mathcal{L}]$ and let $\overline{\mathcal{S}} = \mathcal{S}/(\mathcal{S}\cap\mathcal{Z}(\mathcal{A}))$. Next, let $\mathcal{B}$ be a simple $\mathcal{F}$-algebra with involution and with extended centroid $\mathcal{C}$, let $\mathcal{K}$ be the skew elements of $\mathcal{B}$, let $\mathcal{R} = [\mathcal{K}, \mathcal{K}]$, and let $\overline{\mathcal{R}} = \mathcal{R}/(\mathcal{R}\cap\mathcal{Z}(\mathcal{B}))$. Further, let $\alpha : \overline{\mathcal{S}} \to \overline{\mathcal{R}}$ be an isomorphism of Lie algebras. Suppose that $\mathrm{char}(\mathcal{F}) \neq 2$ and one of the following conditions is fulfilled:*

(a) *Both involutions are of the first kind and $\dim_{\mathcal{C}}(\mathcal{B}) \neq 1, 4, 9, 16, 25, 64$.*
(b) *Both involutions are of the second kind and $\dim_{\mathcal{C}}(\mathcal{B}) > 4$ (if $\mathrm{char}(\mathcal{F}) \neq 3$) or $\dim_{\mathcal{C}}(\mathcal{B}) > 9$ (if $\mathrm{char}(\mathcal{F}) = 3$).*
(c) *$\dim_{\mathcal{C}}(\mathcal{B}) > 400$.*

Then there exists an isomorphism $\phi : \mathcal{A} \to \mathcal{B}$ of $\mathcal{F}$-algebras such that $\overline{x}^{\alpha} = \overline{x^{\phi}}$ for all $x \in \mathcal{S}$.

Theorem 1.3. *Let $\mathcal{B}$ be a simple $\mathcal{F}$-algebra with involution $*$, with centroid $\mathcal{C}$, and with skew elements $\mathcal{K}$. Further, let $\mathcal{R} = [\mathcal{K}, \mathcal{K}]$, let $\overline{\mathcal{R}} = \mathcal{R}/(\mathcal{R}\cap\mathcal{Z}(\mathcal{B}))$, let $\mathrm{Aur}_{\mathcal{F}}(\mathcal{R})$ and $\mathrm{Aut}_{\mathcal{F}}(\overline{\mathcal{R}})$ be the groups of automorphisms of the Lie algebras $\mathcal{R}$ and $\overline{\mathcal{R}}$ respectively, and let*

$$\mathcal{H} = \{\sigma \in \mathrm{Aut}_{\mathcal{F}}(\mathcal{B}) \mid (x^*)^{\sigma} = (x^{\sigma})^* \text{ for all } x \in \mathcal{B}\}.$$

Suppose that $\mathrm{char}(\mathcal{F}) \neq 2$ and one of the following conditions is fulfilled:

(a) *The involution $*$ is of the first kind and $\dim_{\mathcal{C}}(\mathcal{B}) \neq 1, 4, 9, 16, 25, 64$.*

(b) *The involution* $*$ *is of the second kind and* $\dim_{\mathcal{C}}(\mathcal{B}) > 4$ *(if* $\mathrm{char}(\mathcal{F}) \neq 3$*) or* $\dim_{\mathcal{C}}(\mathcal{B}) > 9$ *(if* $\mathrm{char}(\mathcal{F}) = 3$*).*

Then $\mathcal{H} \cong \mathrm{Aut}_{\mathcal{F}}(\mathcal{R}) \cong \mathrm{Aut}_{\mathcal{F}}(\overline{\mathcal{R}})$.

The purpose of our next theorem is to show that in certain situations (that is, those cases not covered in Theorem 1.1) Lie isomorphisms cannot exist in the first place (except for several low-dimensional examples).

Theorem 1.4. *Let* $\mathcal{B}$ *be a prime ring with involution* $*$ *with skew elements* $\mathcal{K}$ *and with Martindale centroid* $\mathcal{C}$*. Further, let* $\mathcal{R}$ *be a noncentral Lie ideal of* $\mathcal{R}$*. Suppose that* $\mathrm{char}(\mathcal{B}) \neq 2$ *and* $\deg(\mathcal{B}) > 20$*. Then the Lie ring* $\overline{\mathcal{R}} = \mathcal{R}/(\mathcal{R} \cap \mathcal{C})$ *is not a homomorphic image of a Lie ideal of any ring.*

The case $\mathcal{R} = \mathcal{K}$ of the above theorem was treated in [12, Theorem 1.6]. Further, one can describe a Lie (Jordan) homomorphism as a map preserving the polynomial $xy - yx$ ($xy + yx$, respectively). In Section 5 we consider a more general problem of characterizing maps that preserve arbitrary polynomials. Some special cases were considered already by Kaplansky [47] (the polynomial xyx), Jacobson and Rickart [45] (the polynomial $[[x, y], z]$) and Herstein [38] (the polynomial x^n). The approach with functional identities has already been proved to be efficient in this problem [13, 14, 27, 32]. In this paper we prove

Theorem 1.5. *Let* $\mathcal{B}$ *be prime ring with* $\mathrm{char}(\mathcal{B}) \neq 2$*, with involution* $*$ *and with extended centroid* $\mathcal{C}$*. Let* $\mathcal{R}$ *be a noncentral Lie ideal of the Lie ring* $\mathcal{K}(\mathcal{B})$*, let* $\mathcal{F}$ *be a subring of* $\mathcal{S}(\mathcal{C})$ *such that* $\frac{1}{2} \in \mathcal{F}$*,* $\mathcal{F}\mathcal{R} = \mathcal{R}$ *and* $\mathcal{F}\mathcal{B} = \mathcal{B}$*. Let* $\mathcal{A}$ *be an* $\mathcal{F}$*-algebra with involution, let* $\mathcal{S}$ *be a Lie ideal of the Lie* $\mathcal{F}$*-algebra* $\mathcal{K}(\mathcal{A})$*, let* $0 \neq f(x_1, x_2, \dots, x_m) \in \mathcal{F}\langle\mathcal{X}\rangle$ *be a multilinear polynomial in* $x_1, x_2, \dots, x_m$*,* $m \geq 2$*, such that* $f(s_1, s_2, \dots, s_m) \in \mathcal{S}$ *for all* $s_i \in \mathcal{S}$ *and let* $\alpha : \mathcal{S} \to \mathcal{R}$ *be an epimorphism of* $\mathcal{F}$*-modules such that*

$$f(s_1, s_2, \dots, s_m)^\alpha = f(s_1^\alpha, s_2^\alpha, \dots, s_m^\alpha) \quad \text{for all } s_1, s_2, \dots, s_m \in \mathcal{S}.$$

Suppose that $\deg(\mathcal{B}) > \max\{4m + 2, 20\}$ *and one of the following conditions is fulfilled:*

(a) $f(x_1, x_2, \dots, x_m)$ *is a Lie polynomial;*
(b) α *is an isomorphism of* $\mathcal{F}$*-modules and* $\mathcal{A}$ *is a prime ring with* $\deg(\mathcal{A}) > \max\{2m + 2, 8\}$*;*
(c) $[\mathcal{S}, \mathcal{S}] = \mathcal{S}$*.*

Then there exist $\zeta \in \mathcal{C}$*, an* $\mathcal{F}$*-linear map* $\mu : \langle\mathcal{S}\rangle \to \mathcal{C}$ *and a homomorphism of* $\mathcal{F}$*-algebras* $\beta : \langle\mathcal{S}\rangle \to \langle\mathcal{R}\rangle\mathcal{C} + \mathcal{C}$ *such that* $\zeta^{m-1} = 1$*,*

$$x^\alpha = \zeta x^\beta + \mu(x) \quad \text{for all } x \in \mathcal{S}$$

and $\mu(f(s_1, s_2, \dots, s_m)) = 0$ *for all* $s_i \in \mathcal{S}$*. Moreover, if* f_{x_i}*, the partial derivative of* f *at* x_i*, is nonzero for some* $1 \leq i \leq m$*, then* $\mu = 0$ *and* $\langle\mathcal{S}\rangle^\phi = \langle\mathcal{R}\rangle$*.*

As a corollary to Theorem 1.5 we will obtain a solution to Herstein's problem on additive automorphisms of $\mathcal{K}$ preserving the polynomial x^{2n+1} [38, Problem 1, p. 528]:

Theorem 1.6. *Let* $\mathcal{B}$ *be a prime* $\mathcal{F}$*-algebra with involution and with skew elements* $\mathcal{K}$*, let* $\mathcal{C}$ *be the Martindale centroid of* $\mathcal{B}$ *and let* $m \geq 3$ *be an odd integer. Further, let* $\mathcal{A}$ *be an* $\mathcal{F}$*-algebra with involution and with skew elements* $\mathcal{L}$ *and let* $\alpha : \mathcal{L} \to \mathcal{K}$ *be a surjective* $\mathcal{F}$*-module map such that* $(x^m)^\alpha = (x^\alpha)^m$ *for all* $x \in \mathcal{L}$*. Suppose that*

$\mathrm{char}(\mathcal{B}) \neq 2$ *and* $\deg(\mathcal{B}) > \max\{4m+2, 20\}$. *Then there exist a homomorphism of* $\mathcal{F}$*-algebras* $\beta : \langle\mathcal{L}\rangle \to \langle\mathcal{K}\rangle\mathcal{C} + \mathcal{C}$, *an* $\mathcal{F}$*-linear map* $\mu : \langle\mathcal{L}\rangle \to \mathcal{C}$ *and an element* $\zeta \in \mathcal{C}$ *such that* $\zeta^{m-1} = 1$ *and* $x^\alpha = \zeta x^\beta + \mu(x)$ *for all* $x \in \mathcal{L}$. *Furthermore, if* $\mathrm{char}(\mathcal{B}) = 0$ *or* $\mathrm{char}(\mathcal{B}) = p > 0$ *and* m *is not a power of* p, *then* $\mu = 0$.

2. Functional identities and d-free sets

First we fix the notation. Throughout the section, $\mathcal{F}$ will be a commutative ring with 1, $\mathcal{Q}$ will be an $\mathcal{F}$-algebra with 1 and $\mathcal{C}$ will be its center. Later on we shall be primarily interested in the case when $\mathcal{Q}$ is the maximal (right or left) ring of quotients of a prime algebra $\mathcal{B}$, and hence $\mathcal{C}$ is the Martindale centroid of $\mathcal{B}$. However, until further notice, $\mathcal{Q}$ may be an arbitrary algebra. By $\mathcal{S}$ we denote an arbitrary set, and by $\mathcal{R}$ we denote a subset of $\mathcal{Q}$. Let $\mathcal{N}^*$ be the set of all positive integers and for $n \in \mathcal{N}^*$ we let $\mathcal{S}^n$ denote the nth Cartesian power of $\mathcal{S}$.

Let $m \in \mathcal{N}^*$ and $E : \mathcal{R}^{m-1} \to \mathcal{Q}$, $p : \mathcal{R}^{m-2} \to \mathcal{Q}$ be arbitrary maps. In the case when $m = 1$ it should be understood that E is an element in $\mathcal{Q}$ and $p = 0$. Let $1 \leq i < j \leq m$, and define $E^i, p^{ij}, p^{ji} : \mathcal{R}^m \to \mathcal{Q}$ by

$$\begin{aligned} E^i(\overline{x}_m) &= E(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \quad \text{and} \\ p^{ij}(\overline{x}_m) &= p^{ji}(\overline{x}_m) = p(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_m). \end{aligned}$$

Here, $\overline{x}_m$ stands for $(x_1, \dots, x_m)$.

Now let $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \dots, m\}$, and for each $i \in \mathcal{I}$, $j \in \mathcal{J}$, let $E_i, F_j : \mathcal{R}^{m-1} \to \mathcal{Q}$ be arbitrary maps. The basic *functional identities* are

$$\sum_{i\in\mathcal{I}} E_i^i(\overline{x}_m)x_i + \sum_{j\in\mathcal{J}} x_j F_j^j(\overline{x}_m) = 0 \quad \text{for all } \overline{x}_m \in \mathcal{R}^m, \tag{1}$$

and a slightly more general one,

$$\sum_{i\in\mathcal{I}} E_i^i(\overline{x}_m)x_i + \sum_{j\in\mathcal{J}} x_j F_j^j(\overline{x}_m) \in \mathcal{C} \quad \text{for all } \overline{x}_m \in \mathcal{R}^m. \tag{2}$$

The goal in the theory of functional identities is to describe the form of the maps appearing in the identity. A natural possibility when (1) (and hence also (2)) is fulfilled is when there exist maps $p_{ij} : \mathcal{R}^{m-2} \to \mathcal{Q}, i \in \mathcal{I}, j \in \mathcal{J}, i \neq j$, $\lambda_k : \mathcal{R}^{m-1} \to \mathcal{C}$, $k \in \mathcal{I} \cup \mathcal{J}$, such that

$$\begin{aligned} E_i^i(\overline{x}_m) &= \sum_{\substack{j\in\mathcal{J},\\ j\neq i}} x_j p_{ij}^{ij}(\overline{x}_m) + \lambda_i^i(\overline{x}_m), \\ F_j^j(\overline{x}_m) &= -\sum_{\substack{i\in\mathcal{I},\\ i\neq j}} p_{ij}^{ij}(\overline{x}_m)x_i - \lambda_j^j(\overline{x}_m) \quad \text{and} \\ \lambda_k &= 0 \qquad \text{if} \qquad k \notin \mathcal{I} \cap \mathcal{J}, \end{aligned} \tag{3}$$

for all $\overline{x}_m \in \mathcal{R}^m$, $i \in \mathcal{I}$, $j \in \mathcal{J}$. Indeed, one can readily check that (3) implies (1). We shall refer to (3) as a *standard solution* of (1) (and of (2)). It should be pointed out that the case when one of the sets $\mathcal{I}$ or $\mathcal{J}$ is empty is not excluded. The sum over the empty set of indices should be simply read as zero. This means that, for example, a standard solution of the functional identity $\sum_{i\in\mathcal{I}} E_i^i(\overline{x}_m)x_i = 0$ is $E_i = 0$, $i \in \mathcal{I}$.

It turns out that often a standard solution is also the only possible solution. This is the reason for introducing the following fundamental concept [10].

Definition 2.1. A nonempty subset $\mathcal{R} \subseteq \mathcal{Q}$ is said to be *d-free*, where $d \in \mathcal{N}^*$, if for all $m \in \mathcal{N}^*$ and $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \ldots, m\}$ the following two conditions are satisfied:

(a) If $\max\{|\mathcal{I}|, |\mathcal{J}|\} \leq d$, (1) implies (3).
(b) If $\max\{|\mathcal{I}|, |\mathcal{J}|\}| \leq d - 1$, (2) implies (3).

Thus, in a loose manner one can say that d-free sets are those subsets $\mathcal{R}$ of $\mathcal{Q}$ that any functional identity on $\mathcal{R}$ in "not too many" variables has only a standard solution. Incidentally, it is easy to see that this standard solution is unique [10, Remark 2.5].

The basic result justifying the introduction of the concept of d-free sets states that a prime ring $\mathcal{B}$ with $\deg(\mathcal{B}) \geq d$ is a d-free subset of $\mathcal{Q}$, where $\mathcal{Q}$ is any of the rings $\mathcal{Q}_{mr}(\mathcal{B})$ or $\mathcal{Q}_{ml}(\mathcal{B})$ (this is explicitly stated in [10], but essentially already proved in [7] and [16]). For this paper, however, the following deeper result is needed.

Theorem 2.2 ([9, Theorem 1.1])**.** *Let $\mathcal{B}$ be a prime ring with involution and let $\mathcal{K}$ be the set of skew elements in $\mathcal{B}$. Let $\mathcal{Q}$ be either $\mathcal{Q}_{mr}(\mathcal{B})$ or $\mathcal{Q}_{ml}(\mathcal{B})$. If* $\mathrm{char}(\mathcal{B}) \neq 2$ *and* $\deg(\mathcal{B}) \geq 2d + 3$*, then any noncentral Lie ideal $\mathcal{R}$ of $\mathcal{K}$ is a d-free subset of $\mathcal{Q}$.*

This theorem makes it possible for us to apply to noncentral Lie ideals of $\mathcal{K}$ all the results on d-free subsets obtained in [10, 11]. We will now review some of those results that will be needed in the following sections. The first one will be used frequently.

