Comment. Math. Helv. 83 (2008), 67–100 **Commentarii Mathematici Helvetici**

On Hilbert's 17th Problem for global analytic functions in dimension 3

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Abstract. Among the invariant factors g of a positive semidefinite analytic function f on $\mathbb{R}^3$, those g whose zero set Y is a curve are called *special*. We show that if each special g is a sum of squares of global meromorphic functions on a neighbourhood of Y, then f is a sum of squares of global meromorphic functions. Here sums can be (convergent) infinite, but we also find some sufficient conditions to get *finite* sums of squares. In addition, we construct several examples of positive semidefinite analytic functions which are infinite sums of squares but maybe could not be finite sums of squares.

Mathematics Subject Classification (2000). 11E25, 32B10, 32S05.

Keywords. Hilbert's 17th Problem, sum of squares, irreducible factors, special factors.

1. Introduction

The representation of positive semidefinite functions on a real variety as a sum of squares has attracted a lot of attention from specialists in number theory, quadratic forms, real algebra and real geometry; the problem goes back to the famous Hilbert's 17th Problem for polynomial functions. The solution of that problem (see [Ar]) was the starting point for the development of real methods in algebra and geometry. Such development led to the theory of the real spectrum due to Coste–Roy (for more details see [BCR]) which has been the suitable technique for an algebraic approach to many problems in real geometry.

This tool has been proved fruitful to understand and solve Hilbert's 17th Problem for polynomial functions, Nash functions, analytic function germs at points and compact sets, …, but it has fallen short to deal with global analytic functions in dimension $n \geq 3$ without compactness assumptions. Maybe this lack of a suitable machinery is the main reason why the problem for general global analytic functions has been apart from any substantial progress until now.

*Author supported by Spanish GAAR BFM2002-04797 and GAAR Grupos UCM 910444.

As it is well known, the problem is whether or not:

H17. Every positive semidefinite analytic function $f:\mathbb{R}^n\to\mathbb{R}$ is a finite sum of squares of meromorphic functions.

Let us mention the best result we can state today: *A positive semidefinite global analytic function whose zero set is discrete off a compact set is a finite sum of squares of meromorphic functions* ([BKS], [Rz], and [Jw2]; see also [ABR]). Such result dates back to the 80s. On the other hand, note that in the analytic setting infinite convergent sums have a meaning, which gives a new viewpoint on the problem (see [ABFR]). Nevertheless, the definition of an infinite sum of squares of analytic functions, which will be recalled later, must be done carefully to keep the analyticity of such sum. It is clear that an infinite sum of squares, whatever it means, is positive semidefinite and the classical Hilbert's 17th Problem can be weakened to ask whether:

h17. Every positive semidefinite analytic function $f:\mathbb{R}^n\to\mathbb{R}$ is an infinite sum of squares of meromorphic functions, that is, there exists a nonzero analytic function $g\in\mathcal{O}(\mathbb{R}^n)$ such that g^2f is an infinite sum of squares of analytic functions.

This is also a qualitative question, and suggests the study of the *finiteness* question for the field $\mathcal{M}(\mathbb{R}^n)$ of meromorphic functions of $\mathbb{R}^n$: *Is every infinite sum of squares in $\mathcal{M}(\mathbb{R}^n)$ also a finite sum of squares?* Obviously, H17 is equivalent to h17 plus finiteness. Quite remarkably, *finiteness is equivalent to the finiteness of the Pythagoras number $p_{\mathbb{R}^n}$ of the field $\mathcal{M}(\mathbb{R}^n)$ of meromorphic functions on $\mathbb{R}^n$* (see [ABFR]). We recall that $p_{\mathbb{R}^n}$ is either the least integer p such that every sum of squares in $\mathcal{M}(\mathbb{R}^n)$ can be written as a sum of p squares in $\mathcal{M}(\mathbb{R}^n)$ or $+\infty$ if such integer does not exists.

