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On higher-power moments of $\Delta(x)$ (II)

by

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1. Introduction and main results

1.1. *Notations.* Throughout this paper, let $d(n)$ denote the Dirichlet divisor function, $r(n)$ the number of ways n can be written as $n = x^2 + y^2$ for $x, y \in \mathbb{Z}$, and $a(n)$ the Fourier coefficients of a holomorphic cusp form of weight $\kappa = 2n \geq 12$ for the full modular group, $\widetilde{a}(n) := a(n)n^{-\kappa/2+1/2}$. For short, we use $d, r, a, \widetilde{a}$ to denote these functions, respectively. $\zeta(s)$ denotes the Riemann zeta-function.

Suppose $x, t > 0$. Define

$$\Delta(x) := \sum_{n\leq x} d(n) - x\log x - (2\gamma - 1)x, \tag{1.1}$$

$$P(x) := \sum_{n\leq x} r(n) - \pi x, \tag{1.2}$$

$$A(x) := \sum_{n\leq x} a(n), \tag{1.3}$$

$$E(t) := \int_0^t |\zeta(1/2 + iu)|^2\, du - t\log(t/2\pi) - (2\gamma - 1)t. \tag{1.4}$$

Suppose $f\colon \mathbb{N} \to \mathbb{R}$ is any function such that $f(n) \ll n^{\varepsilon}$, $k \geq 2$ is a fixed integer. Define

$$s_{k;l}(f) := \sum_{\sqrt{n_1}+\ldots+\sqrt{n_l}=\sqrt{n_{l+1}}+\ldots+\sqrt{n_k}} \frac{f(n_1)\ldots f(n_k)}{(n_1\ldots n_k)^{3/4}} \quad (1 \leq l < k), \tag{1.5}$$

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$$B_k(f) := \sum_{l=1}^{k-1} \binom{k-1}{l} s_{k;l}(f) \cos \frac{\pi(k-2l)}{4}. \tag{1.6}$$

We shall use $s_{k;l}(f)$ to denote both the series (1.5) and its value. We will prove the convergence of $s_{k;l}(f)$ in Section 3.

Suppose $A_0 > 2$ is a real number. Define

$$K_0 := \min\{n \in \mathbb{N} : n \geq A_0,\, 2 \,|\, n\}, \quad b(k) := 2^{k-2} + (k-6)/4,$$

$$\sigma(k, A_0) := \begin{cases} 1/4 & \text{if } k-1 < A_0/2, \\ \dfrac{A_0 - k}{2(A_0-2)} & \text{if } A_0/2 + 1 \leq k < A_0, \end{cases}$$

$$\delta_1(k, A_0) := \frac{\sigma(k, A_0)}{2b(K_0)}, \quad \delta_2(k, A_0) := \frac{\sigma(k, A_0)}{2b(k) + 2\sigma(k, A_0)}.$$

$\mathbb{N}$ denotes the set of all natural numbers; ε always denotes a sufficiently small positive constant which may be different at different places. We will use the inequality $d(n) \ll n^{\varepsilon}$ freely. SC($\sum$) denotes the summation condition of the sum $\sum$; $\mu(n)$ is the Möbius function.

1.2. *Introduction.* In this paper we shall study the higher-power moments of $\Delta(x)$, $P(x)$, $A(x)$ and $E(t)$.

We begin with the Dirichlet divisor problem. Dirichlet first proved that $\Delta(x) = O(x^{1/2})$. The exponent 1/2 was improved by many authors. The latest result reads

$$\Delta(x) \ll x^{23/73} (\log x)^{315/146}, \tag{1.7}$$

which can be found in Huxley [6] (see also "Note added in proof"). It is conjectured that

$$\Delta(x) = O(x^{1/4+\varepsilon}), \tag{1.8}$$

which is supported by the classical mean-square result

$$\int_1^T \Delta^2(x)\, dx = \frac{(\zeta(3/2))^4}{6\pi^2 \zeta(3)} T^{3/2} + O(T \log^5 T) \tag{1.9}$$

proved by Tong [17] and the upper bound estimate

$$\int_1^T |\Delta(x)|^{A_0}\, dx \ll T^{1+A_0/4+\varepsilon}, \tag{1.10}$$

where $A_0 > 2$ is a fixed real number. The estimate of type (1.10) can be found in Ivić [7, Thm. 13.9] with $A_0 = 35/4$ and Heath-Brown [5] with $A_0 = 28/3$. On the other hand, Voronoï [19] proved that

$$\int_1^T \Delta(x)\, dx = T/4 + O(T^{3/4}), \tag{1.11}$$

which in conjunction with (1.9) shows that $\Delta(x)$ has a lot of sign changes and cancellations between the positive and negative portions.

Tsang [18] first studied the third- and fourth-power moments of $\Delta(x)$. He proved that (with notations of Section 1.1)

$$\int_1^T \Delta^3(x)\,dx = \frac{3s_{3;1}(d)}{28\pi^3}\,T^{7/4} + O(T^{7/4-1/14+\varepsilon}), \tag{1.12}$$

$$\int_1^T \Delta^4(x)\,dx = \frac{3s_{4;2}(d)}{64\pi^4}\,T^{2} + O(T^{2-1/23+\varepsilon}). \tag{1.13}$$

Heath-Brown [5] proved that for $k = 3,\dots,9$ the limit

$$\lim_{T\to\infty} T^{-1-k/4}\int_1^T \Delta(x)^k\,dx$$

exists.

In [20] the author improved Tsang's method and proved that

$$\int_1^T \Delta^3(x)\,dx = \frac{3s_{3;1}(d)}{28\pi^3}\,T^{7/4} + O(T^{3/2+\varepsilon}), \tag{1.14}$$

$$\int_1^T \Delta^4(x)\,dx = \frac{3s_{4;2}(d)}{64\pi^4}\,T^{2} + O(T^{2-2/41}), \tag{1.15}$$

$$\int_1^T \Delta^5(x)\,dx = \frac{5(2s_{5;2}(d) - s_{5;1}(d))}{288\pi^5}\,T^{9/4} + O(T^{9/4-5/816}). \tag{1.16}$$

But the argument of [20] fails for $k \ge 6$.

1.3. *New results on higher-power moments of $\Delta(x)$.* In this paper we shall use a different approach to study the higher-power moments of $\Delta(x)$. This leads to the asymptotic formulas for the integral $\int_1^T \Delta^k(x)\,dx$ for $3 \le k \le 9$. Furthermore, if the estimate (1.8) is true, then our approach can give the asymptotic formulas for $\int_1^T \Delta^k(x)\,dx$ for any $k \ge 10$.

THEOREM 1. *Let $A_0 > 9$ be a real number such that* (1.10) *holds. Then for any integer* $3 \le k < A_0$, *we have the asymptotic formula*

$$\int_1^T \Delta^k(x)\,dx = \frac{B_k(d)}{(1+k/4)2^{3k/2-1}\pi^k}\,T^{1+k/4} + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon}). \tag{1.17}$$

REMARK 1.1. From Ivić's argument [7, Thm. 13.9], we know that the value of A_0 for which (1.10) holds depends on the large-value estimate and the upper bound estimate of $\Delta(x)$. If we insert the estimate (1.7) into the argument of Ivić, we find that (1.10) holds with $A_0 = 184/19$. Hence for $k \in \{3,4,5,6,7,8,9\}$, we get the asymptotic formula (1.17). Moreover, if the estimate $\Delta(x) \ll x^{5/16-\delta}$ holds for some small $\delta > 0$, then the asymptotic formula (1.17) holds for $k = 10$.