Theorem 2.3 ([10, Theorem 2.8])**.** *Let $\mathcal{D} \subseteq \mathcal{R} \subseteq \mathcal{Q}$ be nonempty subsets and $d \in \mathcal{N}^*$. Suppose that $\mathcal{D}$ is d-free. Then $\mathcal{R}$ is d-free also.*

The concept of a functional identity can be viewed as a generalization of the concept of a polynomial identity. Indeed, assuming that all the maps E_i and F_j are "monomials" $\lambda x_{i_1} \ldots x_{i_{m-1}}$, the functional identity (1) reduces to a polynomial identity. However, the theory of functional identities can be considered, at this point, more as a complement to that of polynomial identities, rather than its extension. Namely, one can obtain definite results concerning functional identities on d-free sets, which are, in a way, in striking contrast to the sets satisfying polynomial identities. Namely, assume that a nonzero multilinear polynomial $f(X_1, \ldots, X_m) \in \mathcal{C}\langle\mathcal{X}\rangle$, the free algebra over $\mathcal{C}$ on the set of noncommuting variables $\mathcal{X} = \{X_1, X_2, \ldots\}$, is a polynomial identity on a set $\mathcal{R} \subseteq \mathcal{Q}$. That is, $f(x_1, \ldots, x_m) = 0$ for all $x_1, \ldots, x_m \in \mathcal{R}$. But this can be written as $\sum_{i \in \mathcal{I}} E_i^i(\overline{x}_m)x_i = 0$, where each $E_i \in \mathcal{C}\langle\mathcal{X}\rangle$ is a multilinear polynomial in $m-1$ variables. Thus we have arrived at a functional identity with a nonstandard solution, unless $\mathcal{R}$ satisfies a polynomial identity of a smaller degree. But in the latter case we can repeat the argument just given and so, after a finite number of steps, we certainly get a functional identity in $\leq m$ variables with a nonstandard solution. For future reference we record this observation as

Remark 2.4. Let $f(X_1, \ldots, X_m) \in \mathcal{C}\langle\mathcal{X}\rangle$ be a nonzero multilinear polynomial. If $\mathcal{R}$ is a d-free subset of $\mathcal{Q}$ with $d \geq m$, then $f(X_1, \ldots, X_m)$ is not a polynomial identity on $\mathcal{R}$.

Moreover, from [11, Corollary 2.4] we deduce at once the following somewhat less obvious fact.

Remark 2.5. Let $f(X_1, \dots, X_m) \in \mathcal{C}\langle\mathcal{X}\rangle$ be a nonzero multilinear polynomial at least one of whose coefficients is invertible and let $a \in \mathcal{Q}$. If $\mathcal{R}$ is a d-free subset of $\mathcal{Q}$ with $d \geq m-1$, then $f(x_1, \dots, x_{m-1}, a) = 0$ for all $x_1, \dots, x_{m-1} \in \mathcal{R}$ implies $a \in \mathcal{C}$.

As a matter of fact, functional identities that will appear in subsequent sections are not exactly of the same type as (1) and (2). They will involve expressions of the form

$$\pi(\overline{x}_m) = \sum_{i\in\mathcal{I}} E_i^i(\overline{x}_m)x_i^\alpha + \sum_{j\in\mathcal{J}} x_j^\alpha F_j^j(\overline{x}_m),$$

where α is a map from an arbitrary set $\mathcal{S}$ into $\mathcal{Q}$ and $E_i, F_j : \mathcal{S}^{m-1} \to \mathcal{Q}$. Such functional identities may look more complicated than (1) and (2), but fortunately this is only apparently so. One defines standard solutions in this setting in a self–explanatory way (just replace x_i by x_i^α on appropriate places in (3)). Now assume that the range of α, $\mathcal{R} = \mathcal{S}^\alpha$, is a d-free subset of $\mathcal{Q}$. Then [10, Theorem 2.6] tells us that if $\max\{|\mathcal{I}|, |\mathcal{J}|\} \leq d$, then the functional identity $\pi(\overline{x}_m) = 0$ for all $\overline{x}_m \in \mathcal{S}^m$ has only the standard solution. The same is true for the functional identity $\pi(\overline{x}_m) \in \mathcal{C}$ for all $\overline{x}_m \in \mathcal{S}^m$, provided that $\max\{|\mathcal{I}|, |\mathcal{J}|\}| \leq d-1$.

When dealing with functional identities, the so–called *quasi-polynomials* naturally get in our way. Let us give a definition in a somewhat loose manner (see [11] for details). A map $E : \mathcal{S}^2 \to \mathcal{Q}$ is said to be a *multilinear quasi-polynomial* (with respect to a map $\alpha : \mathcal{S} \to \mathcal{Q}$) of degree 2 if there exist $\lambda_1, \lambda_2 \in \mathcal{C}$ and maps $\mu_1, \mu_2 : \mathcal{S} \to \mathcal{C}$, $\nu : \mathcal{S}^2 \to \mathcal{C}$ such that

$$E(x, y) = \lambda_1 x^\alpha y^\alpha + \lambda_2 y^\alpha x^\alpha + \mu_1(x)y^\alpha + \mu_2(y)x^\alpha + \nu(x, y) \quad \text{for all } x, y \in \mathcal{S},$$

and at least one of what we shall call the *coeffients* of E, that is, $\lambda_1, \lambda_2, \mu_1, \mu_2$, ν, is nonzero (map or element). A multilinear quasi-polynomial of degree m would, of course, involve summands such as $\lambda x_1^\alpha \dots x_m^\alpha$, $\mu(x_1)x_2^\alpha \dots x_m^\alpha$, $\nu(x_1, x_2)x_3^\alpha \dots x_m^\alpha$ etc. Adapting the argument establishing Remark 2.4 one easily proves the following statement (which in turn is a special case of [11, Theorem 1.1]).

Lemma 2.6. *Let $E : \mathcal{S}^m \to \mathcal{Q}$ be a multilinear quasi-polynomial (with respect to α) of degree $\leq m$. If $E(\overline{x}_m) = 0$ for all $\overline{x}_m \in \mathcal{S}^m$ and $\mathcal{R} = \mathcal{S}^\alpha$ is an $(m+1)$-free subset of $\mathcal{Q}$, then all the coefficients of E are zero.*

Occasionally we shall have to use a variant of Lemma 2.6 for quasi-polynomials which are not multilinear. Instead of giving a rigorous statement [11, Corollary 2.12] we rather explain what we have in mind by a simple example. Suppose that $\lambda(x^\alpha)^2 y^\alpha x^\alpha + \mu(x, x)y^\alpha x^\alpha + \nu(x, x, x, y) = 0$ for all $x, y \in \mathcal{S}$, where the coefficients μ and ν are multi-additive maps into $\mathcal{C}$ and $\lambda \in \mathcal{C}$. A complete linearization then gives a "multilinear" version of this identity (consisting of 18 terms such as $\lambda x_1^\alpha x_2^\alpha y x_3^\alpha$, $\mu(x_1, x_2)y^\alpha x_3^\alpha$, $\nu(x_1, x_2, x_3, y)$ etc.). Now, Lemma 2.6 tells us that if $\mathcal{R} = \mathcal{S}^\alpha$ is 5-free, then $\lambda = 0$, $\mu(x_1, x_2) + \mu(x_2, x_1) = 0$ and $\sum_{\pi\in S_3} \nu(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, y) = 0$. Consequently, $\lambda = 2\mu(x, x) = 6\nu(x, x, x, y) = 0$ for all $x \in \mathcal{S}$.

Functional identities involving only one map deserve special attention. We shall need the following result on such identities which is a special case of [11, Theorem 1.2].

Theorem 2.7. *Let $\mathcal{S}$ be an $\mathcal{F}$-module, let $\alpha : \mathcal{S} \to \mathcal{Q}$ be an $\mathcal{F}$-module map and let $E : \mathcal{S}^{m-1} \to \mathcal{Q}$ be an $\mathcal{F}$-multilinear map such that*

$$\sum_{i=1}^{m} \lambda_i E^i(\overline{x}_m)x_i^\alpha + \mu_i x_i^\alpha E^i(\overline{x}_m) \in \mathcal{C} \quad \textit{for all } \overline{x}_m \in \mathcal{S},$$

where $\lambda_i, \mu_i \in \mathcal{C}$ and at least one of them is invertible. If $\mathcal{R} = \mathcal{S}^\alpha$ is an $(m+1)$-free subset of $\mathcal{Q}$, then E is a multilinear quasi-polynomial of degree $\leq m$ and each of its coefficient is $\mathcal{F}$-multilinear.

3. Lie homomorphisms and d-freeness

Throughout this section, $\mathcal{F}$ will be a commutative ring with 1, $\mathcal{Q}$ will be an $\mathcal{F}$-algebra with 1 and with center $\mathcal{C}$, and $\mathcal{R}$ will be a subset of $\mathcal{Q}$. Further, we set $\overline{\mathcal{Q}} = \mathcal{Q}/\mathcal{C}$ and $\overline{\mathcal{R}} = \mathcal{R}/(\mathcal{R} \cap \mathcal{C})$. We identify $\overline{\mathcal{R}}$ with $(\mathcal{R} + \mathcal{C})/\mathcal{C} \subseteq \overline{\mathcal{Q}}$ and remark that $\overline{\mathcal{Q}}$ is the factor Lie algebra of $\mathcal{Q}$ by the Lie ideal $\mathcal{C}$.

Our goal is to prove the main result of this section, Theorem 3.5. This will be done in a series of steps. We begin by stating a result from [11].

Theorem 3.1 ([11, Theorem 2.11])**.** *Let $\mathcal{U}$ be a Lie subalgebra of an $\mathcal{F}$-algebra $\mathcal{A}$, let $\alpha : \mathcal{U} \to \mathcal{Q}$ be an $\mathcal{F}$-module map and let $B : \mathcal{U}^m \to \mathcal{Q}$, $m \geq 2$, be any $\mathcal{F}$-multilinear map such that*

(a) $\sum_{i=1}^n B(x_i, \ldots, x_m, x_1, \ldots, x_{i-1}) \in \mathcal{C}$ *for all* $x_1, \ldots, x_m \in \mathcal{U}$,
(b) $B(\overline{x}_{m-1}, [u, v]) - \lambda[B(\overline{x}_{m-1}, u), v^\alpha] - \lambda[u^\alpha, B(\overline{x}_{m-1}, v)] \in \mathcal{C}$ *for all* $\overline{x}_{m-1} \in \mathcal{U}^{m-1}$ *and* $u, v \in \mathcal{U}$,
(c) $[u, v]^\alpha - \lambda[u^\alpha, v^\alpha] \in \mathcal{C}$ *for all* $u, v \in \mathcal{U}$,

where $\lambda \in \mathcal{C}$ is an invertible element. If $\mathcal{R} = \mathcal{U}^\alpha$ is a $(2m+1)$-free subset of $\mathcal{Q}$, then there exist a multilinear quasi-polynomial $q : \mathcal{U}^{m-1} \to \mathcal{Q}$ (with respect to α) of degree $\leq m-1$ and a map $\mu : \mathcal{U}^m \to \mathcal{C}$ such that $B(\overline{x}_m) = [q(\overline{x}_{m-1}), x_m^\alpha] + \mu(\overline{x}_m)$ for all $\overline{x}_m \in \mathcal{U}^m$. Moreover, μ and all the coefficients of q are $\mathcal{F}$-multilinear.

The conditions of this theorem might appear somewhat strange, but actually we shall arrive at them in the proof of the next lemma. This lemma reduces the general case of the Lie homomorphism problem to the case when the domain of the map is closed under the Jordan triple product $xyz + zyx$ (using terms of the classical situation, the case of $[\mathcal{K}, \mathcal{K}]$ shall be reduced to the case of $\mathcal{K}$).

Lemma 3.2. *Let $\mathcal{A}$ be an $\mathcal{F}$-algebra, $\mathcal{D}$ a Lie subalgebra of $\mathcal{A}$ such that $xyz+zyx \in \mathcal{D}$ for all $x, y, z \in \mathcal{D}$ and $\mathcal{S}$ a Lie ideal of $\mathcal{D}$. Further, let $\mathcal{R}$ be a submodule of the $\mathcal{F}$-algebra $\mathcal{Q}$ and $\alpha : \mathcal{S} \to \overline{\mathcal{Q}}$ an $\mathcal{F}$-module map such that $\mathcal{S}^\alpha = \overline{\mathcal{R}}$ and*

$$[x, y]^\alpha = \lambda[x^\alpha, y^\alpha] \quad \textit{for all } x, y \in \mathcal{S},$$

where $\lambda \in \mathcal{C}$ is some fixed invertible element. Suppose that $\mathcal{R}$ is a 9-free subset of $\mathcal{Q}$ and $\mathcal{C}$ is a direct summand of the $\mathcal{C}$-module $\mathcal{Q}$. Then there exists a pair $(\mathcal{U}; \gamma)$ such that

(a) $\mathcal{U}$ *is a Lie ideal of the Lie $\mathcal{F}$-algebra $\mathcal{D}$ containing $\mathcal{S}$,*
(b) $\gamma : \mathcal{U} \to \overline{\mathcal{Q}}$ *is an $\mathcal{F}$-module map,*
(c) $[x, y]^\gamma = \lambda[x^\gamma, y^\gamma]$ *for all* $x, y \in \mathcal{U}$,
(d) $x^\gamma = x^\alpha$ *for all* $x \in \mathcal{S}$,
(e) $xyz + zyx \in \mathcal{U}$ *for all* $x, y, z \in \mathcal{U}$.

Proof. We begin with a simple observation. Let $\mathcal{V}$ be a submodule of $\mathcal{Q}$ containing $\mathcal{R}$. Then $\mathcal{V}$ is 9-free (and hence 2-free) by Theorem 2.3. Let $c \in \mathcal{Z}(\mathcal{V})$. Then $cx - xc = 0$, $x \in \mathcal{V}$, and so $c \in \mathcal{C}$ by Remark 2.5. Therefore $\mathcal{Z}(\mathcal{V}) = \mathcal{C} \cap \mathcal{V}$.

Now consider the class Ω of all pairs $(\mathcal{U}'; \gamma')$ satisfying (a)–(d). Given $(\mathcal{U}'; \gamma')$, $(\mathcal{U}''; \gamma'') \in \Omega$, we shall write $(\mathcal{U}'; \gamma') \geq (\mathcal{U}''; \gamma'')$ provided $\mathcal{U}' \supseteq \mathcal{U}''$ and $x^{\gamma'} = x^{\gamma''}$ for all $x \in \mathcal{U}''$. By Zorn's lemma the class Ω has a maximal element, say $(\mathcal{U}; \gamma)$. It remains to show that condition (e) is satisfied. To this end, we set $\overline{\mathcal{U}}$ to be equal to the $\mathcal{F}$-submodule of $\mathcal{A}$ generated by the subset $\{x, xyz + zyx \mid x, y, z \in \mathcal{U}\}$. Since $\mathcal{U} \subseteq \mathcal{D}$, $\overline{\mathcal{U}} \subseteq \mathcal{D}$ also. Given $t \in \mathcal{D}$ and $x, y, z \in \mathcal{U}$, we have

$$[xyz + zyx, t] = [x, t]yz + zy[x, t] + x[y, t]z + z[y, t]x + xy[z, t] + [z, t]yx \in \overline{\mathcal{U}},$$

showing that $\overline{\mathcal{U}}$ is a Lie ideal of $\mathcal{D}$ containing $\mathcal{U}$ (and so containing $\mathcal{S}$). Now our goal is to extend γ to an $\mathcal{F}$-module map $\overline{\mathcal{U}} \to \overline{\mathcal{Q}}$ satisfying (c). Then the maximality of $(\mathcal{U}; \gamma)$ will imply that $\mathcal{U} = \overline{\mathcal{U}}$ and the proof will be completed.