Now, let us fix some terminology. Given a closed set $Z\subset\mathbb{R}^n$ and an analytic function $f:\mathbb{R}^n\to\mathbb{R}$ we say that f *is a sum of squares at* Z if there exist an open neighbourhood $\Omega\subset\mathbb{R}^n$ of Z in $\mathbb{R}^n$ such that $f|_\Omega$ is a (possible infinite) sum of squares of meromorphic functions on Ω. One of the most relevant results in [ABFR] is the following:

($\bullet$) *To represent a positive semidefinite analytic function f as a sum of squares it suffices to represent it at $X=f^{-1}(0)$.*

In this work we go further and we search the obstructions for a positive semidefinite analytic function f to have the following property:

($*$) *To represent f as a sum of squares it suffices to represent its irreducible factors at their respective zero sets.*

The most satisfactory results hold for dimension 3 and, in fact, we will prove that ($*$) holds for $\mathbb{R}^3$. Furthermore, notice that to represent as sum of squares each irreducible factor at its zero set is much less than to represent f at its zero set.

Of course, first of all we have to define carefully the *irreducible factors* of a real analytic function on $\mathbb{R}^n$. For that, it is crucial to introduce the irreducible factors of a holomorphic function $F\colon U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}^n$ and to recall some of their main properties. Throughout the paper holomorphic functions will refer to the complex case and analytic functions to the real case. In both cases the notion of irreducibility is very similar, however the behaviours are extremely different.

Given an open set $\Omega \subset \mathbb{R}^n$, we will say that an analytic function $f \in \mathcal{O}(\Omega)$ is *irreducible* if it cannot be written as the product of two analytic functions with nonempty zero set. Analogously, a holomorphic function $F \in \mathcal{H}(U)$ on an open set $U \subset \mathbb{C}^n$ of $\mathbb{C}^n$ is *irreducible* if it cannot be written as the product of two holomorphic functions on U with nonempty zero set. We recall also that an analytic set X of U is *irreducible* if it cannot be written as the union of two global analytic sets $X_1, X_2 \subset X$ both different from X.

First, we consider holomorphic functions. Here irreducibility behaves neatly. If every locally principal sheaf of ideals on U is principal (which happens for instance if $H^2(U,\mathbb{Z}) = 0$) then there exists a bijection between the irreducible analytic sets of U of codimension 1 and the principal prime ideals of the ring $\mathcal{H}(U)$ of holomorphic functions on U.

Next, we turn to the real case. The situation for the irreducible functions of $\mathcal{O}(\mathbb{R}^n)$ is completely different and the behaviour of the zero set of an irreducible function is unpredictable. The zero set of an irreducible function of $\mathcal{O}(\mathbb{R}^n)$ can have any dimension; for instance, if $2 \le k \le n$ the analytic function $f_k(x) = f_k(x_1,\dots,x_n) = x_1^2+\dots+x_k^2$ is irreducible in $\mathcal{O}(\mathbb{R}^n)$ but its zero set has dimension $n-k$. Furthermore, there exist irreducible analytic functions of $\mathcal{O}(\mathbb{R}^n)$ with the same zero set but which do not generate the same ideal of $\mathcal{O}(\mathbb{R}^n)$. Take, for instance, $f_1(x) = x_1^2 + x_2^2$ and $f_2(x) = x_1^2 + 4x_2^2$, whose common zero set is $\{x_1 = 0, x_2 = 0\}$. Even more, as we will see in Section 2, we can produce examples of real analytic functions which are irreducible but whose zero set is *reducible*, and which can even have infinitely many irreducible components.

Thus, one is led to define the *irreducible factors* of a real analytic function f through the irreducible factors of a holomorphic extension F of f to a suitable open neighbourhood U of $\mathbb{R}^n$ in $\mathbb{C}^n$. As usual the *uniqueness* of the irreducible factors will be up to multiplication by units of the respective ring, $\mathcal{O}(\mathbb{R}^n)$ or $\mathcal{H}(U)$, that is, never vanishing analytic or holomorphic functions.

The process to construct the irreducible factors of a real analytic function f will be developed carefully in Section 2, but we can describe roughly the main steps.