REMARK 1.2. For $k \geq 10$, Theorem 1 is only a conditional result. However, it tells us that for any $k \geq 10$, the main term in the asymptotic formula for $\int_1^T \Delta^k(x)\,dx$ (if it exists) must have the form stated in (1.17).

REMARK 1.3. We can state the following three conjectures about $\Delta(x)$:

CONJECTURE 1. The estimate (1.8) is true.

CONJECTURE 2. The estimate (1.10) is true for any $A_0 > 2$.

CONJECTURE 3. For any fixed $k \geq 3$, there exists a constant $\delta_k > 0$ such that the following asymptotic formula holds:

$$\int_1^T \Delta^k(x)\,dx = \frac{B_k(d)}{(1+k/4)2^{3k/2-1}\pi^k}\,T^{1+k/4} + O(T^{1+k/4-\delta_k+\varepsilon}).$$

It is well known that Conjectures 1 and 2 are equivalent. From Theorem 1 we know that actually the three conjectures are equivalent. It is easy to deduce Conjecture 2 from Conjecture 3. To deduce Conjecture 3 from Conjecture 2, we take $A_0 = 2(k-1)$ and $\delta_k = \delta_1(k, 2(k-1))$.

REMARK 1.4. From (1.11) we know that the integral $\int_1^T \Delta(x)\,dx$ has many cancellations from the positive and negative portions of $\Delta(x)$. However, from (1.12) Tsang [18] observed that this is not so for $\int_1^T \Delta^3(x)\,dx$. From Theorem 1 we know that this is also not so for $\int_1^T \Delta^k(x)\,dx$ ($k = 5, 7, 9$) since numerical computation tells $B_k(d) > 0$ for $k = 5, 7, 9$. Maybe $B_k(d) > 0$ holds for any odd $k \geq 3$.

The constant $\delta_1(k, A_0)$ is small for k small. If we combine Ivić's argument with the proof of Theorem 1, we get the following Theorem 2 for $3 \leq k \leq 9$. Note that the results for $k = 3, 4$ are weaker than those of [20]. Theorem 2 for $k = 5$ improves (1.16).

THEOREM 2. *For* $3 \leq k \leq 9$, *the asymptotic formula* (1.17) *holds with* $\delta_1(k, A_0)$ *replaced by* $\delta_2(k, 184/19)$.

In particular, *for* $k = 5, 6, 7, 8, 9$, *we have*

$$\int_1^T \Delta^5(x)\,dx = \frac{5(2s_{5;2}(d) - s_{5;1}(d))}{288\pi^5}\,T^{9/4} + O(T^{9/4-1/64+\varepsilon}), \tag{1.18}$$

$$\int_1^T \Delta^6(x)\,dx = \frac{5s_{6;3}(d) - 3s_{6;1}(d)}{320\pi^6}\,T^{5/2} + O(T^{5/2-35/4742+\varepsilon}), \tag{1.19}$$

$$\int_1^T \Delta^7(x)\,dx = \frac{7(5s_{7;3}(d) - 3s_{7;2}(d) - s_{7;1}(d))}{2816\pi^7}\,T^{11/4} + O(T^{11/4-17/6312+\varepsilon}), \tag{1.20}$$

$$(1.21)\qquad \int_1^T \Delta^8(x)\,dx = \frac{7(5s_{8;4}(d)-4s_{8;2}(d))}{6144\pi^8}T^3 + O(T^{3-8/9433+\varepsilon}),$$

$$(1.22)\qquad \int_1^T \Delta^9(x)\,dx = \frac{3(3s_{9;1}(d)-12s_{9;2}(d)-28s_{9;3}(d)+42s_{9;4}(d))}{26624\pi^9}T^{13/4} + O(T^{13/4-13/75216+\varepsilon}).$$

1.4. *Higher-power moments of $P(x)$, $A(x)$ and $E(t)$.* The method of proving Theorems 1 and 2 can also be applied to study the higher-power moments of $P(x), A(x)$ and $E(t)$.

The conjectured bound of $P(x)$ is

$$(1.23)\qquad P(x) = O(x^{1/4+\varepsilon}),$$

which is supported by

$$(1.24)\qquad \int_2^T P^2(x)\,dx = \left(\frac{1}{3\pi^2}\sum_{n=1}^{\infty} r^2(n)n^{-3/2}\right)T^{3/2} + O(T\log^2 T)$$

proved by Kátai [14]. Tsang [18] also studied the third- and fourth-power moments of $P(x)$. His results were improved by the present author [20]. An asymptotic formula for the fifth-power moment of $P(x)$ was also obtained in [20], which is further improved by the following (for $k=5$):

Theorem 3. *Let $A_0 > 9$ be a real number such that*

$$(1.25)\qquad \int_1^T |P(x)|^{A_0}\,dx \ll T^{1+A_0/4+\varepsilon}.$$

Then for any integer $3 \le k < A_0$, the following asymptotic formula holds:

$$(1.26)\qquad \int_1^T P^k(x)\,dx = \frac{(-1)^k B_k(r)}{(1+k/4)2^{k-1}\pi^k}T^{1+k/4} + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon}).$$

In particular, *for $3 \le k \le 9$, (1.26) holds with $\delta_1(k,A_0)$ replaced by $\delta_2(k,184/19)$.*

Remark 1.5. Ivić [7, Thm. 13.12] proved that the estimate (1.25) holds for $A_0 = 35/4$. If we insert the estimate $P(x) = O(x^{23/73+\varepsilon})$ (see Huxley [6]) into his argument, we find that (1.25) holds for $A_0 = 184/19$.

It is well known that $A(x)$ has no main term and $A(x) \ll x^{\kappa/2-1/6+\varepsilon}$. From Deligne [4], we have $|\widetilde{a}(n)| \le d(n)$.