Choose a $\mathcal{C}$-submodule $\mathcal{W}$ of $\mathcal{Q}$ such that $\mathcal{Q} = \mathcal{W} \oplus \mathcal{C}$. Since $\mathcal{F} \cdot 1 \subseteq \mathcal{C}$, both $\mathcal{C}$ and $\mathcal{W}$ are $\mathcal{F}$-submodules of $\mathcal{Q}$. Clearly $\mathcal{F}$-modules $\mathcal{W}$ and $\overline{\mathcal{Q}}$ are isomorphic and so we may assume, without loss of generality, that $\gamma : \mathcal{U} \to \mathcal{W}$. Let π be the canonical projection of the module $\mathcal{Q}$ onto $\mathcal{W}$. Then

$$[x, y]^{\gamma} = \lambda[x^{\gamma}, y^{\gamma}]^{\pi} \quad \text{for all } x, y \in \mathcal{U}.$$

Therefore

$$[x, y]^{\gamma} = \lambda[x^{\gamma}, y^{\gamma}] + \epsilon(x, y) \quad \text{for all } x, y \in \mathcal{U}$$

where $\epsilon(x, y) \in \mathcal{C}$.

Since we do not know whether $\mathcal{U}^{\gamma}$ is 9-free, we do the following. Set $\mathcal{A}' = \mathcal{A} \oplus \mathcal{C}$, $\mathcal{D}' = \mathcal{D} \oplus \mathcal{C} \subseteq \mathcal{A}'$ and $\mathcal{U}' = \mathcal{U} \oplus \mathcal{C} \subseteq \mathcal{D}'$. Clearly $\mathcal{A}'$ is an $\mathcal{F}$-algebra, $\mathcal{D}'$ is a Lie subalgebra of $\mathcal{A}'$ such that $xyz + zyx \in \mathcal{D}'$ for all $x, y, z \in \mathcal{D}'$ and $\mathcal{U}'$ is a Lie ideal of $\mathcal{D}'$. We identify $\mathcal{A}$ and $\mathcal{C}$ with corresponding subalgebras of $\mathcal{A}'$. Further, set $\mathcal{V} = \mathcal{U}^{\gamma} \oplus \mathcal{C} \subseteq \mathcal{Q}$. It follows from (d) that $\mathcal{U}^{\gamma} \supseteq \overline{\mathcal{R}} = \mathcal{R}^{\pi}$ and so $\mathcal{V} \supseteq \mathcal{R}$. By the above observation $\mathcal{Z}(\mathcal{V}) = \mathcal{C} \cap \mathcal{V} = \mathcal{C}$.

Finally, we define maps $\gamma' : \mathcal{U}' \to \mathcal{V}$ and $\epsilon' : \mathcal{U}' \times \mathcal{U}' \to \mathcal{C}$ by the rule

$$\begin{aligned} (x + a)^{\gamma'} &= x^{\gamma} + a \quad \text{for all } x \in \mathcal{U},\ a \in \mathcal{C} \text{ and} \\ \epsilon'(x + a, y + b) &= \epsilon(x, y) \quad \text{for all } x, y \in \mathcal{U},\ a, b \in \mathcal{C}. \end{aligned}$$

Clearly $x^{\gamma'} = x^{\gamma}$ for all $x \in \mathcal{U}$, $(\mathcal{U}')^{\gamma'} = \mathcal{V}$ and so $(\mathcal{U}')^{\gamma'}$ is a 9-free subset of $\mathcal{Q}$.

We have

$$\begin{aligned} &[x + a, y + b]^{\gamma'} = [x, y]^{\gamma} = \lambda[x^{\gamma}, y^{\gamma}]^{\pi} = \lambda[x^{\gamma}, y^{\gamma}] + \epsilon(x, y) \\ &\quad = \lambda[(x + a)^{\gamma'}, (y + b)^{\gamma'}] + \epsilon'(x + a, y + b) \quad \text{for all } x, y \in \mathcal{U},\ a, b \in \mathcal{C} \end{aligned}$$

and so

$$[u, v]^{\gamma'} = \lambda[u^{\gamma'}, v^{\gamma'}]^{\pi} = \lambda[u^{\gamma'}, v^{\gamma'}] + \epsilon'(u, v) \quad \text{for all } u, v \in \mathcal{U}'. \tag{4}$$

Next, setting $\mathcal{U}'' = \overline{\mathcal{U}} + \mathcal{C} \subseteq \mathcal{D}'$, we remark that $\mathcal{F}$-submodule $\mathcal{U}''$ is generated by the subset $\{x, xyz + zyx \mid x, y, z \in \mathcal{U}'\}$. In particular, $\mathcal{U}'' \supseteq \mathcal{U}'$. To complete the proof, it is enough to construct an $\mathcal{F}$-module map $\gamma'' : \mathcal{U}'' \to \mathcal{W}$ such that $x^{\gamma''} = x^{\gamma}$ for all $x \in \mathcal{U}$ and

$$[x, y]^{\gamma''} = \lambda[x^{\gamma''}, y^{\gamma''}]^{\pi} \quad \text{for all } x, y \in \mathcal{U}''. \tag{5}$$

To this end, we define a map $B : (\mathcal{U}')^4 \to \mathcal{Q}$ by the rule

$$B(x, y, z, t) = [xyz + zyx, t]^{\gamma'} \quad \text{for all } x, y, z, t \in \mathcal{U}'.$$

Since $xyz + zyx \in \mathcal{D}$, $[xyz + zyx, t] \in \mathcal{U}'$ and so B is a well-defined $\mathcal{F}$-multilinear map. A straightforward computation shows that

$$[xyz + zyx, t] + [yzt + tzy, x] + [ztx + xtz, y] + [txy + yxt, z] = 0 \tag{6}$$

and whence

$$B(x, y, z, t) + B(y, z, t, x) + B(z, t, x, y) + B(t, x, y, z) = 0 \tag{7}$$

for all $x, y, z, t \in \mathcal{U}'$. It follows from (4) that

$$\begin{aligned} B(x, y, z, [u, v]) &= [xyz + zyx, [u, v]]^{\gamma'} \\ &= \{[[xyz + zyx, u], v] + [u, [xyz + zyx, v]]\}^{\gamma'} \\ &= \lambda[[xyz + zyx, u]^{\gamma'}, v^{\gamma'}]^{\pi} + \lambda[u^{\gamma'}, [xyz + zyx, v]^{\gamma'}]^{\pi} \\ &= \lambda[B(x, y, z, u), v^{\gamma'}]^{\pi} + \lambda[u^{\gamma'}, B(x, y, z, v)]^{\pi} \end{aligned}$$

and so

$$B(x, y, z, [u, v]) - \lambda[B(x, y, z, u), v^{\gamma'}] - \lambda[u^{\gamma'}, B(x, y, z, v)] \in \mathcal{C} \tag{8}$$

for all $x, y, z, u, v \in \mathcal{U}'$. Since $\mathcal{V}$ is a 9-free subset of $\mathcal{Q}$ and $(\mathcal{U}')^{\gamma'} = \mathcal{V}$, it follows from (4), (7) and (8) that all the conditions of Theorem 3.1 are met (with $m = 4$). By Theorem 3.1 there exist a multilinear quasi-polynomial $q(x, y, z)$ (with respect to γ') of degree ≤ 3, all of whose coefficients are $\mathcal{F}$-multilinear maps, and an $\mathcal{F}$-multilinear map $\mu : (\mathcal{U}')^4 \to \mathcal{C}$ such that

$$B(x, y, z, t) = [q(x, y, z), t^{\gamma'}] + \mu(x, y, z, t) \quad \text{for all } x, y, z, t \in \mathcal{U}'. \tag{9}$$

We now define a map $\gamma'' : \mathcal{U}'' \to \mathcal{W}$ by the rule

$$\left(x + \sum_{i=1}^{n}(x_i y_i z_i + z_i y_i x_i)\right)^{\gamma''} = \left(x^{\gamma'} + \lambda^{-1} \sum_{i=1}^{n} q(x_i, y_i, z_i)\right)^{\pi}$$

for all $x, x_i, y_i, z_i \in \mathcal{U}'$, $i = 1, 2, \ldots, n$. We claim that γ'' is a well-defined $\mathcal{F}$-module map. Indeed, suppose that $x + \sum_{i=1}^{n}(x_i y_i z_i + z_i y_i x_i) = 0$. Let $t \in \mathcal{U}'$. Then

$$\begin{aligned} 0 &= \left[x + \sum_{i=1}^{n}(x_i y_i z_i + z_i y_i x_i), t\right]^{\gamma'} = [x, t]^{\gamma'} + \sum_{i=1}^{n} B(x_i, y_i, z_i, t) \\ &= \left[\lambda x^{\gamma'} + \sum_{i=1}^{n} q(x_i, y_i, z_i), t^{\gamma'}\right] + \epsilon'(x, t) + \sum_{i=1}^{n} \mu(x_i, y_i, z_i, t) \end{aligned}$$

and so $[a, t^{\gamma'}] \in \mathcal{C}$ for all $t \in \mathcal{U}'$, where $a = x^{\gamma'} + \lambda^{-1} \sum_{i=1}^{n} q(x_i, y_i, z_i)$. Since $(\mathcal{U}')^{\gamma'} = \mathcal{V}$, $[a, \mathcal{V}] \subseteq \mathcal{C}$ (forcing $[[a, \mathcal{V}], \mathcal{V}] = 0$) and whence $a \in \mathcal{C}$ (Remark 2.5). Therefore $a^{\pi} = 0$ and so γ'' is well-defined. As γ' is an $\mathcal{F}$-module map and all the coefficients of q are $\mathcal{F}$-multilinear maps, we infer that γ'' is an $\mathcal{F}$-module map which proves our claim.

Finally, we claim that (5) is satisfied. Indeed, let $u, v, w, x, y, z, t \in \mathcal{U}'$. Let us first show that $[x, t]^{\gamma''} = \lambda[x^{\gamma''}, t^{\gamma''}]^{\pi}$. Clearly $[x, t] \in \mathcal{U}$ and so $[x, t]^{\gamma'} = [x, t]^{\gamma} \in \mathcal{W}$. Therefore

$$[x, t]^{\gamma''} = ([x, t]^{\gamma'})^{\pi} = [x, t]^{\gamma'} = \lambda[x^{\gamma'}, t^{\gamma'}]^{\pi}$$

by (4). Since $[a^\pi, b^\pi] = [a, b]$ for all $a, b \in \mathcal{Q}$ and $\gamma'' = \gamma'\pi$, we conclude that

$$[x, t]^{\gamma''} = \lambda[x^{\gamma'}, t^{\gamma'}]^\pi = \lambda[x^{\gamma''}, t^{\gamma''}]^\pi.$$

Next we show that $[xyz+zyx, t]^{\gamma''} = [(xyz+zyx)^{\gamma''}, t^{\gamma''}]^\pi$. We have $[xyz+zyx, t] \in \mathcal{U}'$ because $\mathcal{U}'$ is a Lie ideal of $\mathcal{D}'$ and so

$$\begin{aligned} [xyz + zyx, t]^{\gamma''} &= ([xyz + zyx, t]^{\gamma'})^\pi = ([q(x, y, z), t^{\gamma'}] + \mu(x, y, z, t))^\pi \\ &= \lambda[\lambda^{-1}q(x, y, z), t^{\gamma'}]^\pi = \lambda[(xyz + zyx)^{\gamma''}, t^{\gamma''}]^\pi. \end{aligned}$$

Since the $\mathcal{F}$-module $\mathcal{U}''$ is generated by the subset $\{x, xyz + zyx \mid x, y, z \in \mathcal{U}'\}$, we conclude that $[s, t]^{\gamma''} = \lambda[s^{\gamma''}, t^{\gamma''}]^\pi$ for all $s \in \mathcal{U}''$. It remains to show that

$$[xyz + zyx, uvw + wvu]^{\gamma''} = \lambda[(xyz + zyx)^{\gamma''}, (uvw + wvu)^{\gamma''}]^\pi.$$

Clearly

$$[xyz + zyx, uvw + wvu] \in \mathcal{U}'', \quad [xyz + zyx, t], [uvw + wvu, t] \in \mathcal{U}'$$

and so, by what we just proved, we have

$$\begin{aligned} &\lambda[[xyz + zyx, uvw + wvu]^{\gamma''}, t^{\gamma''}]^\pi = [[xyz + zyx, uvw + wvu], t]^{\gamma''} \\ &= ([[xyz + zyx, t], uvw + wvu] + [xyz + zyx, [uvw + wvu, t]])^{\gamma''} \\ &= \lambda[[xyz + zyx, t]^{\gamma''}, (uvw + wvu)^{\gamma''}]^\pi \\ &\quad +\lambda[(xyz + zyx)^{\gamma''}, [uvw + wvu, t]^{\gamma''}]^\pi \\ &= \lambda^2[[(xyz + zyx)^{\gamma''}, t^{\gamma''}]^\pi, (uvw + wvu)^{\gamma''}]^\pi \\ &\quad +\lambda^2[(xyz + zyx)^{\gamma''}, [(uvw + wvu)^{\gamma''}, t^{\gamma''}]^\pi]^\pi \\ &= \lambda^2[[(xyz + zyx)^{\gamma''}, t^{\gamma''}], (uvw + wvu)^{\gamma''}]^\pi \\ &\quad +\lambda^2[(xyz + zyx)^{\gamma''}, [(uvw + wvu)^{\gamma''}, t^{\gamma''}]]^\pi \\ &= \lambda^2[[(xyz + zyx)^{\gamma''}, (uvw + wvu)^{\gamma''}], t^{\gamma''}]^\pi. \end{aligned}$$

Setting

$$a = [xyz + zyx, uvw + wvu]^{\gamma''} - \lambda[(xyz + zyx)^{\gamma''}, (uvw + wvu)^{\gamma''}],$$

we see that $[a, t^{\gamma''}]^\pi = 0$ for all $t \in \mathcal{U}'$. By Remark 2.5, $a \in \mathcal{C}$ and so $a^\pi = 0$. Therefore

$$[xyz + zyx, uvw + wvu]^{\gamma''} = \lambda[(xyz + zyx)^{\gamma''}, (uvw + wvu)^{\gamma''}]^\pi$$

because $\gamma''\pi = \gamma''$. Thus (5) is satisfied and the proof is thereby complete.

We continue with an elementary lemma whose proof is extracted from that of [39, Theorem 2.3].

Lemma 3.3. *Let $\mathcal{A}$ be an $\mathcal{F}$-algebra and let $\mathcal{U}$ be a Lie subalgebra of $\mathcal{A}$ such that $xyz + zyx \in \mathcal{U}$ for all $x, y, z \in \mathcal{U}$. Suppose that $\frac{1}{2} \in \mathcal{F}$. Then $\langle \mathcal{U} \rangle = \mathcal{U} + \mathcal{U} \circ \mathcal{U}$.*

Proof. We proceed by induction on $n \geq 2$ to show that $x_1 x_2 \ldots x_n \in \mathcal{U} + \mathcal{U} \circ \mathcal{U}$ for all $x_1, x_2, \ldots, x_n \in \mathcal{U}$. Let $x_1, x_2 \in \mathcal{U}$. Then

$$x_1 x_2 = \frac{1}{2}([x_1, x_2] + x_1 \circ x_2) \in \mathcal{U} + \mathcal{U} \circ \mathcal{U}.$$

In the inductive case $n > 2$ we have that

$$[x_1, x_2]x_3 \ldots x_n,\ x_2[x_1, x_3]x_4 \ldots x_n \in \mathcal{U} + \mathcal{U} \circ \mathcal{U}$$

by the induction assumption and whence

$$(x_1x_2x_3 - x_2x_3x_1)x_4 \dots x_n = [x_1, x_2]x_3 \dots x_n + x_2[x_1, x_3]x_4 \dots x_n \in \mathcal{U} + \mathcal{U} \circ \mathcal{U}.$$

By assumption $x_1x_3x_2 + x_2x_3x_1 \in \mathcal{U}$. Therefore the induction assumption yields that $(x_1x_3x_2 + x_2x_3x_1)x_4 \dots x_n \in \mathcal{U} + \mathcal{U} \circ \mathcal{U}$ and so

$$\begin{aligned} x_1(x_2 \circ x_3)x_4 \dots x_n &= (x_1x_2x_3 - x_2x_3x_1)x_4 \dots x_n \\ &\quad + (x_1x_3x_2 + x_2x_3x_1)x_4 \dots x_n \in \mathcal{U} + \mathcal{U} \circ \mathcal{U}. \end{aligned}$$

Thus

$$x_1x_2 \dots x_n = \frac{1}{2}\{x_1(x_2 \circ x_3)x_4 \dots x_n + x_1[x_2, x_3]x_4 \dots x_n\} \in \mathcal{U} + \mathcal{U} \circ \mathcal{U}.$$

The proof is now complete.