(i) First, we consider a holomorphic extension $F\colon \mathcal{U} \to \mathbb{C}$ of f to a suitable open neighbourhood $\mathcal{U}$ of $\mathbb{R}^n$ in $\mathbb{C}^n$, invariant under conjugation, and decompose $S = F^{-1}(0)$ as the union of its irreducible components $\{S_i\}_{i\in I}$. We show that we can assume that $S_i \cap \mathbb{R}^n \neq \emptyset$ for all $i \in I$.

(ii) Next, for each S_i we construct a holomorphic function H_i which generates the ideal sheaf $\mathcal{J}_{S_i}$ of S_i. For those S_i which are invariant under conjugation we prove that the holomorphic function H_i can be chosen to restrict on $\mathbb{R}^n$ to a real analytic function.

(iii) The holomorphic function germs $\{H_{i,\mathbb{R}^n}\}_{i\in\mathcal{I}}$ at $\mathbb{R}^n$, which are unique, will be called the *irreducible factors of* f.

Moreover, $f \in \mathcal{O}(\mathbb{R}^n)$ is *irreducible* if either (1) f has one irreducible complex factor, whose zero set germ at $\mathbb{R}^n$ is invariant under conjugation, or (2) two irreducible complex factors whose respective zero set germs at $\mathbb{R}^n$ are conjugated. In case (1) F is irreducible, and in case (2) F is reducible.

Given analytic functions $g, f \in \mathcal{O}(\mathbb{R}^n)$, we say that g *divides* f (*in* $\mathcal{O}(\mathbb{R}^n)$) *with multiplicity* $k \geq 1$ if g^k divides f but g^{k+1} does not. As it can be checked, taking germs at any point of $\mathbb{R}^n$ at which both f and g vanish, if g divides f, there exist an integer $k \geq 1$ with the previous property. An irreducible factor $h \in \mathcal{O}(\mathbb{R})$ of an analytic function $f \in \mathcal{O}(\mathbb{R}^n)$ is *special* if the zero set germ at $\mathbb{R}^n$ of a holomorphic extension of h is invariant under conjugation, it divides f with odd multiplicity and $1 \leq \dim h^{-1}(0) \leq n-2$.

In close relation to the irreducible factors of *positive semidefinite* analytic functions we will prove in Section 2 the following *decomposition* result that will be crucial for our purposes.

Lemma 1.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a positive semidefinite analytic function. Then there exist analytic functions $f_0, f_1, f_2, f_3\colon \mathbb{R}^n \to \mathbb{R}$ such that f_1, f_2, f_3 are positive semidefinite, $f = f_0^2 f_1 f_2 f_3$ and*

(i) *$f_1^{-1}(0)$ is a discrete set (hence, by* [BKS], *f_1 is a finite sum of squares of meromorphic functions on $\mathbb{R}^n$),*

(ii) *f_2 is a sum of two squares of analytic functions on $\mathbb{R}^n$, and*

(iii) *the irreducible factors of f_3 are all special and divide f_3 with multiplicity one.*

In fact, we also see that the irreducible factors of f_3 are the *special irreducible factors of f* or just the *special factors of f*. Moreover, if $n \leq 2$ we may take $f_3 \equiv 1$ in Lemma 1.1. Hence, we get that f is a finite sum of squares of meromorphic functions (this is of course well known: [BKS] and [Jw1]). Thus, in what follows we may assume $n \geq 3$.

Next, we recall the suitable definition of infinite sums of squares introduced in [ABFR]:

Definition 1.2. Let $\Omega \subset \mathbb{R}^n$ be an open set. An *infinite sum of squares of analytic functions on* Ω is a series $\sum_{k\geq 1} f_k^2$ where all $f_k \in \mathcal{O}(\Omega)$, such that:

(i) the f_k's have holomorphic extensions F_k's, all defined on the same neighborhood U of Ω in $\mathbb{C}^n$, and

(ii) for every compact set $L \subset U$, $\sum_{k\geq 1} \sup_L |F_k|^2 < +\infty$.