The conjectured bound of $A(x)$ is $A(x) \ll x^{\kappa/2-1/4+\varepsilon}$. Ivić [9] proved that

$$(1.27)\qquad \int_1^T A^2(x)\,dx = B_2 T^{\kappa+1/2} + O(T^{\kappa}\log^5 T),$$

where

$$B_2 = \frac{1}{4\kappa+2}\sum_{n=1}^{\infty} a^2(n)n^{-\kappa-1/2}.$$

Ivić [9] also proved that

$$\int_1^T |A(x)|^{A_0}\,dx \ll T^{1+A_0(2\kappa-1)/4+\varepsilon} \tag{1.28}$$

for $A_0 = 8$. Cai [3] studied the third- and fourth-power moments of $A(x)$. His results were improved in [20], where an asymptotic formula for the fifth-power moment of $A(x)$ was also obtained, which is further improved by the case $k = 5$ of the following:

THEOREM 4. *Let $A_0 \geq 8$ be a real number such that* (1.28) *is true. Then for any $3 \leq k < A_0$, we have the asymptotic formula*

$$\int_1^T A^k(x)\,dx = \frac{B_k(\widetilde{a})}{\left(1+\frac{k(2\kappa-1)}{4}\right)2^{3k/2-1}\pi^k} T^{1+\frac{k(2\kappa-1)}{4}} + O(T^{1+\frac{k(2\kappa-1)}{4}-\delta_1(k,A_0)+\varepsilon}). \tag{1.29}$$

In particular, for $3 \leq k \leq 7$, (1.29) *holds with $\delta_1(k,A_0)$ replaced by $\delta_2(k,8)$.*

Many results for $E(t)$ parallel to those for $\Delta(x)$ have been obtained; see Ivić [8] for a survey. The conjectured bound for $E(t)$ is $E(t) \ll t^{1/4+\varepsilon}$, which is supported by

$$\int_2^T E^2(t)\,dt = \frac{2\zeta^4(3/2)}{3\zeta(3)\sqrt{2\pi}} T^{3/2} + O(T\log^5 T), \tag{1.30}$$

proved by Meurman [15]. It has been proved (see Huxley [6]) that

$$E(t) \ll t^{72/227}(\log t)^{629/227}, \qquad t > 2. \tag{1.31}$$

Ivić [7, Thm. 15.7] proved that

$$\int_1^T |E(t)|^{A_0}\,dt \ll T^{1+A_0/4+\varepsilon} \tag{1.32}$$

for $A_0 = 35/4$. Inserting (1.31) into Ivić's argument, we find that (1.32) is true for $A_0 = 576/61$.

Tsang [18] studied the third- and fourth-power moment of $E(t)$ by using the Atkinson formula [1]. His results were further improved by Ivić [10] in a different way. The author [20] obtained new results on the third- and fourth-power moments of $E(t)$. An asymptotic formula for the fifth-power moment of $E(t)$ was also obtained in [20], which is further improved by the case $k=5$ of the following:

THEOREM 5. *Let $A_0>9$ be a real number such that the estimates* (1.10) *and* (1.32) *hold. Then for any $3\le k<A_0$, we have the asymptotic formula*

$$\int_1^T E^k(t)\,dt=\frac{B_k(d)}{(1+k/4)2^{3k/4-1}\pi^{k/4}}T^{1+k/4}+O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon}). \tag{1.33}$$

In particular, for $3\le k\le 9$, (1.33) *holds with $\delta_1(k,A_0)$ replaced by $\delta_2(k,576/61)$.*

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2. Some preliminary lemmas. We need the following lemmas.

LEMMA 2.1. *The square roots of squarefree numbers are linearly independent over the integers.*

Proof. This is a classical result of Besicovitch [2]. ■

LEMMA 2.2. *Suppose $k\ge 3$ and $(i_1,\dots,i_{k-1})\in\{0,1\}^{k-1}$ are such that*

$$\sqrt{n_1}+(-1)^{i_1}\sqrt{n_2}+(-1)^{i_2}\sqrt{n_3}+\dots+(-1)^{i_{k-1}}\sqrt{n_k}\ne 0.$$

Then

$$|\sqrt{n_1}+(-1)^{i_1}\sqrt{n_2}+(-1)^{i_2}\sqrt{n_3}+\dots+(-1)^{i_{k-1}}\sqrt{n_k}| \gg \max(n_1,\dots,n_k)^{-(2^{k-2}-2^{-1})}.$$

Proof. The cases $k=3,4$ are Lemmas 1 and 2 of Tsang [18], respectively. The proof for the general case is the same as the proof of Lemma 1 of [18]. We note that Heath-Brown [5] stated a similar result for k even. ■

LEMMA 2.3. *Suppose $A,B\in\mathbb{R}$, $A\ne 0$. Then*

$$\int_T^{2T}\cos(A\sqrt{t}+B)\,dt\ll T^{1/2}|A|^{-1}.$$

LEMMA 2.4. *Suppose $k\ge 3$, $(i_1,\dots,i_{k-1})\in\{0,1\}^{k-1}$, $(i_1,\dots,i_{k-1})\ne(0,\dots,0)$, $N_1,\dots,N_k>1$, $0<\Delta\ll E^{1/2}$, $E=\max(N_1,\dots,N_k)$. Let*

$$\mathcal{A}=\mathcal{A}(N_1,\dots,N_k;i_1,\dots,i_{k-1};\Delta)$$

denote the number of solutions of the inequality

$$|\sqrt{n_1}+(-1)^{i_1}\sqrt{n_2}+(-1)^{i_2}\sqrt{n_3}+\dots+(-1)^{i_{k-1}}\sqrt{n_k}|<\Delta \tag{2.1}$$

with $N_j < n_j \le 2N_j, 1 \le j \le k$. *Then*

$$\mathcal{A} \ll \Delta E^{-1/2} N_1 \dots N_k + E^{-1} N_1 \dots N_k.$$

Proof. Without loss of generality, suppose $E = N_k$. If $(n_1, \dots, n_k)$ satisfies (2.1), then

$$\sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + (-1)^{i_2}\sqrt{n_3} + \dots + (-1)^{i_{k-2}}\sqrt{n_{k-1}} = (-1)^{i_{k-1}+1}\sqrt{n_k} + \theta\Delta$$

for some $|\theta| < 1$, whence we get

$$(\sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + (-1)^{i_2}\sqrt{n_3} + \dots + (-1)^{i_{k-2}}\sqrt{n_{k-1}})^2 = n_k + O(\Delta N_k^{1/2}).$$

Hence for fixed $(n_1, \dots, n_{k-1})$, the number of n_k is $\ll 1 + \Delta N_k^{1/2}$ and thus

$$\mathcal{A} \ll \Delta N_k^{1/2} N_1 \dots N_{k-1} + N_1 \dots N_{k-1}. \blacksquare$$

3. On the series $s_{k;l}(d)$. Suppose $y > 1$ is a large parameter, and define

$$s_{k;l}(d; y) := \sum_{\substack{\sqrt{n_1}+\dots+\sqrt{n_l}=\sqrt{n_{l+1}}+\dots+\sqrt{n_k} \\ n_1,\dots,n_k \le y}} \frac{d(n_1)\dots d(n_k)}{(n_1 \dots n_k)^{3/4}}, \qquad 1 \le l < k.$$

We shall prove

LEMMA 3.1. *We have*

$$|s_{k;l}(d) - s_{k;l}(d; y)| \ll y^{-1/2+\varepsilon}, \qquad 1 \le l < k.$$

REMARK. Lemma 3.1 is still true if the divisor function d is replaced by any function $f : \mathbb{N} \to \mathbb{R}$ with $f(n) \ll n^{\varepsilon}$.

Proof. We shall prove Lemma 3.1 by induction in k. The case $k = 2$ is easy. The case $k = 3$ is contained in [18, p. 70]. Later we suppose $k \ge 4$. Since $s_{k;l}(d) = s_{k;k-l}(d)$, we suppose $l \le k/2$.