Proposition 3.4. *Let $\mathcal{A}$ be an $\mathcal{F}$-algebra and let $\mathcal{S}$ be a Lie subalgebra of $\mathcal{A}$ such that $\mathcal{S} \cap (\mathcal{S} \circ \mathcal{S}) = 0$ and $xyz + zyx \in \mathcal{S}$ for all $x, y \in \mathcal{S}$. Further, let $\mathcal{R}$ be a submodule of the $\mathcal{F}$-algebra $\mathcal{Q}$. Let $\rho : \mathcal{S} \to \overline{\mathcal{Q}}$ be an $\mathcal{F}$-module map with $\mathcal{S}^\rho = \overline{\mathcal{R}}$ and let $\lambda \in \mathcal{C}$ be an invertible element such that*

$$[u, v]^\rho = \lambda[u^\rho, v^\rho] \quad \textit{for all } u, v \in \mathcal{S}.$$

Suppose that $\frac{1}{2} \in \mathcal{F}$, $\mathcal{C}$ is a direct summand of the $\mathcal{C}$-module $\mathcal{Q}$ and $\mathcal{R}$ is a 7-free subset of $\mathcal{Q}$. Then there exists a homomorphism of algebras $\phi : \langle \mathcal{S} \rangle \to \langle \mathcal{R} \rangle \mathcal{C} + \mathcal{C}$ such that $x^\rho = \lambda^{-1}\overline{x^\phi}$ for all $x \in \mathcal{S}$.

Proof. Let $\mathcal{W} \subseteq \mathcal{Q}$ be a $\mathcal{C}$-submodule such that $\mathcal{Q} = \mathcal{W} \oplus \mathcal{C}$ and let $\pi : \mathcal{Q} \to \mathcal{W}$ be a canonical projection of modules. Since $\mathcal{F} \cdot 1 \subseteq \mathcal{C}$, we conclude that both $\mathcal{W}$ and $\mathcal{C}$ are $\mathcal{F}$-submodules of $\mathcal{Q}$ and π is an $\mathcal{F}$-module map. Clearly $\mathcal{F}$-modules $\mathcal{W}$ and $\overline{\mathcal{Q}}$ are isomorphic and so we may assume, without loss of generality, that $\rho : \mathcal{S} \to \mathcal{W}$ is an $\mathcal{F}$-module map such that

$$[u, v]^\rho = \lambda[u^\rho, v^\rho]^\pi \quad \text{for all } u, v \in \mathcal{S} \quad \text{and} \quad \mathcal{S}^\rho = \mathcal{R}^\pi.$$

Set $\mathcal{A}' = \mathcal{A} \oplus \mathcal{C}$, $\mathcal{S}' = \mathcal{S} \oplus \mathcal{C}$ and $\mathcal{R}' = \mathcal{R}^\pi + \mathcal{C} = \mathcal{R}^\pi \oplus \mathcal{C}$. Clearly $\mathcal{S}'$ is a Lie subalgebra of the algebra $\mathcal{A}'$ such that $xyz + zyx \in \mathcal{S}'$ for all $x, y, z \in \mathcal{S}'$ and $\mathcal{R}'$ is an $\mathcal{F}$-submodule of $\mathcal{Q}$ containing $\mathcal{R}$. We identify $\mathcal{A}$ and $\mathcal{C}$ with corresponding ideals of the algebra $\mathcal{A}'$. Since $\mathcal{R}$ is 7-free, Theorem 2.3 yields that $\mathcal{R}'$ is 7-free as well. We now define a map $\alpha : \mathcal{S}' \to \mathcal{R}'$ by the rule $(u + c)^\alpha = u^\rho + c$ for all $u \in \mathcal{S}$, $c \in \mathcal{C}$. Clearly α is an epimorphism of $\mathcal{F}$-modules and

$$[u + a, v + b]^\alpha = [u, v]^\rho = \lambda[u^\rho, v^\rho]^\pi = \lambda[(u + a)^\alpha, (v + b)^\alpha]^\pi$$

for all $u, v \in \mathcal{S}$, $a, b \in \mathcal{C}$. Therefore $[x, y]^\alpha - \lambda[x^\alpha, y^\alpha] \in \mathcal{C}$ for all $x, y \in \mathcal{S}'$. Setting $\epsilon(x, y) = [x, y]^\alpha - \lambda[x^\alpha, y^\alpha]$, $x, y \in \mathcal{S}'$, we conclude that $\epsilon : (\mathcal{S}')^2 \to \mathcal{C}$ is an $\mathcal{F}$-bilinear map such that

$$[x, y]^\alpha = \lambda[x^\alpha, y^\alpha] + \epsilon(x, y) \quad \text{for all } x, y \in \mathcal{S}'. \tag{10}$$

Define a map $B : (\mathcal{S}')^3 \to \mathcal{Q}$ by the rule

$$B(x, y, z) = \lambda^{-2}(xyz + zyx)^\alpha - (x^\alpha y^\alpha z^\alpha + z^\alpha y^\alpha x^\alpha) \quad \text{for all } x, y, z \in \mathcal{S}'.$$

We first remark that $B(x, y, z)$ is a symmetric map "modulo $\mathcal{C}$". Indeed, obviously $B(x, y, z) = B(z, y, x)$ for all $x, y, z \in \mathcal{S}'$. Further,

$$B(x, y, z) - B(y, x, z) = \lambda^{-2}[[x, y], z]^\alpha - [[x^\alpha, y^\alpha], z^\alpha] \in \mathcal{C}$$

which proves our remark. It now follows from both (6) and (10) that

$$[B(x,y,z),t^\alpha]+[B(y,z,t),x^\alpha]+[B(z,t,x),y^\alpha]+[B(t,x,y),z^\alpha]\in\mathcal{C}$$

for all $x,y,z\in\mathcal{S}'$. By Theorem 2.7, $B(x,y,z)$ is a multilinear quasi-polynomial, so that

$$\begin{aligned}B(x,y,z)&=\lambda_1x^\alpha y^\alpha z^\alpha+\lambda_2x^\alpha z^\alpha y^\alpha+\lambda_3y^\alpha x^\alpha z^\alpha+\lambda_4y^\alpha z^\alpha x^\alpha\\&+\lambda_5z^\alpha x^\alpha y^\alpha+\lambda_6z^\alpha y^\alpha x^\alpha+\nu_1(x)y^\alpha z^\alpha+\nu_2(x)z^\alpha y^\alpha+\nu_3(y)x^\alpha z^\alpha\\&+\nu_4(y)z^\alpha x^\alpha+\nu_5(z)x^\alpha y^\alpha+\nu_6(z)y^\alpha x^\alpha+\mu_1(x,y)z^\alpha+\mu_2(x,z)y^\alpha\\&+\mu_3(y,z)x^\alpha+\omega(x,y,z)\quad\text{for all } x,y,z\in\mathcal{S}',\end{aligned}$$

where $\lambda_i\in\mathcal{C}$, $\nu_i:\mathcal{S}'\to\mathcal{C}$ are $\mathcal{F}$-linear maps, $\mu_j:(\mathcal{S}')^2\to\mathcal{C}$ are $\mathcal{F}$-bilinear maps and $\omega:(\mathcal{S}')^3\to\mathcal{C}$ is an $\mathcal{F}$-trilinear map. Since $B(x,y,z)=B(z,y,x)$, Lemma 2.6 implies that $\lambda_1=\lambda_6$, $\lambda_2=\lambda_5$, $\lambda_3=\lambda_4$, $\nu_1=\nu_6$, $\nu_2=\nu_5$, $\nu_3=\nu_4$, $\mu_1(x,y)=\mu_3(y,x)$, $\mu_2(x,z)=\mu_2(z,x)$ and $\omega(x,y,z)=\omega(z,y,x)$. As $B(x,y,z)-B(y,x,z)\in\mathcal{C}$, we obtain that

$$\lambda_1=\lambda_3,\ \lambda_2=\lambda_4,\ \lambda_5=\lambda_6,\ \nu_1=\nu_3,\ \nu_2=\nu_4,\ \nu_5=\nu_6,\ \mu_1(x,y)=\mu_1(y,x)$$

and $\mu_2(x,z)=\mu_3(x,z)$. Set $a=\lambda_1$, $\nu=\nu_1$, $\mu=\mu_1$. We now have

$$\begin{aligned}&B(x,y,z)\\&\quad=\ a\{x^\alpha y^\alpha z^\alpha+x^\alpha z^\alpha y^\alpha+y^\alpha x^\alpha z^\alpha+y^\alpha z^\alpha x^\alpha+z^\alpha x^\alpha y^\alpha+z^\alpha y^\alpha x^\alpha\}\\&\quad+\ \nu(x)(y^\alpha z^\alpha+z^\alpha y^\alpha)+\nu(y)(x^\alpha z^\alpha+z^\alpha x^\alpha)+\nu(z)(x^\alpha y^\alpha+y^\alpha x^\alpha)\\&\quad+\ \mu(x,y)z^\alpha+\mu(x,z)y^\alpha+\mu(y,z)x^\alpha+\omega(x,y,z)\end{aligned}\tag{11}$$

for all $x,y,z\in\mathcal{S}'$. Recalling the definition of B and making use of (11), we get

$$\begin{aligned}(xyx)^\alpha&=\lambda^2(x^\alpha y^\alpha x^\alpha+\frac{1}{2}B(x,y,x))\\&=\lambda^2\{(1+a)x^\alpha y^\alpha x^\alpha+a(x^\alpha x^\alpha y^\alpha+y^\alpha x^\alpha x^\alpha)+\nu(x)(x^\alpha y^\alpha+y^\alpha x^\alpha)\\&+\nu(y)x^\alpha x^\alpha+\mu(x,y)x^\alpha+\frac{1}{2}\mu(x,x)y^\alpha+\frac{1}{2}\omega(x,y,x)\}\end{aligned}\tag{12}$$

for all $x,y\in\mathcal{S}'$. Analogously

$$\begin{aligned}(xyz+zyx)^\alpha&=\lambda^2\{(1+a)(x^\alpha y^\alpha z^\alpha+z^\alpha y^\alpha x^\alpha)\\&+a(x^\alpha z^\alpha y^\alpha+z^\alpha x^\alpha y^\alpha+y^\alpha x^\alpha z^\alpha+y^\alpha z^\alpha x^\alpha)+\nu(x)(z^\alpha y^\alpha+y^\alpha z^\alpha)\\&+\nu(z)(x^\alpha y^\alpha+y^\alpha x^\alpha)+\nu(y)(x^\alpha z^\alpha+z^\alpha x^\alpha)+\mu(x,y)z^\alpha+\mu(z,y)x^\alpha\\&+\mu(x,z)y^\alpha+\omega(x,y,z)\}\quad\text{for all } x,y,z\in\mathcal{S}'.\end{aligned}\tag{13}$$

It follows from both (12) and (13) that

$$\begin{aligned}2(xyxyx)^\alpha&=\{(xyx)yx+xy(xyx)\}^\alpha\\&=\lambda^2\Big\{(1+a)\{x^\alpha y^\alpha(xyx)^\alpha+(xyx)^\alpha y^\alpha x^\alpha\}\\&+a\{x^\alpha(xyx)^\alpha y^\alpha+(xyx)^\alpha x^\alpha y^\alpha+y^\alpha x^\alpha(xyx)^\alpha+y^\alpha(xyx)^\alpha x^\alpha\}\\&+\nu(x)\{(xyx)^\alpha y^\alpha+y^\alpha(xyx)^\alpha\}+\nu(xyx)(x^\alpha y^\alpha+y^\alpha x^\alpha)\\&+\nu(y)\{x^\alpha(xyx)^\alpha+(xyx)^\alpha x^\alpha\}+\mu(x,y)(xyx)^\alpha+\mu(xyx,y)x^\alpha\\&+\mu(x,xyx)y^\alpha+\omega(x,y,xyx)\Big\}=\lambda^2a\{(xyx)^\alpha x^\alpha y^\alpha+y^\alpha x^\alpha(xyx)^\alpha\}+q_1\\&=2\lambda^4a^2y^\alpha x^\alpha x^\alpha x^\alpha y^\alpha+q_1'\end{aligned}\tag{14}$$

for all $x, y \in \mathcal{S}'$, where both q_1 and q_1' are quasi-polynomials which do not involve the monomial $y^\alpha x^\alpha x^\alpha x^\alpha y^\alpha$.

On the other hand, taking into account (12) we now have that

$$\begin{aligned}(x(yxy)x)^\alpha = \lambda^2\Big\{&(1+a)x^\alpha(yxy)^\alpha x^\alpha + a(x^\alpha x^\alpha)\circ(yxy)^\alpha \\ &+ \nu(x)x^\alpha\circ(yxy)^\alpha + \nu(yxy)x^\alpha x^\alpha + \mu(x,yxy)x^\alpha \\ &+ \frac{1}{2}\mu(x,x)(yxy)^\alpha + \frac{1}{2}\omega(x,yxy,x)\Big\} = q_2 \quad \text{for all } x,y\in\mathcal{S}',\end{aligned} \tag{15}$$

where q_2 is a quasi-polynomial not involving the monomial $y^\alpha x^\alpha x^\alpha x^\alpha y^\alpha$. Comparing (14) and (15) we get that the coefficient $\lambda^4 a^2$ of $y^\alpha x^\alpha x^\alpha x^\alpha y^\alpha$ is equal to 0 (see Remark 2.4 and the comment that follows). Therefore $a = 0$.