Accordingly, the infinite sum of squares $\sum_{k\geq 1} f_k^2$ defines well an analytic function f on $\Omega = U \cap \mathbb{R}^n$ and we write $f = \sum_{k\geq 1} f_k^2 \in \mathcal{O}(\Omega)$; of course, this trivially includes finite sums. Hence, it makes sense to say that an element of the ring $\mathcal{O}(\Omega)$ is a sum of p squares in $\mathcal{O}(\Omega)$, even for $p = +\infty$. We recall that an analytic function $f : \Omega \to \mathbb{R}$ is a *sum of* $p \leq +\infty$ *squares* (of meromorphic functions on Ω) if there is $g \in \mathcal{O}(\Omega)$ such that $g^2 f$ is a sum of p squares of analytic functions on Ω. The zero set $\{g = 0\}$ is called the *bad set* of that representation as a sum of squares. The choice of a suitable sum of squares representation will be a crucial matter and we will need often to have a *controlled bad set,* that is, a bad set contained in the zero set $\{f = 0\}$. Concerning the difference between arbitrary and controlled bad sets, we recall this

Proposition 1.3 ([ABFR, 4.1]). *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \to \mathbb{R}$ be an analytic function which is a sum of $p \leq +\infty$ squares of meromorphic functions. Then f is a sum of $q \leq 2^n p$ squares with controlled bad set. Moreover, on a smaller neighborhood of $\{f = 0\}$ we can assume $q \leq 2^{n-1} p$.*

Our main result here is the following:

Theorem 1.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positive semidefinite analytic function and let $\{h_j\}_{j\in J}$ be the special irreducible factors of f. Assume that for each $j \in J$ the positive semidefinite analytic function h_j is a (possibly infinite) sum of squares of meromorphic functions at $X_j = h_j^{-1}(0)$. Let $\{Y_i\}_{i\in I}$ be the family of the irreducible components of the global analytic set $X = \bigcup_{j\in J} X_j$. Suppose that one of the two following conditions holds true:*

(a) *$Y_i \cap Y_k$ is a discrete set for $i \neq k$.*

(b) *Y_i is a compact set for all $i \in I$.*

Then f is a possibly infinite sum of squares of meromorphic functions on $\mathbb{R}^n$ with controlled bad set.

The proof of the previous result goes along the same lines of the one of [ABFR, 1.5], but there are several aspects that go far beyond a mere updating of [ABFR, 1.5]. Moreover, one of the main difficulties for the proof of Theorem 1.4 and the reason why the hypotheses (a) and/or (b) appear in its statement is that it cannot exist a *general formula to multiply infinitely many sums of squares*; even, if these sums of squares are finite.

On the other hand, if $n = 3$, then the condition (a) in the statement of Theorem 1.4 is always satisfied, since $\dim X = 1$, and we get the following relevant consequence:

(∗∗) *To represent a positive semidefinite analytic function* $f\colon \mathbb{R}^3 \to \mathbb{R}$ *as a sum of squares it is enough to represent its special factors at their zero sets.*

Thus, the problem h17 for $\mathbb{R}^3$ is reduced to study if every *special* positive semidefinite irreducible analytic function on $\mathbb{R}^3$ is a sum of squares of meromorphic functions. Moreover, from Theorem 1.4 (b) and [ABR, VIII.5.8] one gets almost straightforwardly the following:

Corollary 1.5. *Let* $f\colon \mathbb{R}^n \to \mathbb{R}$ *be a positive semidefinite analytic function and let* $\{h_j\}_{j\in J}$ *be its special factors. Suppose that for all* $j \in J$*, the set* $X_j = h_j^{-1}(0)$ *is compact. Then* f *is a (possibly infinite) sum of squares of meromorphic functions in* $\mathbb{R}^n$.

The previous result points out that the obstruction to be an infinite sum of squares concentrates on the special irreducible factors whose zero set is not compact.

Concerning finite sums of squares the situation is quite more delicate and we only have some partial results.