By symmetry, we get

$$(3.1) \qquad |s_{k;l}(d) - s_{k;l}(d; y)| \ll \sum_{\substack{\sqrt{n_1}+\dots+\sqrt{n_l}=\sqrt{n_{l+1}}+\dots+\sqrt{n_k} \\ n_1 > y}} \frac{d(n_1)\dots d(n_k)}{(n_1 \dots n_k)^{3/4}}$$
$$\ll U_1(d; y) + U_2(d; y),$$

say, where

$$U_1(d; y) := \sum_{j=l+1}^{k} \sum_{\substack{\sqrt{n_1}+\dots+\sqrt{n_l}=\sqrt{n_{l+1}}+\dots+\sqrt{n_k} \\ n_1 = n_j > y}} \frac{d(n_1)\dots d(n_k)}{(n_1 \dots n_k)^{3/4}},$$

$$U_2(d; y) := \sum_{\substack{\sqrt{n_1}+\dots+\sqrt{n_l}=\sqrt{n_{l+1}}+\dots+\sqrt{n_k} \\ n_1 > y,\, n_1 \ne n_j,\, l+1 \le j \le k}} \frac{d(n_1)\dots d(n_k)}{(n_1 \dots n_k)^{3/4}}.$$

If $l = 1$, then obviously $U_1(d;y) = 0$. If $l > 1$, then by induction we get

$$U_1(d;y) \ll \sum_{n>y} \frac{d^2(n)}{n^{3/2}}\, s_{k-2;l-1}(d) \ll y^{-1/2+\varepsilon}. \tag{3.2}$$

Now we estimate $U_2(d;y)$. Let $I = \{1,\ldots,l\}$, $J = \{l+1,\ldots,k\}$. Suppose $(n_1,\ldots,n_k) \in \mathbb{N}^k$ are such that

$$(*)\quad \sqrt{n_1} + \ldots + \sqrt{n_l} = \sqrt{n_{l+1}} + \ldots + \sqrt{n_k}, \qquad n_1 \neq n_j, \qquad l+1 \le j \le k.$$

Then there exist two sets $I_0 \subset I$, $J_0 \subset J$ with the following properties:

(1) $1 \in I_0$;
(2) $\sum_{i\in I_0} \sqrt{n_i} = \sum_{j\in J_0} \sqrt{n_j}$;
(3) For any real subset $I_0' \subset I_0$, $J_0' \subset J_0$, we have

$$\sum_{i\in I_0'} \sqrt{n_i} \neq \sum_{j\in J_0'} \sqrt{n_j}.$$

If $(I_0,J_0) = (I,J)$, then we say $(n_1,\ldots,n_k)$ is a *primitive* (k,l)-*point*. Let $\mathbb{N}_{k;l}$ denote the set of all points in $\mathbb{N}^k$ which satisfy $(*)$ and $\mathbb{N}^*_{k;l}$ the set of all primitive (k,l)-points. Let $\mathcal{G}_{k;l}$ denote the set of all possible pairs (I_0,J_0) when $(n_1,\ldots,n_k)$ runs through $\mathbb{N}_{k;l}$. Note that if $l = 1$, then $\mathcal{G}_{k;l} = \{(I,J)\}$.

Suppose $(I_0,J_0) \in \mathcal{G}_{k;l}$. Let $l_1 = \#I_0, l_2 = l - l_1$, $k_1 = \#I_0 + \#J_0, k_2 = k - k_1$. From $(*)$, we know that $k_1 \ge 3$. Define

$$R_1^{(I_0,J_0)}(d;y) := \sum_{\substack{\sqrt{n_1}+\ldots+\sqrt{n_{l_1}}=\sqrt{n_{l_1+1}}+\ldots+\sqrt{n_{k_1}} \\ n_1>y,\, (n_1,\ldots,n_{k_1})\in\mathbb{N}^*_{k_1;l_1}}} \frac{d(n_1)\ldots d(n_{k_1})}{(n_1\ldots n_{k_1})^{3/4}}.$$

If $(I_0,J_0) \neq (I,J)$, then $l_1 < l, k_1 < k$ and we define

$$R_2^{(I_0,J_0)}(d) := \sum_{\sqrt{m_1}+\ldots+\sqrt{m_{l_2}}=\sqrt{m_{l_2+1}}+\ldots+\sqrt{m_{k_2}}} \frac{d(m_1)\ldots d(m_{k_2})}{(m_1\ldots m_{k_2})^{3/4}}.$$

By the induction assumption, $R_2^{(I_0,J_0)}(d) \ll 1$.

If $(n_1,\ldots,n_{k_1}) \in \mathbb{N}^*_{k_1;l_1}$, then by Lemma 2.1 we have

$$n_j = s_j^2 h, \qquad s_1 + \ldots + s_{l_1} = s_{l_1+1} + \ldots + s_{k_1}, \qquad \mu(h) \neq 0.$$

Now $n_1 > y$ implies that there exists at least one n_j $(l_1+1 \le j \le j_1)$ such that $n_j \gg y$. We suppose $n_{k_1} \gg y$. So we have

$$R_1^{(I_0,J_0)}(d;y) \ll \sum_h \sum_{\substack{s_1+\ldots+s_{l_1}=s_{l_1+1}+\ldots+s_{k_1} \\ s_1^2h>y,\, s_{k_1}^2h\gg y}} \frac{d(s_1^2h)\ldots d(s_{k_1}^2h)}{h^{3k_1/4}(s_1\ldots s_{k_1})^{3/2}}$$

$$\ll \sum_h \sum_{\substack{s_1+\ldots+s_{l_1}=s_{l_1+1}+\ldots+s_{k_1}\\ s_1^2h>y,\, s_{k_1}^2h\gg y}} \frac{d^2(s_1)\ldots d^2(s_{k_1})d^{k_1}(h)}{h^{3k_1/4}(s_1\ldots s_{k_1})^{3/2}}$$
$$\ll \sum_h \frac{d^{k_1}(h)}{h^{3k_1/4}} \sum_{s_1>(y/h)^{1/2}} \frac{d^2(s_1)}{s_1^{3/2}} \sum_{s_{k_1}\gg(y/h)^{1/2}} \frac{d^2(s_{k_1})}{s_{k_1}^{3/2}}$$
$$\ll \sum_h \frac{d^{k_1}(h)}{h^{3k_1/4}}\left(\frac{y}{h}\right)^{-1/2+\varepsilon} \ll y^{-1/2+\varepsilon}$$

if we notice $k_1 \ge 3$.

If $\mathcal{G}_{k;l} = (I, J)$, we have

$$U_2(d;y) \ll R_1^{(I,J)}(d;y) \ll y^{-1/2+\varepsilon}. \tag{3.3}$$

If $\mathcal{G}_{k;l} \ne (I, J)$, we have

$$U_2(d;y) \ll R_1^{(I,J)}(d;y) + \sum_{\substack{(I_0,J_0)\in\mathcal{G}_{k;l}\\ (I_0,J_0)\ne(I,J)}} R_1^{(I_0,J_0)}(d;y)R_2^{(I_0,J_0)}(d) \ll y^{-1/2+\varepsilon}. \tag{3.4}$$

Now Lemma 3.1 follows from (3.1)–(3.4). ∎

4. Proofs of Theorems 1 and 2. Suppose $T \ge 10$ is a real number. It suffices to evaluate the integral $\int_T^{2T} \Delta^k(x)\,dx$. Suppose y is a parameter such that $T^\varepsilon < y \le T^{1/3}$. For any $T \le x \le 2T$, define

$$\mathcal{R}_1 = \mathcal{R}_1(x,y) := (\sqrt{2}\,\pi)^{-1}x^{1/4}\sum_{n\le y}\frac{d(n)}{n^{3/4}}\cos(4\pi\sqrt{xn}-\pi/4),$$
$$\mathcal{R}_2 = \mathcal{R}_2(x,y) := \Delta(x) - \mathcal{R}_1.$$

We shall show that the higher-power moment of $\mathcal{R}_2$ is small and hence the integral $\int_T^{2T} \Delta^k(x)\,dx$ can be well approximated by $\int_T^{2T} \mathcal{R}_1^k\,dx$, which is easy to evaluate.