We see that (12) can be now rewritten as

$$\begin{aligned}(xyx)^\alpha = \lambda^2\{&x^\alpha y^\alpha x^\alpha + \nu(x)x^\alpha\circ y^\alpha + \nu(y)x^\alpha x^\alpha + \mu(x,y)x^\alpha \\ &+\frac{1}{2}\mu(x,x)y^\alpha + \frac{1}{2}\omega(x,y,x)\}\end{aligned} \tag{16}$$

for all $x, y \in \mathcal{S}'$. Next, (15) now yields

$$\begin{aligned}&\{x(yxy)x\}^\alpha \\ &= \lambda^2\Big\{x^\alpha(yxy)^\alpha x^\alpha + \nu(x)x^\alpha\circ(yxy)^\alpha + \nu(yxy)x^\alpha x^\alpha \\ &\qquad + \mu(x,yxy)x^\alpha + \frac{1}{2}\mu(x,x)(yxy)^\alpha + \frac{1}{2}\omega(x,yxy,x)\Big\} \\ &= \lambda^2\{x^\alpha(yxy)^\alpha x^\alpha + \nu(x)x^\alpha\circ(yxy)^\alpha\} + q_3 \\ &= \lambda^4\{\mu(x,y) + 2\nu(x)\nu(y)\}x^\alpha y^\alpha x^\alpha + q_3' \quad \text{for all } x,y\in\mathcal{S}',\end{aligned} \tag{17}$$

where both q_3 and q_3' are quasi-polynomials which do not involve the monomial $x^\alpha y^\alpha x^\alpha$. Analogously, it follows from (14) that

$$\begin{aligned}2(xyxyx)^\alpha &= \{(xyx)yx + xy(xyx)\}^\alpha \\ &= \lambda^2\{x^\alpha y^\alpha(xyx)^\alpha + (xyx)^\alpha y^\alpha x^\alpha + \nu(x)(xyx)^\alpha\circ y^\alpha \\ &\quad + \nu(xyx)x^\alpha\circ y^\alpha + \nu(y)x^\alpha\circ(xyx)^\alpha + \mu(x,y)(xyx)^\alpha + \mu(xyx,y)x^\alpha \\ &\qquad + \mu(x,xyx)y^\alpha + \omega(x,y,xyx)\} \\ &= \lambda^2\{x^\alpha y^\alpha(xyx)^\alpha + (xyx)^\alpha y^\alpha x^\alpha + \nu(y)x^\alpha\circ(xyx)^\alpha + \mu(x,y)(xyx)^\alpha\} \\ &\quad + q_4 = \lambda^4\{3\mu(x,y) + 2\nu(x)\nu(y)\}x^\alpha y^\alpha x^\alpha + q_4' \quad \text{for all } x,y\in\mathcal{S}',\end{aligned} \tag{18}$$

where both q_4 and q_4' are quasi-polynomials which do not involve the monomial $x^\alpha y^\alpha x^\alpha$. Comparing (17) and (18) we conclude from Lemma 2.6 that the coefficients of $x^\alpha y^\alpha x^\alpha$ are equal, that is to say,

$$2\lambda^4[\mu(x,y) + 2\nu(x)\nu(y)] = \lambda^4[3\mu(x,y) + 2\nu(x)\nu(y)] \quad \text{for all } x,y\in\mathcal{S}'.$$

Therefore $\mu(x,y) = 2\nu(x)\nu(y)$ and (13) can be rewritten as

$$\begin{aligned}&(xyz + zyx)^\alpha \\ &= \lambda^2\{x^\alpha y^\alpha z^\alpha + z^\alpha y^\alpha x^\alpha + \nu(x)z^\alpha\circ y^\alpha + \nu(z)x^\alpha\circ y^\alpha + \nu(y)x^\alpha\circ z^\alpha \\ &\qquad + 2\nu(x)\nu(y)z^\alpha + 2\nu(z)\nu(y)x^\alpha + 2\nu(x)\nu(z)y^\alpha + \omega(x,y,z)\}\end{aligned} \tag{19}$$

for all $x, y, z \in \mathcal{S}'$. Define a map $\beta : \mathcal{S}' \to \mathcal{R}'$ by the rule $x^\beta = x^\alpha + \nu(x)$. We now have

$$(xyz + zyx)^\beta = \lambda^2\{x^\beta y^\beta z^\beta + z^\beta y^\beta x^\beta + 2\tau(x, y, z)\} \quad \text{for all } x, y, z \in \mathcal{S}', \tag{20}$$

for some $\mathcal{F}$-multilinear map $\tau : (\mathcal{S}')^3 \to \mathcal{C}$. Clearly $\tau(x, y, z) = \tau(z, y, x)$ for all $x, y, z \in \mathcal{S}'$. Taking $z = x$ in (20), we get

$$(xyx)^\beta = \lambda^2\{x^\beta y^\beta x^\beta + \tau(x, y, x)\} \quad \text{for all } x, y \in \mathcal{S}'. \tag{21}$$

Our aim is to show that $\tau = 0$. By (21) we have

$$\begin{aligned}(x^2yx^2)^\beta &= \{x(xyx)x\}^\beta = \lambda^2\{x^\beta(xyx)^\beta x^\beta + \tau(x, xyx, x)\}\\ &= \lambda^4\{x^\beta(x^\beta y^\beta x^\beta + \tau(x, y, x))x^\beta\} + \lambda^2\tau(x, xyx, x) \quad \text{and}\\ \{x(x^2yx^2)x\}^\beta &= \lambda^2 x^\beta\{\lambda^4 x^\beta(x^\beta y^\beta x^\beta + \tau(x, y, x))x^\beta\\ &\qquad + \lambda^2\tau(x, xyx, x)\}x^\beta + \lambda^2\tau(x, x^2yx^2, x)\\ &= \lambda^6(x^\beta)^3 y^\beta (x^\beta)^3 + \lambda^6\tau(x, y, x)(x^\beta)^4\\ &\qquad + \lambda^4\tau(x, xyx, x)(x^\beta)^2 + \lambda^2\tau(x, x^2yx^2, x)\end{aligned} \tag{22}$$

for all $x, y \in \mathcal{S}'$. On the other hand,

$$\begin{aligned}(x^3yx^3)^\beta &= \lambda^2\{(x^3)^\beta y^\beta (x^3)^\beta + \tau(x^3, y, x^3)\}\\ &= \lambda^6\{((x^\beta)^3 + \tau(x, x, x))y^\beta((x^\beta)^3 + \tau(x, x, x))\} + \lambda^2\tau(x^3, y, x^3)\\ &= \lambda^6(x^\beta)^3 y^\beta (x^\beta)^3 + \lambda^6\tau(x, x, x)(x^\beta)^3 \circ y^\beta + \lambda^6\tau^2(x, x, x)\\ &\qquad + \lambda^2\tau(x^3, y, x^3) \quad \text{for all } x, y \in \mathcal{S}'.\end{aligned} \tag{23}$$

Subtracting (22) from (23) we obtain

$$\begin{aligned}&\lambda^6\tau(x, y, x)(x^\beta)^4 - \lambda^6\tau(x, x, x)(x^\beta)^3 \circ y^\beta + \lambda^4\tau(x, xyx, x)(x^\beta)^2\\ &\qquad + \lambda^2\tau(x, x^2yx^2, x) - \lambda^6\tau^2(x, x, x) - \lambda^2\tau(x^3, y, x^3) = 0\end{aligned} \tag{24}$$

for all $x, y \in \mathcal{S}'$. Since $\mathcal{R}' = (\mathcal{S}')^\alpha$ is 7-free, substituting of $x^\alpha + \nu(x)$ for x^β in (24) we conclude that $\lambda^6\tau(x, y, x)(x^\alpha)^4$ can be written as a quasi-polynomial not involving $(x^\alpha)^4$ and so $\tau(x, y, x) = 0$ by Lemma 2.6. Since $\tau(x, y, z) = \tau(z, y, x)$, we now conclude that $\tau = 0$.

Define a map $\gamma : \mathcal{S}' \to \mathcal{R}'\mathcal{C}$ by the rule $x^\gamma = \lambda x^\beta$. It now follows from (20) that

$$(xyz + zyx)^\gamma = x^\gamma y^\gamma z^\gamma + z^\gamma y^\gamma x^\gamma \quad \text{for all } x, y, z \in \mathcal{S}'. \tag{25}$$

We also have that

$$\begin{aligned}[x, y]^\gamma - [x^\gamma, y^\gamma] &= \lambda[x, y]^\beta - \lambda^2[x^\beta, y^\beta]\\ &= \lambda[x, y]^\alpha + \lambda\nu([x, y]) - \lambda^2[x^\alpha, y^\alpha] = \lambda\epsilon(x, y) + \lambda\nu([x, y]) \in \mathcal{C}\end{aligned}$$

for all $x, y \in \mathcal{S}'$. Setting $\theta(x, y) = [x, y]^\gamma - [x^\gamma, y^\gamma]$ for all $x, y \in \mathcal{S}'$, we remark that $\theta : (\mathcal{S}')^2 \to \mathcal{C}$ is a biadditive map and

$$[x, y]^\gamma = [x^\gamma, y^\gamma] + \theta(x, y) \quad \text{for all } x, y \in \mathcal{S}'. \tag{26}$$

Our aim is to show that $\theta = 0$. Since

$$[xzx, y] = [x, y]zx + x[y, z]x + xz[x, y],$$

we get from both (25) and (26)

$$\begin{aligned}[x^\gamma z^\gamma x^\gamma, y^\gamma] + \theta(xzx, y) &= [xzx, y]^\gamma = ([x,y]zx + x[y,z]x + xz[x,y])^\gamma \\ &= \{[x^\gamma, y^\gamma] + \theta(x,y)\}z^\gamma x^\gamma + x^\gamma\{[y^\gamma, z^\gamma] + \theta(y,z)\}x^\gamma \\ &\quad + x^\gamma z^\gamma\{[x^\gamma, y^\gamma] + \theta(x,y)\} \quad \text{for all } x,y,z \in \mathcal{S}'\end{aligned}$$

and so

$$\theta(x,y)z^\gamma x^\gamma + \theta(y,z)(x^\gamma)^2 + \theta(x,y)x^\gamma z^\gamma - \theta(xzx, y) = 0 \quad \text{for all } x,y,z \in \mathcal{S}'.$$

Substituting $\lambda x^\alpha + \lambda\nu(x)$ for x^γ and making use of Lemma 2.6, we conclude that $\theta = 0$ and so γ is a Lie homomorphism.

Setting $z = y$ in (25), we get

$$(y^2 \circ x)^\gamma = (y^\gamma)^2 \circ x^\gamma \quad \text{for all } x,y \in \mathcal{S}'. \tag{27}$$

We now remark that $\mathcal{S}' \circ \mathcal{S}' = \mathcal{S} \circ \mathcal{S} + \mathcal{C}$ because $\mathcal{S}\mathcal{C} = 0$ and $\mathcal{C} \circ \mathcal{C} = \mathcal{C}$. It follows from Lemma 3.3 that every element a of the algebra $\langle \mathcal{S}' \rangle$ can be written in the form $a = b + \sum d_i^2 + c$ for some $b, d_i \in \mathcal{S}$, $c \in \mathcal{C}$. We define a map $\phi : \langle \mathcal{S}' \rangle \to \langle \mathcal{R}'\mathcal{C} \rangle$ by the rule

$$a^\phi = b^\gamma + \sum_i (d_i^\gamma)^2 + c^\gamma, \quad a \in \langle \mathcal{S} \rangle.$$

Suppose that $b + \sum_i d_i^2 + c = 0$. Since $\mathcal{A} \cap \mathcal{C} = 0$ and $b + \sum_i d_i^2 \in \mathcal{A}$, we conclude that $b + \sum_i d_i^2 = 0 = c$. As $\mathcal{S} \cap (\mathcal{S} \circ \mathcal{S}) = 0$, $b = 0 = \sum d_i^2$ and so (27) implies that

$$0 = \left\{\left(\sum d_i^2\right) \circ z\right\}^\gamma = e \circ z^\gamma \quad \text{for all } z \in \mathcal{S}',$$

where $e = \sum (d_i^\gamma)^2$. Substituting $[u,v]$ for z, we conclude that

$$e \circ [u^\gamma, v^\gamma] = 0 \quad \text{for all } u, v \in \mathcal{S}'.$$

Recalling that $w^\gamma = \lambda w^\alpha + \lambda\nu(w)$ for all $w \in \mathcal{S}'$, we get $e \circ [u^\alpha, v^\alpha] = 0$ for all $u, v \in \mathcal{S}'$. Since $(\mathcal{S}')^\alpha = \mathcal{R}'$, we see that $e \circ [x,y] = 0$ for all $x, y \in \mathcal{R}'$. As $\mathcal{R}'$ is 7-free, Remark 2.5 implies that $e \in \mathcal{C}$ and so $2e[x,y] = 0$ is a polynomial identity on $\mathcal{R}'$. It now follows from Remark 2.4 that $e = 0$ and so ϕ is a well-defined map of $\mathcal{F}$-modules.

Given $x, y \in \mathcal{S}'$ we have $xy = \frac{1}{2}\{(x+y)^2 - x^2 - y^2 + [x,y]\}$ and so

$$\begin{aligned}(xy)^\phi &= \frac{1}{2}\{[(x+y)^2]^\phi - (x^2)^\phi - (y^2)^\phi + [x,y]^\phi\} \\ &= \frac{1}{2}\{(x^\gamma + y^\gamma)^2 - (x^\gamma)^2 - (y^\gamma)^2 + [x^\gamma, y^\gamma]\} \\ &= x^\gamma y^\gamma = x^\phi y^\phi. \end{aligned} \tag{28}$$

The identity $x^2 y = \frac{1}{2}\{x \circ [x,y] + x^2 \circ y\}$ together with (27) yields

$$\begin{aligned}(x^2y)^\phi &= \frac{1}{2}\{(x \circ [x,y])^\phi + (x^2 \circ y)^\gamma\} = \frac{1}{2}\{x^\gamma \circ [x^\gamma, y^\gamma] + (x^\gamma)^2 \circ y^\gamma\} \\ &= (x^\gamma)^2 y^\gamma = (x^2)^\phi y^\phi \quad \text{for all } x, y \in \mathcal{S}'. \end{aligned} \tag{29}$$

In view of Lemma 3.3, both (28) and (29) imply that

$$(ux)^\phi = u^\phi x^\phi \quad \text{for all } u \in \langle \mathcal{S}' \rangle,\ x \in \mathcal{S}'. \tag{30}$$

Since the algebra $\langle \mathcal{S}' \rangle$ is generated by $\mathcal{S}'$, it follows from (30) that ϕ is a homomorphism of the algebra $\langle \mathcal{S}' \rangle$ into $\langle \mathcal{R}'\mathcal{C} \rangle$. From the definition of ϕ we see that $x^\rho = \lambda^{-1}\,\overline{x^\phi} \in \overline{\mathcal{Q}}$ for all $x \in \mathcal{S}$. The proof is complete.

We are now in a position to prove the main result of this section.

Theorem 3.5. *Let $\mathcal{F}$ be a commutative ring with $1, \frac{1}{2} \in \mathcal{F}$, let $\mathcal{A}$ be an $\mathcal{F}$-algebra, let $\mathcal{D}$ be a Lie subalgebra of $\mathcal{A}$ such that $\mathcal{D} \cap (\mathcal{D} \circ \mathcal{D}) = 0$ and $xyz + zyx \in \mathcal{D}$ for all $x, y, z \in \mathcal{D}$ and let $\mathcal{S}$ be a Lie ideal of $\mathcal{D}$. Further, let $\mathcal{R}$ be a submodule of an $\mathcal{F}$-algebra $\mathcal{Q}$ with unity and let $\mathcal{C}$ be the center of $\mathcal{Q}$. Let $\overline{\mathcal{R}} = \mathcal{R}/(\mathcal{R} \cap \mathcal{C})$ be the factor module of $\mathcal{R}$ by the submodule $\mathcal{R} \cap \mathcal{C}$, let $\overline{\mathcal{Q}} = \mathcal{Q}/\mathcal{C}$ be the factor algebra of the Lie algebra $\mathcal{Q}$ by the Lie ideal $\mathcal{C}$ and let $\alpha : \mathcal{S} \to \overline{\mathcal{Q}}$ be an $\mathcal{F}$-module map such that $\mathcal{S}^\alpha = \overline{\mathcal{R}}$ and*

$$[x, y]^\alpha = \lambda[x^\alpha, y^\alpha] \quad \text{for all } x, y \in \mathcal{S},$$

where $\lambda \in \mathcal{C}$ is some fixed invertible element. Suppose that $\mathcal{R}$ is a 9-free subset of $\mathcal{Q}$ and $\mathcal{C}$ is a direct summand of the $\mathcal{C}$-module $\mathcal{Q}$. Then there exists a homomorphism of $\mathcal{F}$-algebras $\phi : \langle \mathcal{S} \rangle \to \langle \mathcal{R} \rangle \mathcal{C} + \mathcal{C}$ such that

$$x^\alpha = \lambda^{-1}\, \overline{x^\phi} \in \overline{\mathcal{Q}} \quad \text{for all } x \in \mathcal{S}.$$

Moreover, if $\lambda = 1$, $(\mathcal{RC}) \cap \mathcal{C} = 0$ and one of the following conditions is fulfilled:

(a) *$[\mathcal{S}, \mathcal{S}] = \mathcal{S}$;*
(b) *$\mathcal{R} \subseteq \mathcal{K} \subseteq \mathcal{Q}$, where $\mathcal{K}$ is a $\mathcal{C}$-submodule of $\mathcal{Q}$ such that $xyz + zyx \in \mathcal{K}$ for all $x, y, z \in \mathcal{K}$ and $\{(\mathcal{K} \circ \mathcal{K}) + \mathcal{C}\} \cap \mathcal{K} = 0$,*

then identifying $\mathcal{R}$ and $\overline{\mathcal{R}}$ we have that $x^\alpha = x^\phi$ for all $x \in \mathcal{S}$ and $\langle \mathcal{S} \rangle^\phi = \langle \mathcal{R} \rangle$.