Theorem 1.6. *Let* $r \geq 0$ *be an integer and let* $f\colon \mathbb{R}^n \to \mathbb{R}$ *be a positive semidefinite analytic function. Let* $\{h_k\}_{k\in K}$ *be the special factors of* f *and let* $X_k = h_k^{-1}(0)$. *Suppose that each proper intersection* $X_k \cap X_\ell$ *is a discrete set. If* h_k *is a sum of* 2^r *squares at* X_k *for all* $k \in K$*, then* f *is a sum of* 2^{r+n} *squares.*

In fact, the previous result can be improved if we find a suitable distribution of the special factors. Namely,

Corollary 1.7. *Let* $r \geq 0$ *be an integer and let* $f\colon \mathbb{R}^n \to \mathbb{R}$ *be a positive semidefinite analytic function. Let* $\{h_k\}_{k\in K}$ *be the special factors of* f *and let* $X_k = h_k^{-1}(0)$. *Suppose that there exists a partition* $\mathcal{P} = \{A_1, \dots, A_m\}$ *of* K *such that each proper intersection* $X_k \cap X_\ell$*, where* k, ℓ *belong to the same* A_j*, is a discrete set. If* h_k *is a sum of* 2^r *squares at* X_k *for all* $k \in K$*, then* f *is a sum of* 2^{r+n} *squares.*

As we will show in Section 3, this more technical statement allows us to represent as finite sum of squares certain positive semidefinite analytic functions to which Theorem 1.6 does not apply. Although the situation described in Corollary 1.7 is quite general for positive semidefinite functions on $\mathbb{R}^3$, we also construct in Section 3 two examples of positive semidefinite analytic functions on $\mathbb{R}^3$ to which we can apply Theorem 1.4 (hence, they are infinite sums of squares) but to which we cannot even apply our best result Corollary 1.7 about finite sum of squares:

- The first function f has the following properties: (1) *the zero sets of all its special factors, which are infinitely many, have all infinitely many irreducible components;* (2) *its special factors are sums of four squares in* $\mathcal{O}(\mathbb{R}^3)$*;* and

(3) *it has a holomorphic extension F to an open neighbourhood U of* $\mathbb{R}^3$ *in* $\mathbb{C}^3$ *which is the uniform limit of a sequence* $\{F_k\}_k$ *of sums of four squares of holomorphic functions on U whose restrictions to* $\mathbb{R}^3$ *are real analytic functions*.[1]

However, any two of the special factors of this function share infinitely many irreducible components, which makes Corollary 1.7 useless.

- The second function f has the following properties: (1) *the zero sets of all its special factors, which are infinitely many, have all finitely many irreducible components;* (2) *its special factors are sums of four squares in* $\mathcal{O}(\mathbb{R}^3)$*;* (3) *each irreducible component of the zero set of f is contained in no more than two of the zero sets of its special factors;* and (4) *it is the uniform limit of sums of four squares in* $\mathcal{O}(\mathbb{R}^3)$ *in the sense of the previous example.*

  Nevertheless, there does not exist an integer s such that the number of irreducible components of the zero set of each special factor is bounded by s. Again, as we will see later, Corollary 1.7 is useless here.

For the moment, we do not know whether or not the previous examples are finite sums of squares of meromorphic functions on $\mathbb{R}^3$. Both examples have been constructed with the purpose of having a measure of the limitations of Corollary 1.7. This is done avoiding the hypothesis of Corollary 1.7 about the *distribution* of the zero sets of the special factors, while keeping all the other hypotheses that seem essential to have a finite sum. In fact, the second example avoids the hypothesis about the distribution of the zero sets in a quite subtle way.

The relevance of such examples arises from the way and the purpose for what they have been constructed. In fact, they seem to be at the *border* between finite and infinite sums of squares. Thus, they are suitable candidates to be counterexamples to Hilbert's 17th Problem for global analytic functions. On the other hand, if one is able to prove that some or both of them are finite sums of squares, it seems plausible to find relevant information to prove some version of Theorem 1.4 for finite sums of squares.