4.1. *Evaluation of the integral* $\int_T^{2T} \mathcal{R}_1^h\,dx$. Suppose $h \ge 3$ is any fixed integer. By the elementary formula

$$\cos a_1 \ldots \cos a_h = \frac{1}{2^{h-1}} \sum_{(i_1,\ldots,i_{h-1})\in\{0,1\}^{h-1}} \cos(a_1 + (-1)^{i_1}a_2 + (-1)^{i_2}a_3 + \ldots + (-1)^{i_{h-1}}a_h),$$

we have

$$\mathcal{R}_1^h = (\sqrt{2}\,\pi)^{-h} x^{h/4} \sum_{n_1\le y} \dots \sum_{n_h\le y} \frac{d(n_1)\dots d(n_h)}{(n_1\dots n_h)^{3/4}} \prod_{j=1}^{h} \cos(4\pi\sqrt{n_j x} - \pi/4)$$

$$= \frac{x^{h/4}}{(\sqrt{2}\,\pi)^h 2^{h-1}} \sum_{(i_1,\dots,i_{h-1})\in\{0,1\}^{h-1}} \sum_{n_1\le y} \dots \sum_{n_h\le y} \frac{d(n_1)\dots d(n_h)}{(n_1\dots n_h)^{3/4}}$$

$$\times \cos\left(4\pi\sqrt{x}\,\alpha(n_1,\dots,n_h;i_1,\dots,i_{h-1}) - \frac{\pi}{4}\,\beta(i_1,\dots,i_{h-1})\right),$$

where

$$\alpha(n_1,\dots,n_h;i_1,\dots,i_{h-1}) := \sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + (-1)^{i_2}\sqrt{n_3} + \dots + (-1)^{i_{h-1}}\sqrt{n_h},$$

$$\beta(i_1,\dots,i_{h-1}) := 1 + (-1)^{i_1} + (-1)^{i_2} + \dots + (-1)^{i_{h-1}}.$$

Thus we can write

$$\mathcal{R}_1^h = \frac{1}{(\sqrt{2}\,\pi)^h 2^{h-1}}\,(S_1(x) + S_2(x)), \tag{4.1}$$

where

$$S_1(x) := x^{h/4} \sum_{(i_1,\dots,i_{h-1})\in\{0,1\}^{h-1}} \cos\left(-\frac{\pi\beta}{4}\right) \sum_{\substack{n_j\le y,\,1\le j\le h\\ \alpha=0}} \frac{d(n_1)\dots d(n_h)}{(n_1\dots n_h)^{3/4}},$$

$$S_2(x) := x^{h/4} \sum_{(i_1,\dots,i_{h-1})\in\{0,1\}^{h-1}} \sum_{\substack{n_j\le y,\,1\le j\le h\\ \alpha\ne 0}} \frac{d(n_1)\dots d(n_h)}{(n_1\dots n_h)^{3/4}} \times \cos(4\pi\alpha\sqrt{x} - \pi\beta/4),$$

$$\alpha := \alpha(n_1,\dots,n_h;i_1,\dots,i_{h-1}), \qquad \beta := \beta(i_1,\dots,i_{h-1}).$$

First consider the contribution of $S_1(x)$. We have

$$\int_T^{2T} S_1(x)\,dx = \sum_{(i_1,\dots,i_{h-1})\in\{0,1\}^{h-1}} \cos\left(-\frac{\pi\beta}{4}\right) \sum_{\substack{n_j\le y,\,1\le j\le h\\ \alpha=0}} \frac{d(n_1)\dots d(n_h)}{(n_1\dots n_h)^{3/4}} \int_T^{2T} x^{h/4}\,dx. \tag{4.2}$$

It is easily seen that if $\alpha=0$, then $1\in\{i_1,\dots,i_{h-1}\}$. Let $l = i_1+\dots+i_{h-1}$. Then

$$\sum_{\substack{n_j\le y,\,1\le j\le h\\ \alpha=0}} \frac{d(n_1)\dots d(n_h)}{(n_1\dots n_h)^{3/4}} = s_{h;l}(d;y),$$

where $s_{h;l}(d;y)$ was defined in the last section.

By Lemma 3.1 we get

$$(4.3)\qquad \int_T^{2T} S_1(x)\,dx = B_h^*(d)\int_T^{2T} x^{h/4}\,dx + O(T^{1+h/4+\varepsilon}y^{-1/2}),$$

where

$$B_h^*(d) := \sum_{(i_1,\ldots,i_{h-1})\in\{0,1\}^{h-1}} \cos\left(-\frac{\pi\beta}{4}\right) \sum_{\substack{(n_1,\ldots,n_h)\in\mathbb{N}^h\\ \alpha=0}} \frac{d(n_1)\ldots d(n_h)}{(n_1\ldots n_h)^{3/4}}.$$

For any $(i_1,\ldots,i_{h-1})\in\{0,1\}^{h-1}\setminus\{(0,\ldots,0)\}$, let

$$S(d;i_1,\ldots,i_{h-1}) := \sum_{\substack{(n_1,\ldots,n_h)\in\mathbb{N}^h\\ \alpha=0}} \frac{d(n_1)\ldots d(n_h)}{(n_1\ldots n_h)^{3/4}},$$
$$l(i_1,\ldots,i_{h-1}) := i_1+\ldots+i_{h-1}.$$

It is easily seen that if $l(i_1,\ldots,i_{h-1}) = l(i'_1,\ldots,i'_{h-1})$ or $l(i_1,\ldots,i_{h-1}) + l(i'_1,\ldots,i'_{h-1}) = h$, then

$$S(d;i_1,\ldots,i_{h-1}) = S(d;i'_1,\ldots,i'_{h-1}) = s_{h;l(i_1,\ldots,i_{h-1})}(d).$$

From $(-1)^i = 1-2i$ $(i=0,1)$ we also have

$$\beta(i_1,\ldots,i_{h-1}) = h-2l(i_1,\ldots,i_{h-1}).$$

So we get

$$(4.4)\qquad \begin{aligned} B_h^*(d) &= \sum_{l=1}^{h-1}\sum_{l(i_1,\ldots,i_{h-1})=l}\cos\left(-\frac{\pi\beta}{4}\right)S(d;i_1,\ldots,i_{h-1})\\ &= \sum_{l=1}^{h-1} s_{h;l}(d)\cos\frac{\pi(h-2l)}{4}\sum_{l(i_1,\ldots,i_{h-1})=l} 1\\ &= \sum_{l=1}^{h-1}\binom{h-1}{l} s_{h;l}(d)\cos\frac{\pi(h-2l)}{4} = B_h(d). \end{aligned}$$