Proof. Clearly all the assumptions of Lemma 3.2 are satisfied. Let $(\mathcal{U}; \gamma)$ be as in Lemma 3.2 and let $\widehat{\mathcal{R}}$ be the preimage in $\mathcal{Q}$ of $\mathcal{U}^\gamma \subseteq \overline{\mathcal{Q}}$. Clearly $\widehat{\mathcal{R}}$ contains $\mathcal{R}$ and so it is 9-free by Theorem 2.3. It is now clear that all the assumptions of Proposition 3.4 are fulfilled (with $\mathcal{U}$, γ and $\widehat{\mathcal{R}}$ for $\mathcal{S}$, ρ and $\mathcal{R}$). Let $\phi : \langle \mathcal{U} \rangle \to \langle \widehat{\mathcal{R}} \rangle \mathcal{C} + \mathcal{C}$ be as in Proposition 3.4. Clearly $x^\alpha = \lambda^{-1}\, \overline{x^\phi} \in \overline{\mathcal{Q}}$ for all $x \in \mathcal{S}$.

Finally, assume that $\lambda = 1$ and $(\mathcal{RC}) \cap \mathcal{C} = 0$. Since $\mathcal{R} \cap \mathcal{C} = 0$, we may assume, without loss of generality, that $\alpha : \mathcal{S} \to \mathcal{R}$. We have $[\mathcal{R}, \mathcal{R}] = [\mathcal{S}^\alpha, \mathcal{S}^\alpha] = [\mathcal{S}, \mathcal{S}]^\alpha \subseteq \mathcal{S}^\alpha$ and so $\mathcal{R}$ is a Lie subalgebra of $\mathcal{Q}$. Setting $\eta(x) = x^\alpha - x^\phi$, $x \in \mathcal{S}$, we see that $\eta : \mathcal{S} \to \mathcal{C}$ is an $\mathcal{F}$-linear map. Clearly $x^\alpha = x^\phi + \eta(x)$, $x \in \mathcal{S}$. To complete the proof of the theorem it is enough to show that $\eta = 0$ because then $\mathcal{S}^\phi = \mathcal{S}^\alpha = \mathcal{R}$ and so $\langle \mathcal{S} \rangle^\phi = \langle \mathcal{R} \rangle$.

First, assume that the condition (a) is satisfied. Given $x, y \in \mathcal{S}$, we have

$$[x^\alpha, y^\alpha] = [x, y]^\alpha = [x, y]^\phi + \eta([x, y]) = [x^\phi, y^\phi] + \eta([x, y]) = [x^\alpha, y^\alpha] + \eta([x, y])$$

and so $\eta([x, y]) = 0$ for all $x, y \in \mathcal{S}$. Since $[\mathcal{S}, \mathcal{S}] = \mathcal{S}$, we get at once that $\eta = 0$.

Next, assume that the condition (b) is fulfilled. Given $x, y, z \in \mathcal{S}$, we have

$$(xyz + zyx)^\alpha = (xyz + zyx)^\phi + \eta(xyz + zyx) = x^\phi y^\phi z^\phi + z^\phi y^\phi x^\phi + \eta(xyz + zyx).$$

Since $(xyz + zyx)^\alpha, x^\alpha y^\alpha z^\alpha + z^\alpha y^\alpha x^\alpha \in \mathcal{K}$, substituting $x^\alpha - \eta(x)$, $y^\alpha - \eta(y)$ and $z^\alpha - \eta(z)$ for x^ϕ, y^ϕ and z^ϕ respectively, we get that

$$\begin{aligned} &\eta(x)(y^\alpha \circ z^\alpha) + \eta(y)(x^\alpha \circ z^\alpha) \\ &\quad + \eta(z)(x^\alpha \circ y^\alpha) + \sigma(x, y, z) \in \{(\mathcal{K} \circ \mathcal{K}) + \mathcal{C}\} \cap \mathcal{K} = 0 \end{aligned}$$

for all $x, y, z \in \mathcal{S}$, where $\sigma : \mathcal{S}^3 \to \mathcal{C}$ is some map. Lemma 2.6 now yields that $\eta = 0$ and the proof is thereby complete.

Theorems 1.1, 1.2, 1.3 and 1.4 will be obtained as corollaries to Theorem 3.5.

Proof of Theorem 1.1. By Theorem 2.2, $\mathcal{R}$ is a 9-free subset of $\mathcal{Q}$. Clearly $xyz + zyx \in \mathcal{K}(\mathcal{A})$ for all $x, y, z \in \mathcal{K}(\mathcal{A})$. Next, $\mathcal{K}(\mathcal{A}) \circ \mathcal{K}(\mathcal{A}) \subseteq \mathcal{S}(\mathcal{A})$ and so $(\mathcal{K}(\mathcal{A}) \circ \mathcal{K}(\mathcal{A})) \cap \mathcal{K}(\mathcal{A}) \subseteq \mathcal{S}(\mathcal{A}) \cap \mathcal{K}(\mathcal{A}) = 0$. Further, recall that $\mathcal{C}$ is a field and so $\mathcal{C}$ is a direct summand of the $\mathcal{C}$-module $\mathcal{Q}$. Finally, suppose that $*$ is an involution of the first kind. Let $\mathcal{H}$ be a symmetric ring of quotients of $\mathcal{A}$. It is well-known that the involution $*$ can be uniquely extended to $\mathcal{H}$ (see [18, Chapter 2]). Clearly $\mathcal{K}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{H}) = \mathcal{K}$, $\mathcal{C}\mathcal{K} \subseteq \mathcal{K}$ and $\mathcal{K} \cap (\mathcal{K} \circ \mathcal{K} + \mathcal{C}) \subseteq \mathcal{K} \cap \mathcal{S}(\mathcal{H}) = 0$. In particular, $\mathcal{K}(\mathcal{A})\mathcal{C} \cap \mathcal{C} \subseteq \mathcal{K} \cap \mathcal{C} = 0$. The result now follows from Theorem 3.5.

If $\mathcal{B}$ is a simple ring with $\deg(\mathcal{B}) = n < \infty$, then it satisfies the standard polynomial identity of degree $2n$ and so it is a finite dimensional algebra over $\mathcal{Z}(\mathcal{B})$ by Kaplansky's theorem. Moreover, $\dim_{\mathcal{Z}(\mathcal{B})}(\mathcal{B}) \leq n^2$.

Proof of Theorem 1.2. First assume that (c) is fulfilled. Since $\dim_{\mathcal{C}}(\mathcal{B}) > 400$, $\deg(\mathcal{B}) \geq 20$. Denote by π the canonical projection of Lie algebras $\mathcal{S} \to \overline{\mathcal{L}}$ and set $\beta = \pi\alpha$. It follows from Theorem 1.1 that there exists a surjective homomorphism $\phi : \langle \mathcal{S} \rangle \to \langle \mathcal{R} \rangle$ of algebras such that $x^\beta = \overline{x^\phi}$ for all $x \in \mathcal{S}$. That is to say, $\overline{x}^{\,\alpha} = \overline{x^\phi}$ for all $x \in \mathcal{S}$. Since $\dim_{\mathcal{C}}(\mathcal{B}) > 400$, $\langle \mathcal{R} \rangle = \mathcal{B}$ by [39, Theorem 2.13]. Clearly $\mathcal{B}$ does not satisfy St_4, the standard identity of degree 4. Therefore $\langle \mathcal{S} \rangle$ (and so $\mathcal{A}$) does not satisfy St_4. It follows from Kaplansky's theorem that $\dim_{\mathcal{T}}(\mathcal{A}) > 4$. But then [39, Theorem 2.13] tells us that $\langle \mathcal{S} \rangle = \mathcal{A}$. As $\mathcal{A}$ is a simple algebra and ϕ is a surjective map, we conclude that ϕ is an isomorphism.

Next, assume that (a) is satisfied. By the above result we may also suppose that $\dim_{\mathcal{C}}(\mathcal{B}) \leq 400$. Replacing α with α^{-1} we reduce the proof to the case $\dim_{\mathcal{T}}(\mathcal{A}) \leq 400$. It is well-known that $\mathcal{S} = \mathcal{L}$ and $\mathcal{R} = \mathcal{K}$ (see, for example [38, p. 529]). If $\mathrm{char}(\mathcal{F}) \neq 3$, the result follows from [17, Corollary]. The case $\mathrm{char}(\mathcal{F}) = 3$ follows from [35, Theorem 1.1].

Finally, suppose that (b) is fulfilled. Then we again may suppose that $\dim_{\mathcal{C}}(\mathcal{B}) \leq 400$ and so the result follows from [25, Theorem 1.1]. The proof is now complete.

Proof of Theorem 1.3. Let $\phi \in \mathrm{Aut}_{\mathcal{F}}(\mathcal{B})$ with $(x^*)^\phi = (x^\phi)^*$ for all $x \in \mathcal{B}$. Clearly $\mathcal{K}(\mathcal{B})^\phi = \mathcal{K}(\mathcal{B})$ and so $\mathcal{R}^\phi = \mathcal{R}$. If $\phi|_{\mathcal{R}} = \mathrm{id}_{\mathcal{R}}$, then $\phi = \mathrm{id}_{\mathcal{B}}$ because $\mathcal{B} = \langle \mathcal{R} \rangle$ by [39, Theorem 2.13]. Therefore

$$\mathrm{Aut}_{\mathcal{F}}(\mathcal{R}) \supseteq \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(\mathcal{B}) \mid (x^*)^\alpha = (x^\alpha)^* \quad \text{for all } x \in \mathcal{B}\}.$$

Further, suppose that $\psi \in \mathrm{Aut}_{\mathcal{F}}(\mathcal{R})$ induces an identical automorphism on $\overline{\mathcal{R}}$. Then $x^\psi - x \in \mathcal{Z}(\mathcal{B})$ for all $x \in \mathcal{R}$. Therefore

$$[x, y]^\psi = [x^\psi, y^\psi] = [x, y] \quad \text{for all } x, y \in \mathcal{R}. \tag{31}$$

If $\dim_{\mathcal{C}}(\mathcal{B}) > 16$, then [39, Theorem 2.15] implies that $[\mathcal{R}, \mathcal{R}] = \mathcal{R}$. In this case (31) implies that $x^\phi = x$ for all $x \in \mathcal{R}$ forcing $\psi = \mathrm{id}_{\mathcal{R}}$. Now assume that $\dim_{\mathcal{C}}(\mathcal{B}) \leq 16$. Then it follows from our assumption that the involution is of the second kind and either $\dim_{\mathcal{C}}(\mathcal{B}) = 9, 16$ and $\mathrm{char}(\mathcal{F}) \neq 3$, or $\dim_{\mathcal{C}}(\mathcal{B}) = 16$. In both cases $[\mathcal{B}, \mathcal{B}]$ is a simple Lie algebra. Let $\overline{\mathcal{C}}$ be the algebraic closure of $\mathcal{C}$ and let $\mathcal{T} = \{c \in \mathcal{C} \mid c^* = c\}$. It is well-known that $\mathcal{K}(\mathcal{B}) \otimes_{\mathcal{T}} \overline{\mathcal{C}} \cong \mathcal{B} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ as Lie algebras and so $\mathcal{R} = [\mathcal{K}(\mathcal{B}), \mathcal{K}(\mathcal{B})]$ is a simple Lie algebra. Therefore (31) implies that $\psi = \mathrm{id}_{\mathcal{R}}$ in this case as well. We conclude that canonical homomorphisms $\mathcal{G} \to \mathrm{Aut}_{\mathcal{F}}(\mathcal{R})$ and $\mathrm{Aut}_{\mathcal{F}}(\mathcal{R}) \to \mathrm{Aut}_{\mathcal{F}}(\overline{\mathcal{R}})$ are monomorphisms.

Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(\overline{\mathcal{R}})$. By Theorem 1.2 there exists an automorphism ϕ of the algebra $\mathcal{B}$ such that

$$\overline{x}^{\,\alpha} = \overline{x^{\phi}} \quad \text{for all } x \in \mathcal{R}.$$

Therefore $\mathcal{R}^{\phi} \subseteq \mathcal{R} + \mathcal{Z}(\mathcal{B})$. As $[\mathcal{R}, \mathcal{R}] = \mathcal{R}$ according to the above discussion, we conclude that

$$\mathcal{R}^{\phi} = [\mathcal{R}^{\phi}, \mathcal{R}^{\phi}] \subseteq [\mathcal{R} + \mathcal{Z}(\mathcal{B}), \mathcal{R} + \mathcal{Z}(\mathcal{B})] = \mathcal{R} \subseteq \mathcal{K}(\mathcal{B}).$$

Hence $(xyz + zyx)^{\phi} \in \mathcal{K}(\mathcal{B})$ for all $x, y, z \in \mathcal{R}$. Let $\mathcal{U}$ be an $\mathcal{F}$-submodule generated by $\{x,\ xyz + zyx \mid x, y, z \in \mathcal{R}\}$. As we note at the beginning of the proof of Lemma 3.2, $\mathcal{U}$ is a Lie ideal of $\mathcal{K}(\mathcal{B})$ containing $\mathcal{R}$. Clearly $\mathcal{U}^{\phi} \subseteq \mathcal{K}(\mathcal{B})$. Continuing in this way and applying Zorn's lemma we get that there exists a Lie ideal $\mathcal{V}$ of $\mathcal{K}(\mathcal{B})$ containing $\mathcal{R}$ such that $xyz + zyx \in \mathcal{V}$ for all $x, y, z \in \mathcal{V}$ and $\mathcal{V}^{\phi} \subseteq \mathcal{K}(\mathcal{B})$. Therefore $(\mathcal{V} \circ \mathcal{V})^{\phi} \subseteq \mathcal{S}(\mathcal{B})$. By Lemma 3.3, $\langle \mathcal{V} \rangle = \mathcal{V} + \mathcal{V} \circ \mathcal{V}$. As $\langle \mathcal{R} \rangle = \mathcal{B}$ and $\mathcal{R} \subseteq \mathcal{V}$, we conclude that $\mathcal{B} = \mathcal{V} + \mathcal{V} \circ \mathcal{V}$ and so $\mathcal{V} = \mathcal{K}(\mathcal{B})$ and $\mathcal{V} \circ \mathcal{V} = \mathcal{S}(\mathcal{B})$. It follows that $\mathcal{K}(\mathcal{B})^{\phi} \subseteq \mathcal{K}(\mathcal{B})$ and $\mathcal{S}(\mathcal{B})^{\phi} \subseteq \mathcal{S}(\mathcal{B})$. Since ϕ is an automorphism, we have $\mathcal{K}(\mathcal{B})^{\phi} = \mathcal{K}(\mathcal{B})$ and $\mathcal{S}(\mathcal{B})^{\phi} = \mathcal{S}(\mathcal{B})$. One can now easily check that $(x^*)^{\phi} = (x^{\phi})^*$ for all $x \in \mathcal{B}$. The proof is complete.