The paper is organized as follows. In Section 2 we prove some key results concerning the definition and computation of irreducible factors of a real analytic function and the decomposition of positive semidefinite analytic functions described in Lemma 1.1. Section 3 is devoted to the introduction and development of the examples previously mentioned. Finally, Theorems 1.4 and 1.6, are proved in Section 4.

The author would like to thank Prof. J. M. Ruiz for friendly helpful discussions during the preparation of this work.

[1] As it is well known, the best way to guarantee the analyticity of the uniform limit of a sequence of real analytic functions on $\mathbb{R}^n$ is to consider only sequences of such functions which have holomorphic extensions to a common open neighbourhood of $\mathbb{R}^n$ in $\mathbb{C}^n$.

2. Irreducible factors

We gather here some notations and technical lemmas for later purposes. Although our problem concerns real analytic functions, we will of course use some complex analysis. For holomorphic functions we refer the reader to the classical [GuRo].

2.1 General terminology. Denote the coordinates in $\mathbb{C}^n$ by $z = (z_1, \dots, z_n)$, with $z_i = x_i + \sqrt{-1}y_i$, where $x_i = \mathrm{Re}(z_i)$ and $y_i = \mathrm{Im}(z_i)$ are respectively the *real* and the *imaginary parts* of z_i. Consider the usual conjugation $\sigma\colon \mathbb{C}^n \to \mathbb{C}^n$, $z \mapsto \overline{z} = (\overline{z_1}, \dots, \overline{z_n})$, whose fixed points are $\mathbb{R}^n$. A subset $A \subset \mathbb{C}^n$ is (σ-) *invariant* if $\sigma(A) = A$; obviously, $A \cap \sigma(A)$ is the biggest invariant subset of A. Thus, we see real spaces as subsets of complex spaces. The notations Int and Cl stand for topological interiors and closures, respectively.

Let $U \subset \mathbb{C}^n$ be an invariant open set and let $F\colon U \to \mathbb{C}$ be a holomorphic function. We say that F is (σ-) *invariant* if $F(z) = \overline{F(\overline{z})}$. This implies that F restricts to a real analytic function on $U \cap \mathbb{R}^n$. In general, we denote by

$$\Re(F)\colon U \to \mathbb{C}, \qquad z \mapsto \frac{F(z) + \overline{F(\overline{z})}}{2},$$

and

$$\Im(F)\colon U \to \mathbb{C}, \qquad z \mapsto \frac{F(z) - \overline{F(\overline{z})}}{2\sqrt{-1}},$$

the *real* and the *imaginary* parts of F, which satisfy $F = \Re(F) + \sqrt{-1}\,\Im(F)$. Note that both are invariant holomorphic functions.

Given a closed set $Z \subset \mathbb{C}^n$, germs (of sets or of holomorphic functions) at Z are defined exactly as germs at a point, through neighborhoods of Z in $\mathbb{C}^n$; we will denote by F_Z the germ at Z of a holomorphic function F defined in some neighborhood of Z. For instance, if $F\colon U \to \mathbb{C}$ is an invariant holomorphic function such that $\mathbb{R}^n \subset U$ and $Z = \mathbb{R}^n$, then the germ F_Z is the same as the real analytic function $f = F|_{\mathbb{R}^n}$.

In [ABFR, 2.3] we showed how to extend a convergent sum of squares of holomorphic functions modulo another. Here such result will be again a powerful tool and we recall the precise statement for the sake of the reader.

Proposition 2.2. *Let $\mathcal{U}$ be an invariant open Stein neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ and let $\Phi\colon \mathcal{U} \to \mathbb{C}$ be an invariant holomorphic function. Let V be an open invariant neighborhood of the connected components of $\Phi^{-1}(0)$ that meet $\mathbb{R}^n$, and suppose that V does not meet the other connected components of $\Phi^{-1}(0)$. Let $C_k\colon V \to \mathbb{C}$ be a family of invariant holomorphic functions such that $\sum_k \sup_L |C_k|^2 < +\infty$ for every compact set $L \subset V$. Then, there exist invariant holomorphic functions $A_k\colon \mathcal{U} \to \mathbb{C}$, such that $\sum_k \sup_K |A_k|^2 < +\infty$ for every compact set $K \subset \mathcal{U}$ and $\Phi|_V$ divides all the differences $A_k|_V - C_k$.*

2.3 Complex irreducible factors of a real analytic function. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be an analytic function. We define the *irreducible complex factors* of f as follows.