Now we consider the contribution of $S_2(x)$. By Lemma 2.3 we get

$$(4.5)\qquad \int_T^{2T} S_2(x)\,dx \ll T^{1/2+h/4} \sum_{(i_1,\ldots,i_{h-1})\in\{0,1\}^{h-1}} \sum_{\substack{n_j\le y,\,1\le j\le h\\ \alpha\ne 0}} \frac{d(n_1)\ldots d(n_h)}{(n_1\ldots n_h)^{3/4}|\alpha|}.$$

It suffices to estimate the sum

$$\Sigma(y; i_1, \ldots, i_{h-1}) = \sum_{\substack{n_j \le y,\, 1 \le j \le h \\ \alpha \ne 0}} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4} |\alpha|}$$

for fixed $(i_1, \ldots, i_{h-1}) \in \{0,1\}^{h-1}$. If $(i_1, \ldots, i_{h-1}) = (0, \ldots, 0)$, then

$$\begin{aligned} \Sigma(y; 0, \ldots, 0) &\ll \sum_{n_j \le y,\, 1 \le j \le h} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4} (\sqrt{n_1} + \ldots + \sqrt{n_h})} \\ &\ll \sum_{n_j \le y,\, 1 \le j \le h} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4 + 1/2h}} \ll y^{(h-2)/4} \log^h y, \end{aligned}$$

where we used the estimates

$$\sum_{n \le u} d(n) \ll u \log u, \qquad x_1 + \ldots + x_h \gg (x_1 \ldots x_h)^{1/h}.$$

For $(i_1, \ldots, i_{h-1}) \ne (0, \ldots, 0)$, by a splitting argument we deduce that there exist a collection of numbers $1 < N_1, \ldots, N_h < y$ such that

$$\Sigma(y; i_1, \ldots, i_{h-1}) \ll \Sigma_1^* \log^h y,$$

where

$$\Sigma_1^* = \sum_{\substack{N_j < n_j \le 2N_j,\, 1 \le j \le h \\ \alpha \ne 0}} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4} |\alpha|}.$$

Without loss of generality, we suppose $N_1 \le \ldots \le N_h \le y$. By Lemma 2.2 we have $|\alpha| \gg N_h^{-(2^{h-2} - 2^{-1})}$. Then by a splitting argument and Lemma 2.4, for some $N_h^{-(2^{h-2} - 2^{-1})} \ll \Delta < y^{1/2}$ we get

$$\begin{aligned} \Sigma_1^* &\ll \frac{y^{\varepsilon}}{(N_1 \ldots N_h)^{3/4} \Delta} \mathcal{A}(N_1, \ldots, N_h; i_1, \ldots, i_{h-1}; \Delta) \\ &\ll \frac{y^{\varepsilon}}{(N_1 \ldots N_h)^{3/4} \Delta} (\Delta N_h^{1/2} N_1 \ldots N_{h-1} + N_1 \ldots N_{h-1}) \\ &\ll y^{\varepsilon} \bigg(\frac{(N_1 \ldots N_{h-1})^{1/4}}{N_h^{1/4}} + \frac{(N_1 \ldots N_{h-1})^{1/4}}{N_h^{3/4} \Delta} \bigg) \\ &\ll y^{\varepsilon} (N_h^{(h-2)/4} + N_h^{b(h)}) \ll y^{b(h) + \varepsilon}, \end{aligned}$$

where $b(h)$ was defined in Section 1.1. Thus we get

$$\int_T^{2T} S_2(x)\, dx \ll T^{1/2 + h/4 + \varepsilon} y^{b(h)}. \tag{4.6}$$

Hence from (4.1)–(4.6) we get

LEMMA 4.1. *For any fixed $h \geq 3$, we have*

$$\int_T^{2T} \mathcal{R}_1^h\,dx = \frac{B_h(d)}{(\sqrt{2}\,\pi)^h 2^{h-1}} \int_T^{2T} x^{h/4}\,dx + O(T^{1+h/4+\varepsilon} y^{-1/2} + T^{1/2+h/4+\varepsilon} y^{b(h)}). \tag{4.7}$$

4.2. *Higher-power moments of $\mathcal{R}_2$.* We first study the mean-square of $\mathcal{R}_2$. We begin with the truncated Voronoï formula [9, (2.25)]

$$\Delta(x) = (\pi\sqrt{2})^{-1} x^{1/4} \sum_{n\leq N} \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon} N^{-1/2}), \tag{4.8}$$

where $1 < N \ll x$. Taking $N = T$, we get

$$\begin{aligned}\mathcal{R}_2 &= (\pi\sqrt{2})^{-1} x^{1/4} \sum_{y<n\leq T} \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nx} - \pi/4) + O(T^{\varepsilon})\\ &\ll \left| x^{1/4} \sum_{y<n\leq T} \frac{d(n)}{n^{3/4}} e(2\sqrt{nx}) \right| + T^{\varepsilon},\end{aligned}$$

which implies

$$\begin{aligned}\int_T^{2T} \mathcal{R}_2^2\,dx &\ll T^{1+\varepsilon} + \int_T^{2T} \left| x^{1/4} \sum_{y<n\leq T} \frac{d(n)}{n^{3/4}} e(2\sqrt{nx}) \right|^2 dx\\ &\ll T^{1+\varepsilon} + T^{3/2} \sum_{y<n\leq T} \frac{d^2(n)}{n^{3/2}}\\ &\quad + T \sum_{y<m<n\leq T} \frac{d(n)d(m)}{(mn)^{3/4}(\sqrt{n}-\sqrt{m})}\\ &\ll T^{1+\varepsilon} + \frac{T^{3/2}\log^3 T}{y^{1/2}} \ll \frac{T^{3/2}\log^3 T}{y^{1/2}},\end{aligned} \tag{4.9}$$

where we used the estimates

$$\sum_{n\leq u} d^2(n) \ll u \log^3 u, \qquad \sum_{y<m<n\leq T} \frac{d(n)d(m)}{(mn)^{3/4}(\sqrt{n}-\sqrt{m})} \ll T^{\varepsilon}.$$

Now suppose y satisfies $y^{2b(K_0)} \leq T$. Hence from Lemma 4.1 we get

$$\int_T^{2T} |\mathcal{R}_1|^{K_0}\,dx \ll T^{1+K_0/4+\varepsilon},$$

which implies

$$\int_T^{2T} |\mathcal{R}_1|^{A_0}\,dx \ll T^{1+A_0/4+\varepsilon} \tag{4.10}$$

since $A_0 \le K_0$. From (1.10) and (4.10) we get

$$\int_T^{2T} |\mathcal{R}_2|^{A_0}\,dx \ll \int_T^{2T} (|\Delta(x)|^{A_0} + |\mathcal{R}_1|^{A_0})\,dx \ll T^{1+A_0/4+\varepsilon}. \tag{4.11}$$