Proof of Theorem 1.4. Suppose to the contrary that $\overline{\mathcal{R}}$ is a homomorphic image of a Lie ideal $\mathcal{U}$ of a ring $\mathcal{H}$. Let $\alpha : \mathcal{U} \to \overline{\mathcal{R}}$ be a surjective Lie map. Localizing both rings $\mathcal{H}$ and $\mathcal{B}$ by the multiplicatively closed subset $S = \{1, 2, 4, \dots, 2^n, \dots\}$ of the ring of integers $\mathcal{Z}$, replacing $\mathcal{H}$, $\mathcal{U}$, $\mathcal{B}$ and $\mathcal{R}$ by $S^{-1}\mathcal{H}$, $S^{-1}\mathcal{U}$, $S^{-1}\mathcal{B}$ and $S^{-1}\mathcal{R}$ respectively and setting $\mathcal{F} = S^{-1}\mathcal{Z}$, we reduce the proof to the case when both $\mathcal{H}$ and $\mathcal{B}$ are algebras over $\mathcal{F}$, $\mathcal{U}$ is a Lie ideal of the algebra $\mathcal{H}$, $\mathcal{R}$ is a noncentral Lie ideal of $\mathcal{K}(\mathcal{B})$ and $\frac{1}{2} \in \mathcal{F}$. Next, we set $\mathcal{A} = \mathcal{H} \oplus \mathcal{H}^{op}$, where $\mathcal{H}^{op}$ is the opposite algebra of $\mathcal{H}$, and $\mathcal{S} = \{(x, -x) \mid x \in \mathcal{U}\}$. Letting $\#$ denote the exchange involution on $\mathcal{A}$ (i.e., $(h_1, h_2)^{\#} = (h_2, h_1)$, $h_1, h_2 \in \mathcal{H}$), we see that $\mathcal{S}$ is a Lie ideal of $\mathcal{K}(\mathcal{A})$. We define a map $\beta : \mathcal{S} \to \overline{\mathcal{R}}$ by the rule $(x, -x)^{\beta} = x^{\alpha}$, $x \in \mathcal{U}$. Clearly β is a surjective Lie map. Since all the conditions of Theorem 1.1 are now met, we conclude that there exists a map $\phi : \langle \mathcal{S} \rangle \to \langle \mathcal{R} \rangle \mathcal{C} + \mathcal{C}$ of $\mathcal{F}$-algebras such that $s^{\beta} = s^{\phi} + \mathcal{C} \in \mathcal{Q}_{mr}(\mathcal{B})/\mathcal{C}$ for all $s \in \mathcal{S}$. It now follows that $\mathcal{R} \subseteq \mathcal{S}^{\phi} + \mathcal{C}$ and $\mathcal{S}^{\phi} \subseteq \mathcal{R} + \mathcal{C}$ because $\overline{\mathcal{R}} = \mathcal{S}^{\beta}$. Consequently, $[\mathcal{R}, \mathcal{R}] \subseteq [\mathcal{S}^{\phi}, \mathcal{S}^{\phi}]$ and $[\mathcal{S}^{\phi}, \mathcal{S}^{\phi}] \subseteq [\mathcal{R}, \mathcal{R}]$. Therefore

$$[\mathcal{R}, \mathcal{R}] = [\mathcal{S}^{\phi}, \mathcal{S}^{\phi}] \subseteq \mathcal{S}^{\phi}. \tag{32}$$

Since $\mathcal{R}$ is a noncentral Lie ideal of $\mathcal{K}(\mathcal{B})$, $[\mathcal{R}, \mathcal{R}]$ is a nonzero Lie ideal of $\mathcal{K}(\mathcal{B})$. As $\deg(\mathcal{B}) > 20$, [59, Theorems 3.4 and 5.6] implies that $\langle [\mathcal{R}, \mathcal{R}] \rangle$ contains a nonzero ideal $\mathcal{V}$ of the algebra $\mathcal{B}$. Since the subalgebra $\langle [\mathcal{R}, \mathcal{R}] \rangle$ is $*$-invariant, we may assume that $\mathcal{V}^* = \mathcal{V}$. Clearly

$$\mathcal{V} \subseteq \langle [\mathcal{R}, \mathcal{R}] \rangle \subseteq \langle \mathcal{S} \rangle^{\phi} \subseteq \mathcal{Q}_s(\mathcal{B}) = \mathcal{Q}_s(\mathcal{V}) \tag{33}$$

where $\mathcal{Q}_s(\mathcal{B})$ is the symmetric ring of quotients of the algebra $\mathcal{B}$ (see [18, Chapter 2]). It now follows at once that the algebra $\langle \mathcal{S} \rangle^{\phi}$ is prime.

Set $\mathcal{W}_1 = \langle \mathcal{S} \rangle \cap (\mathcal{H}, 0)$ and $\mathcal{W}_2 = \mathcal{W}_1^{\#}$. Clearly both $\mathcal{W}_1$ and $\mathcal{W}_2$ are ideals of the algebra $\langle \mathcal{S} \rangle$ and $\mathcal{W}_1 \mathcal{W}_2 = 0$. Since $\langle \mathcal{S} \rangle^{\phi}$ is prime, either $\mathcal{W}_1^{\phi} = 0$ or $\mathcal{W}_2^{\phi} = 0$. Say $\mathcal{W}_2^{\phi} = 0$. Replacing $\mathcal{H}$ by $\langle \mathcal{U} \rangle$ we may assume that $\mathcal{H} = \langle \mathcal{U} \rangle$ and so the canonical projection $\pi : \mathcal{A} \to (\mathcal{H}, 0)$ maps $\langle \mathcal{S} \rangle$ onto $(\mathcal{H}, 0)$. Clearly $\mathcal{W}_2$ is the kernel of π restricted to $\langle \mathcal{S} \rangle$. Therefore $\langle \mathcal{S} \rangle / W_2 \cong (\mathcal{H}, 0)$. As $\mathcal{U}$ is a Lie ideal of $\mathcal{H}$ and $\mathcal{S}^{\pi} = (\mathcal{U}, 0)$, we conclude that $(\mathcal{S} + \mathcal{W}_2)/\mathcal{W}_2$ is a Lie ideal of $\langle \mathcal{S} \rangle / W_2$. Recalling that $\mathcal{W}_2^\phi = 0$, we see that $\mathcal{S}^\phi$ is a Lie ideal of $\langle\mathcal{S}\rangle^\phi$. Taking into account (32), we conclude that $[\mathcal{R},\mathcal{R}] = [\mathcal{S}^\phi,\mathcal{S}^\phi]$ is a Lie ideal of the algebra $\langle\mathcal{S}\rangle^\phi$. In particular, $[[\mathcal{R},\mathcal{R}],\mathcal{V}] \subseteq [\mathcal{R},\mathcal{R}]$ by (33). As $\mathcal{R} \subseteq \mathcal{K}(\mathcal{B})$ and $[[\mathcal{R},\mathcal{R}],\mathcal{S}(\mathcal{V})] \subseteq \mathcal{S}(\mathcal{B})$, we conclude that $[[\mathcal{R},\mathcal{R}],\mathcal{S}(\mathcal{V})] = 0$. Recalling that $\mathcal{V} \subseteq \langle[\mathcal{R},\mathcal{R}]\rangle$, we get that $\mathcal{S}(\mathcal{V}) \subseteq \mathcal{Z}(\mathcal{V})$ and whence $\deg(\mathcal{V}) \leq 2$ (see [39, 40]). It follows at once that $\deg(\mathcal{B}) \leq 2$ (see, for example [1]), a contradiction. The proof is thereby complete.

4. Polynomial preservers

We first set some further notation in place. In what follows, $\mathcal{Q}$ is a ring with 1 and with center $\mathcal{C}$, $\mathcal{F}$ is a subring of $\mathcal{C}$ containing 1, $\mathcal{X} = \{x_1, x_2, \dots, x_m, \dots\}$ is an infinite set and $\mathcal{F}\langle\mathcal{X}\rangle$ is the free $\mathcal{F}$-algebra on $\mathcal{X}$ with unity. Let $d : \mathcal{B} \to \mathcal{B}$ be a derivation of an $\mathcal{F}$-algebra $\mathcal{B}$ and let $f(\overline{x}_m) \in \mathcal{F}\langle\mathcal{X}\rangle$ be a multilinear polynomial in $x_1, x_2, \dots, x_m$. Clearly

$$f(\overline{a}_m)^d = \sum_{i=1}^m f(a_1, \dots, a_{i-1}, a_i^d, a_{i+1}, \dots, a_m)$$

for all $\overline{a}_m \in \mathcal{B}^m$. In particular,

$$[f(\overline{a}_m), b] = \sum_{i=1}^m f(a_1, \dots, a_{i-1}, [a_i, b], a_{i+1}, \dots, a_m) \tag{34}$$

for all $\overline{a}_m \in \mathcal{B}^m$, $b \in \mathcal{B}$. Given $1 \leq i \leq m$, we set

$$f_{x_i} = f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_m).$$

If $\mathcal{G} \subseteq \mathcal{F}\langle\mathcal{X}\rangle \subseteq \mathcal{C}\langle\mathcal{X}\rangle$, we put $\mathrm{Ann}_\mathcal{C}(\mathcal{G}) = \{c \in \mathcal{C} \mid cg = 0 \quad \text{for all} \quad g \in \mathcal{G}\}$. Finally, a nonzero polynomial $g \in \mathcal{F}\langle\mathcal{X}\rangle$ is called *special* if all its nonzero coefficients are invertible in $\mathcal{C}$.

Taking $B(u, v)$ in [11, Theorem 2.9] to be equal to $[u, v]^\alpha$ and making use of (34), we immediately get the following result.

Theorem 4.1. *Let $f(\overline{x}_m) \in \mathcal{F}\langle\mathcal{X}\rangle$, $m \geq 2$, be a special multilinear polynomial. Further, let $\mathcal{S}$ be a subset of an $\mathcal{F}$-algebra $\mathcal{A}$ such that $f(\overline{s}_m) \in \mathcal{S}$ for all $\overline{s}_m \in \mathcal{S}^m$, and let $\alpha : \mathcal{S} \to \mathcal{Q}$ be an $\mathcal{F}$-module map such that*

$$f(s_1, s_2, \dots, s_m)^\alpha = f(s_1^\alpha, s_2^\alpha, \dots, s_m^\alpha) \text{ for all } s_1, \dots, s_m \in \mathcal{S}.$$

If $\mathcal{R} = \mathcal{S}^\alpha$ is a $(2m)$-free subset of an $\mathcal{F}$-algebra $\mathcal{Q}$, then there exist $\lambda \in \mathcal{C}$ and an $\mathcal{F}$-bilinear map $\epsilon : \mathcal{S} \times \mathcal{S} \to \mathcal{C}$ such that $[u, v]^\alpha = \lambda[u^\alpha, v^\alpha] + \epsilon(u, v)$ for all $u, v \in \mathcal{S}$. Furthermore, $\mathrm{Im}(\epsilon) \subseteq \mathrm{Ann}_\mathcal{C}(\{f_{x_i} \mid 1 \leq i \leq m\})$.

Proof of Theorem 1.5. Since $\deg(\mathcal{B}) > \max\{4m + 2, 20\}$, Theorem 2.2 tells us that $\mathcal{R}$ is a 9-free as well as a $2m$-free subset of $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{B})$. Therefore all the assumptions of Theorem 4.1 are fulfilled and so there exist $\lambda \in \mathcal{C}$ and a bilinear map $\epsilon : \mathcal{S} \to \mathcal{C}$ such that $[x, y]^\alpha = \lambda[x^\alpha, y^\alpha] + \epsilon(x, y)$ for all $x, y \in \mathcal{S}$. We claim that $\lambda \neq 0$. Assume to the contrary that $\lambda = 0$. Then $[\mathcal{S},\mathcal{S}]^\alpha \subseteq \mathcal{C}$. First, assume that the condition (a) is satisfied. We now get at once that $f(s_1^\alpha, s_2^\alpha, \dots, s_m^\alpha) \in \mathcal{C}$ for all $\overline{s}_m \in \mathcal{S}$. Since $\mathcal{S}^\alpha = \mathcal{R}$, we conclude that $[f(\overline{x}_m), x_{m+1}]$ is a polynomial identity on $\mathcal{R}$, contradicting Remark 2.4.

Next, assume that (b) is fulfilled. By Theorem 2.2, $\mathcal{S}$ is both an m-free subset of $\mathcal{Q}_{mr}(\mathcal{A})$ and a 3-free subset. In particular, $[[\mathcal{S},\mathcal{S}],\mathcal{S}] \neq 0$ by Remark 2.4. Since $[x, y]^\alpha = \epsilon(x, y) \in \mathcal{C}$ for all $x, y \in \mathcal{S}$, we conclude that $\epsilon \neq 0$ because α is bijective.

It follows at once from Theorem 4.1 that each $f_{x_i} = 0$. Take $\overline{u}_m \in \mathcal{S}^m$ such that $u_1 \in [\mathcal{S}, \mathcal{S}]$. Then $u_1^\alpha \in \mathcal{C}$ and so

$$f(\overline{u}_m)^\alpha = f(u_1^\alpha, u_2^\alpha, \ldots, u_m^\alpha) = u_1^\alpha f_{x_1}(u_2^\alpha, u_3^\alpha, \ldots, u_m^\alpha) = 0. \tag{35}$$

Since α is bijective, we conclude that $f(\overline{u}_m) = 0$ for all $u_2, u_3, \ldots, u_m \in \mathcal{S}$ and so u_1 belongs to the extended centroid of $\mathcal{A}$ by Remark 2.5. Therefore $[[\mathcal{S}, \mathcal{S}], \mathcal{S}] = 0$, a contradiction.

Finally, assume that (c) is satisfied. Then $\mathcal{S}^\alpha = [\mathcal{S}, \mathcal{S}]^\alpha \subseteq \mathcal{C}$. Again we conclude that $\epsilon \neq 0$ and so each $f_{x_i} = 0$. Therefore (35) together with surjectivity of α yields that $f(\overline{x}_m)$ is a polynomial identity on $\mathcal{R}$, contradicting Remark 2.4.

Thus $\lambda \neq 0$ and so it is invertible in $\mathcal{C}$ because $\mathcal{C}$ is a field. We now see that the composition of α with the canonical projection $\mathcal{Q} \mapsto \mathcal{Q}/\mathcal{C}$ satisfies all the assumptions of Theorem 3.5 and so there exists a homomorphism of $\mathcal{F}$-algebras $\beta : \langle \mathcal{S} \rangle \to \langle \mathcal{R} \rangle \mathcal{C} + \mathcal{C}$ such that

$$x^\alpha - \zeta x^\beta \in \mathcal{C} \quad \text{for all } x \in \mathcal{S}$$

where $\zeta = \lambda^{-1}$. Setting $\mu(x) = x^\alpha - \zeta x^\beta$, $x \in \mathcal{S}$, we see that $\mu : \mathcal{S} \to \mathcal{C}$ is an $\mathcal{F}$-linear map such that $x^\alpha = \zeta x^\beta + \mu(x)$.

We claim that $\mathcal{S}^\beta$ is an $2m$-free subset of $\mathcal{Q}$. Indeed,

$$\mathcal{S}^\beta \supseteq [\mathcal{S}, \mathcal{S}]^\beta = [\mathcal{S}^\beta, \mathcal{S}^\beta] = \zeta^2 [\mathcal{S}^\alpha, \mathcal{S}^\alpha] = \zeta^2 [\mathcal{R}, \mathcal{R}].$$

Since $\mathcal{R}$ is 9-free, $[[\mathcal{R}, \mathcal{R}], \mathcal{R}] \neq 0$ by Remark 2.4 and so $[\mathcal{R}, \mathcal{R}]$ is a noncentral Lie ideal of $\mathcal{K}(\mathcal{B})$. Therefore $[\mathcal{R}, \mathcal{R}]$ is m-free by Theorem 2.2 and now Theorem 2.3 implies that $\mathcal{S}^\beta$ is m-free as well.