Let $F\colon U_0 \to \mathbb{R}$ be a holomorphic extension of f to a small enough open neighborhood U_0 of $\mathbb{R}^n$ in $\mathbb{C}^n$. By [WBh, 6. Proposition 9], there exists a unique locally finite family of irreducible germs $\{S_i\}_{i\geq 1}$ at $\mathbb{R}^n$ such that $S_i \not\subset S_j$ if $i \neq j$ and $F^{-1}(0)_{\mathbb{R}^n} = \bigcup_{i\in I} S_i$. By [WBh, 6. Prop. 8, Cor. 2], for each $i \in I$ there exists an open neighborhood U_i of $\mathbb{R}^n$ in $\mathbb{C}^n$ and an irreducible analytic set T_i in U_i such that $T_{i,\mathbb{R}^n} = S_i$. Shrinking the open sets U_i (if necessary), we may assume that the family $\{T_i\}_{i\in I}$ is locally finite in $\mathbb{C}^n$. In fact, we can take the U_i's arbitrarily small. Consider the open set in $\mathbb{C}^n$:

$$U = \Big(\mathbb{C}^n \setminus \bigcup_{j\geq 1} \mathrm{Cl}_{\mathbb{C}^n}(T_j)\Big) \cup \bigcup_{j\geq 1} \Big(U_j \setminus \bigcup_{i\neq j} (\mathrm{Cl}_{\mathbb{C}^n}(T_i) \setminus U_i)\Big).$$

A straightforward computation shows that $\mathbb{R}^n \subset U$ and that $T_i \cap U$ is closed in U for all $i \in I$. Hence, for each $i \in I$ there exists an analytic set $T_i' \subset T_i$ in U, which for simplicity we denote again by T_i, such that $T_{i,\mathbb{R}^n} = S_i$. Let $\mathcal{U} \subset U$ be an open invariant Stein neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ such that $\mathbb{R}^n$ is a deformation retract of $\mathcal{U}$ ([Ca]). Denote again by T_i the intersection $T_i \cap \mathcal{U}$. Taking the connected component of T_i that intersects $\mathbb{R}^n$ instead of T_i we may also assume that T_i is irreducible.

Fix $i \in I$. Since the $\dim T_i = n-1$, for each $z \in T_i$ there exists a holomorphic function germ $h_{i,z} \in \mathcal{O}(\mathbb{C}^n_z)$ that generates the ideal of the analytic germ $T_{i,z}$. Consider the subsheaf $\mathcal{J}^{(i)}$ of the structure sheaf $\mathcal{O}_{\mathcal{U}}$ defined by

$$\mathcal{J}^{(i)}_z = \begin{cases} h_{i,z}\mathcal{O}(\mathbb{C}^n_z) & \text{if } z \in T_i, \\ \mathcal{O}(\mathbb{C}^n_z) & \text{if } z \notin T_i. \end{cases}$$

Note that $\mathcal{J}^{(i)}$ is a locally principal coherent ideal sheaf. Hence, it defines a cocycle in $H^1(\mathcal{U}, \mathcal{O}^*_{\mathbb{C}})$. Since $\mathcal{U}$ is a Stein manifold, this group is isomorphic to $H^2(\mathcal{U}, \mathbb{Z})$, which is 0 since $\mathbb{R}^n$ is a deformation retract of $\mathcal{U}$. Hence, $\mathcal{J}^{(i)}$ is in fact a principal ideal sheaf, say generated by a holomorphic function $H_i\colon \mathcal{U} \to \mathbb{C}$. We also have, by the definition of $\mathcal{J}^{(i)}$, that $H_i^{-1}(0) = T_i$ and that $h_{i,z}\mathcal{O}(\mathbb{C}^n)_z = H_{i,z}\mathcal{O}(\mathbb{C}^n)_z$ for all $z \in T_i$. Thus, the germ $S_i = T_{i,\mathbb{R}^n}$ is determined by the holomorphic function H_i. Moreover, since $F^{-1}(0) = \bigcup_{i\in I} T_i$, each H_i divides F.