For any $2 < A < A_0$, from (4.9), (4.11) and Hölder's inequality we get

$$\begin{aligned} \int_T^{2T} |\mathcal{R}_2|^A\,dx &= \int_T^{2T} |\mathcal{R}_2|^{\frac{2(A_0-A)}{A_0-2}+\frac{A_0(A-2)}{A_0-2}}\,dx \\ &\ll \Big(\int_T^{2T} \mathcal{R}_2^2\,dx\Big)^{\frac{A_0-A}{A_0-2}} \Big(\int_T^{2T} |\mathcal{R}_2|^{A_0}\,dx\Big)^{\frac{A-2}{A_0-2}} \\ &\ll T^{1+A/4+\varepsilon} y^{-\frac{A_0-A}{2(A_0-2)}}. \end{aligned} \tag{4.12}$$

Thus, we have proved the following

LEMMA 4.2. *Suppose $T^\varepsilon \le y \le T^{1/2b(K_0)}$, $2 < A < A_0$. Then*

$$\int_T^{2T} |\mathcal{R}_2|^A\,dx \ll T^{1+A/4+\varepsilon} y^{-(A_0-A)/2(A_0-2)}. \tag{4.13}$$

4.3. *Proof of Theorem 1.* Suppose $3 \le k \le K(A_0)$ and $T^\varepsilon \le y \le T^{1/2b(K_0)}$. By the elementary formula $(a+b)^k - a^k \ll |a^{k-1}b| + |b|^k$, we get

$$\int_T^{2T} \Delta^k(x)\,dx = \int_T^{2T} \mathcal{R}_1^k\,dx + O\Big(\int_T^{2T} |\mathcal{R}_1^{k-1}\mathcal{R}_2|\,dx\Big) + O\Big(\int_T^{2T} |\mathcal{R}_2|^k\,dx\Big). \tag{4.14}$$

If $k-1 < A_0/2$, then from (4.9), (4.10) and Cauchy's inequality we get

$$\int_T^{2T} |\mathcal{R}_1^{k-1}\mathcal{R}_2|\,dx \ll \Big(\int_T^{2T} |\mathcal{R}_1|^{2(k-1)}\,dx\Big)^{1/2} \Big(\int_T^{2T} |\mathcal{R}_2|^2\,dx\Big)^{1/2} \ll T^{1+k/4+\varepsilon} y^{-1/4}.$$

If $k-1 \ge A_0/2$, then from (4.10), Lemma 4.2 and Hölder's inequality we get

$$\begin{aligned} \int_T^{2T} |\mathcal{R}_1^{k-1}\mathcal{R}_2|\,dx &\ll \Big(\int_T^{2T} |\mathcal{R}_1|^{A_0}\,dx\Big)^{(k-1)/A_0} \Big(\int_T^{2T} |\mathcal{R}_2|^{A_0/(A_0-k+1)}\,dx\Big)^{(A_0-k+1)/A_0} \\ &\ll T^{1+k/4+\varepsilon} y^{-(A_0-k)/2(A_0-2)}. \end{aligned}$$

Thus we have

$$\int_T^{2T} |\mathcal{R}_1^{k-1}\mathcal{R}_2|\,dx + \int_T^{2T} |\mathcal{R}_2|^k\,dx \ll T^{1+k/4+\varepsilon}y^{-\sigma(k,A_0)}, \tag{4.15}$$

where $\sigma(k, A_0)$ was defined in Section 1.1.

From (4.14) and (4.15) we get

$$\int_T^{2T} \Delta^k(x)\,dx = \int_T^{2T} \mathcal{R}_1^k\,dx + O(T^{1+k/4+\varepsilon}y^{-\sigma(k,A_0)}). \tag{4.16}$$

Now take $y = T^{1/2b(K_0)}$. From Lemma 4.1 and (4.16) we get

$$\begin{aligned}\int_T^{2T} \Delta^k(x)\,dx &= \frac{B_k(d)}{(\sqrt{2}\,\pi)^k 2^{k-1}} \int_T^{2T} x^{k/4}\,dx + O(T^{1+k/4-\sigma(k,A_0)/2b(K_0)+\varepsilon}) \\ &= \frac{B_k(d)}{(\sqrt{2}\,\pi)^k 2^{k-1}} \int_T^{2T} x^{k/4}\,dx + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon}).\end{aligned} \tag{4.17}$$

Theorem 1 follows from (4.17) immediately.

4.4. *Proof of Theorem 2.* Suppose $T^\varepsilon \le y \le T^{1/3}$. By the truncated Voronoï formula (4.8), we have

$$\mathcal{R}_2 = (\sqrt{2}\,\pi)^{-1}x^{1/4} \sum_{y<n\le N} \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon}N^{-1/2}),$$

where $y < N \ll T$. Using Ivić's large-value technique directly to $\mathcal{R}_2$ without modifications, we get the estimate

$$\int_T^{2T} |\mathcal{R}_2|^{A_0}\,dx \ll T^{1+A_0/4+\varepsilon} \tag{4.18}$$

with $A_0 = 184/19, T^\varepsilon \le y \le T^{1/3}$. We omit the details since the argument is completely the same as that of Ivić. Combining (4.18) and (1.10) we get

$$\int_T^{2T} |\mathcal{R}_1|^{A_0}\,dx \ll T^{1+A_0/4+\varepsilon} \tag{4.19}$$

with $A_0 = 184/19$, $T^\varepsilon \le y \le T^{1/3}$.

By the same argument as in the last subsection, we deduce that for $T^\varepsilon \le y \le T^{1/3}$,

$$\int_T^{2T} \Delta^k(x)\,dx = \int_T^{2T} \mathcal{R}_1^k\,dx + O(T^{1+k/4+\varepsilon}y^{-\sigma(k,184/19)}). \tag{4.20}$$

Take $y = T^{1/(2b(k)+2\sigma(k,184/19))}$. From Lemma 4.1 again we get

$$(4.21)\quad \int_T^{2T} \Delta^k(x)\,dx = \frac{B_k(d)}{(\sqrt{2}\,\pi)^k 2^{k-1}} \int_T^{2T} x^{k/4}\,dx + O(T^{1+k/4-\frac{\sigma(k,184/19)}{2b(k)+2\sigma(k,184/19)}+\varepsilon})$$
$$= \frac{B_k(d)}{(\sqrt{2}\,\pi)^k 2^{k-1}} \int_T^{2T} x^{k/4}\,dx + O(T^{1+k/4-\delta_2(k,184/19)+\varepsilon}),$$

and Theorem 2 follows.

5. Proofs of other theorems. $P(x)$ has the following truncated Voronoï formula:

$$(5.1)\quad P(x) = -\frac{1}{\pi}\sum_{n\le N} r(n)n^{-3/4}x^{1/4}\cos(4\pi\sqrt{nx}+\pi/4)+O(x^{1/2+\varepsilon}N^{-1/2})$$

for $1 \le N \ll x$, which follows from Lemma 3 of Müller [16]. Moreover, $A(x)$ has the following truncated Voronoï formula:

$$(5.2)\quad A(x) = \frac{1}{\pi\sqrt{2}}x^{\kappa/2-1/4}\sum_{n\le N} a(n)n^{-\kappa/2-1/4}\cos(4\pi\sqrt{nx}-\pi/4) + O(x^{\kappa/2+\varepsilon}N^{-1/2})$$

for $1 \le N \ll x$, which is a special case of Theorem 1.1 of Jutila [13]. So in the same way as in the last section, we get Theorems 3 and 4.