We now have that

$$\begin{aligned} &\zeta f(\overline{s}_m)^\beta + \mu(f(\overline{s}_m)) \\ &\quad = f(\overline{s}_m)^\alpha = f(s_1^\alpha, \ldots, s_m^\alpha) = f(\zeta s_1^\beta + \mu(s_1), \ldots, \zeta s_m^\beta + \mu(s_m)) \\ &\quad = \zeta^m f(\overline{s}_m)^\beta + \sum_{k=1}^{m} \sum_{1 \leq i_1 < \ldots < i_k \leq m} \zeta^{m-k} \mu(s_{i_1}) \ldots \mu(s_{i_k}) f_{x_{i_1} \ldots x_{i_k}}(\overline{s}_m^\beta) \end{aligned} \tag{36}$$

for all $\overline{s}_m \in \mathcal{S}$. Recalling that $\mathcal{S}^\beta$ is m-free, we conclude from Lemma 2.6 together with (36) that in particular,

$$\begin{aligned} \zeta f(\overline{x}_m) &= \zeta^m f(\overline{x}_m) \in \mathcal{F}\langle \mathcal{X} \rangle, \\ \mu(s_i) f_{x_i}(\overline{s}_m^\beta) &= 0, \quad i = 1, 2, \ldots, m, \ \overline{s}_m \in \mathcal{S}^m, \\ \mu(f(\overline{s}_m)) &= 0, \quad \overline{s}_m \in \mathcal{S}^m, \end{aligned}$$

and so $\zeta^{m-1} = 1$. Since $\mathcal{S}^\beta$ does not satisfy any nonzero polynomial identities of degree $\leq 2m$ by Remark 2.4, we see that either each $f_{x_i} = 0$ or $\mu = 0$. The proof is complete.

Proof of Theorem 1.6. Linearizing x^m, we get

$$\left(\sum_{\sigma \in S_m} x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(m)} \right)^\alpha = \sum_{\sigma \in S_m} x_{\sigma(1)}^\alpha x_{\sigma(2)}^\alpha \ldots x_{\sigma(m)}^\alpha$$

for all $\overline{x}_n \in \mathcal{A}^m$, where S_m is the symmetric group of degree m. By Theorem 1.5 there exist a homomorphism of $\mathcal{F}$-algebras $\beta : \langle \mathcal{L} \rangle \to \langle \mathcal{K} \rangle$, an $\mathcal{F}$-linear map $\mu : \langle \mathcal{L} \rangle \to \mathcal{C}$ and an element $\zeta \in \mathcal{C}$ such that $\zeta^{m-1} = 1$ and

$$x^\alpha = \zeta x^\beta + \mu(x) \quad \text{for all } x \in \mathcal{L}.$$

Set $f(\overline{x}_m) = \sum_{\sigma\in S_m} x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(m)}$. If $f_{x_1} \neq 0$, then $\mu = 0$ by Theorem 1.5. If $f_{x_1} = 0$, then $\mathrm{char}(\mathcal{B}) = p > 0$ and p divides m. Write $m = p^k n$ where p does not divide n. Suppose that $n > 1$. Then we have

$$\begin{aligned}\zeta(x^\beta)^m + \mu(x^m) = (x^m)^\alpha = (x^\alpha)^m = [\zeta x^\beta + \mu(x)]^m = \{[\zeta x^\beta + \mu(x)]^{p^k}\}^n \\ = \sum_{i=0}^{n}\binom{n}{i}\zeta^{ip^k}(x^\beta)^{ip^k}\mu(x)^{(n-i)p^k} \qquad (37)\end{aligned}$$

for all $x \in \mathcal{L}$. Since $x^\alpha = \zeta x^\beta + \mu(x)$ and $\zeta \neq 0$, $\deg(x^\alpha) = \deg(x^\beta)$ for all $x \in \mathcal{L}$. Recall that $\mathcal{L}^\alpha = \mathcal{K}$. By [9, Lemma 2.1] $\deg(\mathcal{B}) = \deg(\mathcal{K})$ and so $\mathcal{L}$ contains an element x with $\deg(x^\beta) = \deg(x^\alpha) > m$. It now follows from (37) that $\mu = 0$. The proof is now complete.

References

[1] S.A. Amitsur, Identities in rings with involutions, *Israel J. Math.* **7** (1969), 63–68. MR **39:**4216

[2] S.A. Ayupov, Anti–automorphisms of factors and Lie operator algebras, *Quart. J. Math. Oxford* **46** (1995), 129–140. MR **94j:**46058

[3] S.A. Ayupov, Skew commutators and Lie isomorphisms in real von Neumann algebras, *J. Funct. Anal.* **138** (1996), 170–187. MR **97h:**46110

[4] S.A. Ayupov and N.A. Azamov, Commutators and Lie isomorphisms of skew elements in prime operator algebras, *Comm. Algebra* **24** (1996), 1501–1520. MR **97a:**46096

[5] S.A. Ayupov, A. Rakhimov and S. Usmanov, *Jordan, Real and Lie Structures in Operator Algebras*, Kluwer Academic Publishers, Dordrecht–Boston–London, 1997. MR **99g:**46100

[6] R. Banning and M. Mathieu, Commutativity preserving mappings on semiprime rings, *Comm. Algebra* **25** (1997), 247–265. MR **97j:**16033

[7] K.I. Beidar, On functional identities and commuting additive mappings, *Comm. Algebra* **26** (1998), 1819–1850. MR **99f:**16023

[8] K.I. Beidar, M. Brešar and M.A. Chebotar, Generalized functional identities with (anti)automorphisms and derivations on prime rings, I, *J. Algebra* **215** (1999), 644–665. MR **2000d:**16033

[9] K.I. Beidar, M. Brešar, M.A. Chebotar and W.S. Martindale III, On functional identities in prime rings with involution, II, *Comm. Algebra*, **28** (2000), 3169–3183. CMP 2000:14

[10] K.I. Beidar and M.A. Chebotar, On functional identities and d-free subsets of rings, *Comm. Algebra* **28** (2000), 3925–3951. CMP 2000:15

[11] K.I. Beidar and M.A. Chebotar, On functional identities and d-free subsets of rings II, *Comm. Algebra* **28** (2000), 3953–3972. CMP 2000:15

[12] K.I. Beidar and M.A. Chebotar, On surjective Lie homomorphisms onto Lie ideals of prime rings, submitted.

[13] K.I. Beidar and M.A. Chebotar, On additive maps onto Lie ideals of prime rings preserving a polynomial, submitted.

[14] K.I. Beidar and Y. Fong, On additive isomorphisms of prime rings preserving polynomials, *J. Algebra* **217** (1999), 650–667. MR **2000k:**16025

[15] K.I. Beidar, Y. Fong, P.-H. Lee and T.-L. Wong, On additive maps of prime rings satisfying the Engel condition, *Comm. Algebra* **25** (1997), 3889–3902. MR **98i:**16022

[16] K.I. Beidar and W.S. Martindale III, On functional identities in prime rings with involution, *J. Algebra* **203** (1998), 491–532. MR **99f:**16024

[17] K.I. Beidar, W.S. Martindale III and A.V. Mikhalev, Lie isomorphisms in prime rings with involution, *J. Algebra* **169** (1994), 304–327. MR **95m:**16021

[18] K.I. Beidar, W.S. Martindale III and A.V. Mikhalev, *Rings with Generalized Identities*, Marcel Dekker, Inc., 1996. MR **97g:**16035

[19] M.I. Berenguer and A.R. Villena, Continuity of Lie derivations on Banach algebras, *Proc. Edinburgh Math. Soc.* **41** (1998), 625–630. MR **2000i:**46040

[20] M.I. Berenguer and A.R. Villena, Continuity of Lie mappings of the skew elements of Banach algebras with involution, *Proc. Amer. Math. Soc.* **126** (1998), 2717–2720. MR **98k:**46083

[21] M.I. Berenguer and A.R. Villena, Continuity of Lie isomorphisms of Banach algebras, *Bull. London Math. Soc.* **31** (1999), 6–10. MR **2000a:**46071

[22] P.S. Blau, *Lie Isomorphisms of Prime Rings Satisfying* S_4, PhD Thesis, Univ. of Massachusetts, 1996.

[23] P.S. Blau, Lie isomorphisms of non-GPI rings with involution, *Comm. Algebra* **27** (1999), 2345–2373. MR **2000d:**16049

[24] P.S. Blau, Lie isomorphisms of prime rings satisfying S_4, *Southeast Asian Math. Bull.*

[25] P.S. Blau and W.S. Martindale III, Lie isomorphisms in *-prime GPI rings with involution, *Taiwanese J. Math.* **4** (2000), 215–252. CMP 2000:13

[26] M. Brešar, Centralizing mappings and derivations in prime rings, *J. Algebra* **156** (1993), 385–394. MR **94f:**16042

[27] M. Brešar, Commuting traces of biadditive mappings, commutativity preserving mappings, and Lie mappings, *Trans. Amer. Math. Soc.* **335** (1993), 525–546. MR **93d:**16044

[28] M. Brešar, On generalized biderivations and related maps, *J. Algebra* **172** (1995), 764–786. MR **96c:**16046

[29] M. Brešar, Functional identities of degree two, *J. Algebra* **172** (1995), 690–720. MR **96k:**16022

[30] M. Brešar and M.A. Chebotar, On a certain functional identity in prime rings, *Comm. Algebra* **26** (1998), 3765–3782. MR **99h:**16037

[31] M. Brešar, W.S. Martindale III and C.R. Miers, Centralizing maps in prime rings with involution, *J. Algebra* **161** (1993), 342–357. MR **94j:**16021

[32] M. Brešar, W.S. Martindale III and C.R. Miers, Maps preserving n^{th} powers, *Comm. Algebra* **26** (1998), 117–138. MR **98k:**16035

[33] M.A. Chebotar, On Lie automorphisms of simple rings of characteristic 2, *Fundamentalnaya i Prikladnaya Matematika* **2** (1996), 1257–1268 (Russian).

[34] M.A. Chebotar, On generalized functional identities in prime rings, *J. Algebra* **202** (1998), 655–670. MR **99e:**16028

[35] M.A. Chebotar, On Lie isomorphisms in prime rings with involution, *Comm. Algebra* **27** (1999), 2767–2777. MR **2000c:**16028

[36] P. de la Harpe, *Classical Banach–Lie Algebras and Banach–Lie Groups of Operators in Hilbert Spaces*, Lecture Notes Math. 285, Springer–Verlag, 1972. MR **57:**16372

[37] I.N. Herstein, Jordan homomorphisms, *Trans. Amer. Math. Soc.* **81** (1956), 331–351. MR **17:**938f

[38] I.N. Herstein, Lie and Jordan structures in simple, associative rings, *Bull. Amer. Math. Soc.* **67** (1961), 517–531. MR **25:**3072

[39] I.N. Herstein, *Topics in Ring Theory*, The University of Chicago Press, 1969. MR **42:**6018

[40] I.N. Herstein, *Rings with Involution*, The University of Chicago Press, 1976. MR **56:**406

[41] R.A. Howland, Lie isomorphisms of derived rings of simple rings, *Trans. Amer. Math. Soc.* **145** (1969), 383–396. MR **40:**5661

[42] L.K. Hua, A theorem on matrices over an sfield and its applications, *J. Chinese Math. Soc. (N.S.)* **1** (1951), 110–163. MR **17:**123a

[43] N. Jacobson, *Lie Algebras*, Interscience, New York, 1962. MR **26:**1345

[44] N. Jacobson, Structure groups and Lie algebras of Jordan algebras of symmetric elements of associative algebras with involution, *Adv. Math.* **20** (1976), 106–150. MR **53:**10886

[45] N. Jacobson and C.E. Rickart, Jordan homomorphisms of rings, *Trans. Amer. Math. Soc.* **69** (1950), 479–502. MR **12:**387h

[46] C. Jiang and D. Meng, The derivation algebra of the associative algebra $C_q[X, Y, X^{-1}, Y^1]$, *Comm. Algebra* **26** (1998), 1723–1736. MR **99c:**16032

[47] I. Kaplansky, Semi-automorphisms of rings, *Duke Math. J.* **14** (1947), 521–527. MR **9:**172e

[48] P.-H. Lee, An example of division rings with involution, *J. Algebra* **74** (1982), 282–283. MR **83f:**16017

[49] P.-H. Lee, J.-S. Lin, R.-J. Wang and T.-L. Wong, Commuting traces of multiadditive mappings, *J. Algebra* **193** (1997), 709–723. MR **98e:**16020

[50] W.S. Martindale III, Lie isomorphisms of primitive rings, *Proc. Amer. Math. Soc.* **14** (1963), 909–916. MR **28:**4008

[51] W.S. Martindale III, Lie derivations of primitive rings, *Michigan J. Math.* **11** (1964), 183–187. MR **29:**3511

[52] W.S. Martindale III, Jordan homomorphisms of the symmetric elements of a ring with involution, *J. Algebra* **5** (1967), 232–249. MR **35:**1636
[53] W.S. Martindale III, Lie isomorphisms of simple rings, *J. London Math. Soc.* **44** (1969), 213–221. MR **38:**2166
[54] W.S. Martindale III, Lie isomorphisms of prime rings, *Trans. Amer. Math. Soc.* **142** (1969), 437–455. MR **40:**4308
[55] W.S. Martindale III, Lie isomorphisms of the skew elements of a simple ring with involution, *J. Algebra* **36** (1975), 408–415. MR **52:**3218
[56] W.S. Martindale III, Lie isomorphisms of the skew elements of a prime ring with involution, *Comm. Algebra* **4** (1976), 927–977. MR **54:**10344
[57] W.S. Martindale III, Lie and Jordan mappings, *Contemp. Math.* **13** (1982), 173–177.
[58] W.S. Martindale III, Jordan homomorphisms onto nondegenerate Jordan algebras, *J. Algebra* **133** (1990), 500–511. MR **91m:**17045
[59] W.S. Martindale III and C.R. Miers, Herstein's Lie theory revisited, *J.Algebra* **98** (1986), 14–37. MR **87e:**16083
[60] K. McCrimmon, The Zel'manov approach to Jordan homomorphisms of associative algebras, *J.Algebra* **123** (1989), 457–477. MR **90j:**17053
[61] D. Meng and C. Jiang, The derivation algebra and the universal central extensions of the q-analog of the Virasoro-like algebra, *Comm. Algebra* **26** (1998), 1335–1346. MR **98m:**17027
[62] C.R. Miers, Lie isomorphisms of factors, *Trans Amer. Math. Soc.* **147** (1970), 55–63. MR **42:**8302
[63] C.R. Miers, Lie $*-$triple homomorphisms into von Neumann algebras, *Proc. Amer. Math. Soc.* **58** (1976), 169–172. MR **53:**14156
[64] C.R. Miers, Lie triple derivations of von Neumann algebras, *Proc. Amer. Math. Soc.* **71** (1978), 57–61. MR **58:**7109
[65] S. Montgomery, Constructing simple Lie superalgebras from associative graded algebras, *J. Algebra* **195** (1997), 558–579. MR **99g:**17043
[66] C. Procesi, *Rings with Polynomial Identities*, Marcel Decker, 1973. MR **51:**3214
[67] M.P. Rosen, Isomorphisms of a certain class of prime Lie rings, *J. Algebra* **89** (1984), 291–317. MR **85m:**16019
[68] L.H. Rowen, *Polynomial Identities in Ring Theory*, Academic Press, 1980. MR **82a:**16021
[69] G.A. Swain, Lie derivations of the skew elements of prime rings with involution, *J. Algebra* **184** (1996), 679–704. MR **97f:**16059
[70] G.A. Swain and P.S. Blau, Lie derivations in prime rings with involution, *Canad. Math. Bull.* **42** (1999), 401–411. MR **2000g:**16039
[71] E.I. Zelmanov, On prime Jordan algebras II, *Siberian Math. J.* **24** (1983), 89–104. MR **85d:**17011

Department of Mathematics, National Cheng-Kung University, Tainan, Taiwan
E-mail address: beidar@mail.ncku.edu.tw

Department of Mathematics, PF, University of Maribor, Maribor, Slovenia
E-mail address: bresar@uni-mb.si

Department of Mechanics and Mathematics, Tula State University, Tula, Russia
E-mail address: mchebotar@tula.net

Department of Mathematics, University of Massachusetts, Amherst, Massachusetts 01003
E-mail address: jmartind@chapline.net