Now we prove Theorem 5. We shall follow Ivić [10]. Define

$$\Delta^*(x) := \frac{1}{2}\sum_{n\le 4x}(-1)^n d(n) - x(\log x + 2\gamma - 1), \qquad x > 0.$$

Jutila [12] proved that

$$(5.3)\quad \int_0^T \left(E(t) - 2\pi\Delta^*\left(\frac{t}{2\pi}\right)\right)^2 dt \ll T^{4/3}\log^3 T,$$

which means that $E(t)$ is well approximated by $2\pi\Delta^*(t/2\pi)$ at least in the mean square sense.

Suppose $A_0 > 9$ is a real number such that both (1.10) and (1.32) hold. Since (see Jutila [11])

$$\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x),$$

from (1.10) we get

$$(5.4)\quad \int_0^T |\Delta^*(t)|^{A_0}\,dt \ll T^{1+A_0/4+\varepsilon}.$$

Then from (1.32), (5.3), (5.4) and Hölder's inequality, for any $3 \le k < A_0$ we get

$$(5.5)\qquad \int_0^T E^k(t)\,dt - (2\pi)^{k+1}\int_0^{T/2\pi}(\varDelta^*(t))^k\,dt$$
$$= \int_0^T\left(E^k(t) - \left(2\pi\varDelta^*\left(\frac{t}{2\pi}\right)\right)^k\right)dt$$
$$\ll \int_0^T\left|E(t) - 2\pi\varDelta^*\left(\frac{t}{2\pi}\right)\right|\left(|E(t)|^{k-1} + \left|\varDelta^*\left(\frac{t}{2\pi}\right)\right|^{k-1}\right)dt$$
$$\ll T^{1+k/4-\sigma(k,A_0)/3+\varepsilon},$$

where $\sigma(k, A_0)$ was defined in Section 1.1. By (5.5) the problem is reduced to evaluating the integral $\int_0^T(\varDelta^*(t))^k\,dt$. For $1 \ll N \ll x$, we have [10, (7)]

$$(5.6)\qquad \varDelta^*(x) = \frac{1}{\pi\sqrt{2}}\sum_{n\le N}(-1)^n d(n)n^{-3/4}x^{1/4}\cos(4\pi\sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon}N^{-1/2}),$$

which is similar to (4.8). Let $d^*(n) = (-1)^n d(n)$. Then in the same way as in the proof of Theorem 1, we get the asymptotic formula

$$(5.7)\qquad \int_1^T(\varDelta^*(t))^k\,dt = \frac{B_k(d^*)}{(1+k/4)2^{3k/2-1}\pi^k}\,T^{1+k/4} + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon})$$

for any $3 \le k < A_0$.

We shall use

Lemma 5.1. *Suppose $1 \le l < k$ are fixed integers and $(n_1, \ldots, n_k) \in \mathbb{N}^k$. If*

$$\sqrt{n_1} + \ldots + \sqrt{n_l} = \sqrt{n_{l+1}} + \ldots + \sqrt{n_k},$$

then $2\,|\,(n_1 + \ldots + n_k)$.

Proof. For any $n \in \mathbb{N}$, let $h(n)$ denote the squarefree part of n. Let $\mathcal{S} = \{h(n_1), \ldots, h(n_k)\} \cap \mathbb{N}$ and $s = \#\mathcal{S}$. For convenience, write

$$\mathcal{S} = \{h_1, \ldots, h_s\}, \quad I = \{1, \ldots, l\}, \quad J = \{l+1, \ldots, k\}.$$

From Lemma 2.1 we can write $I = \bigcup_{e=1}^s I_e$, $J = \bigcup_{e=1}^s J_e$ so that for each $1 \le e \le s$,

$$\sum_{i\in I_e}\sqrt{n_i} = \sum_{j\in J_e}\sqrt{n_j}$$

and all n_i $(i \in I_e)$ and n_j $(j \in J_e)$ have the same squarefree part h_e. Namely we have $(1 \le e \le s)$

$$n_i = m_i^2 h_e \ (i \in I_e), \quad n_j = m_j^2 h_e \ (j \in I_e), \quad \sum_{i\in I_e} m_i = \sum_{j \in J_e} m_j.$$

Thus we get

$$\begin{aligned} n_1 + \ldots + n_k &= \sum_{e=1}^{s} \Big(\sum_{i\in I_e} n_i + \sum_{j\in J_e} n_j \Big) \\ &= \sum_{e=1}^{s} \Big(\sum_{i\in I_e} m_i^2 h_e + \sum_{j\in J_e} m_j^2 h_e \Big) \equiv \sum_{e=1}^{s} \Big(\sum_{i\in I_e} m_i + \sum_{j\in J_e} m_j \Big) h_e \\ &= 2\sum_{e=1}^{s} h_e \sum_{i\in I_e} m_i \equiv 0 \ (\mathrm{mod}\, 2), \end{aligned}$$

where we used the simple congruence $n^2 \equiv n \ (\mathrm{mod}\, 2)$. ∎

From Lemma 5.1, for any $1 \le l < k$ we get

$$\begin{aligned} s_{k;l}(d^*) &= \sum_{\sqrt{n_1}+\ldots+\sqrt{n_l}=\sqrt{n_{l+1}}+\ldots+\sqrt{n_k}} (-1)^{n_1+\ldots+n_k} \frac{d(n_1)\ldots d(n_k)}{(n_1\ldots n_k)^{3/4}} \\ &= \sum_{\sqrt{n_1}+\ldots+\sqrt{n_l}=\sqrt{n_{l+1}}+\ldots+\sqrt{n_k}} \frac{d(n_1)\ldots d(n_k)}{(n_1\ldots n_k)^{3/4}} = s_{k;l}(d). \end{aligned}$$

Hence we conclude that

$$B_k(d^*) = B_k(d). \tag{5.8}$$

From (5.5), (5.7) and (5.8) we get (1.33).

Similarly to Theorem 2, we can prove the asymptotic formula

$$\int_1^T (\Delta^*(t))^k \, dt = \frac{B_k(d)}{(1+k/4)2^{3k/2-1}\pi^k} T^{1+k/4} + O(T^{1+k/4-\delta_2(k,576/61)+\varepsilon}) \tag{5.9}$$

for any $3 \le k \le 9$, which combined with (5.5) yields the second part of Theorem 3.

Note added in proof. Recently M. N. Huxley, *Exponential sums and lattice points III*, Proc. London Math. Soc. 87 (2003), 591–609, proved

$$\Delta(x) \ll x^{131/416} (\log x)^{26957/8320},$$

which implies that the exponent 184/19 for which the formula (1.10) holds can be improved to $A_0 = 262/27$. Correspondingly, the exponent $\delta_2(k, 184/19)$ in Theorem 2 can be improved to $\delta_2(k, 262/27)$ for $k = 6, 7, 8, 9$. The author deeply thanks Professor A. Schinzel for informing him about M. N. Huxley's new result.

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