Furthermore, since the germs S_i are uniquely determined by F and the function germ at $\mathbb{R}^n$ of each H_i is uniquely determined by S_i, the holomorphic function germs $H_{i,\mathbb{R}^n}$ are uniquely determined by $f \equiv F_{\mathbb{R}^n}$. Thus, we will say that $\{H_{i,\mathbb{R}^n}\}$ are the (*complex*) *irreducible factors of* f.

Next we claim: *If S_i is invariant, we may assume that H_i is also invariant.*

Indeed, since S_i is invariant, the function $\overline{H_i \circ \sigma}$ has the same properties as H_i. Hence, there exists a holomorphic function $A_i\colon \mathcal{U} \to \mathbb{C}$ not vanishing on $\mathcal{U}$ such that $\overline{H_i \circ \sigma} = H_i A_i$. Thus, we have that

$$H_i = \overline{\overline{H_i \circ \sigma}}\ \overline{\overline{A_i \circ \sigma}} = H_i A_i \overline{A_i \circ \sigma}$$

and therefore $A_i \overline{A_i \circ \sigma} = 1$.

Now, let $B_i : \mathcal{U} \to \mathbb{C}$ be a holomorphic function such that $B_i^2 = A_i$ and $B_i \overline{B_i \circ \sigma} = 1$. Indeed, $(B_i \overline{B_i \circ \sigma})^2 = A_i \overline{A_i \circ \sigma} = 1$; hence, $B_i \overline{B_i \circ \sigma} = \pm 1$. Since the function $B_i \overline{B_i \circ \sigma}$ restricts on $\mathbb{R}^n$ to a sum of two squares in $\mathcal{O}(\mathbb{R}^n)$, we deduce that $B_i \overline{B_i \circ \sigma} = 1$.

Then $H_i' = H_i B_i : \mathcal{U} \to \mathbb{C}$ is invariant:

$$\overline{H_i' \circ \sigma} = \overline{H_i \circ \sigma}\ \overline{B_i \circ \sigma} = \overline{H_i \circ \sigma}\ \overline{B_i \circ \sigma}^2 B_i = \overline{H_i \circ \sigma}\ \overline{A_i \circ \sigma} B_i = H_i B_i = H_i'.$$

Moreover, it is clear that H_i' satisfies the same properties as H_i with respect to the germ S_i. To simplify notations we denote again H_i' by H_i. Recall that, H_i being invariant, its restriction to $\mathbb{R}^n$ is a real analytic function.

Definitions 2.4. (a) We say that $H_{i,\mathbb{R}^n}$ is a *special irreducible* (*complex*) *factor* of f or just a *special factor* of f if the germ of $H_i^{-1}(0)$ at $\mathbb{R}^n$ is invariant, the dimension d of the real analytic set $H_i^{-1}(0) \cap \mathbb{R}^n$ satisfies the inequalities $1 \leq d \leq n-2$ (which for $n = 3$ gives $d = 1$) and H_i divides F with odd multiplicity. Moreover, since $H_i^{-1}(0) \cap \mathbb{R}^n$ has dimension $\leq n-2$, we may also assume that the special factors H_i of f are invariant and that $h_i = H_{i,\mathbb{R}^n}$ is a real positive semidefinite analytic function.

(b) If f has only one irreducible (complex) factor and it is special, we say that f is a *special* analytic function.

We recall that a global analytic set (in an open set Ω of $\mathbb{R}^n$) is *irreducible* if it cannot be written as the union of two global analytic sets different from itself. By [WBh, §8. Prop. 11] any global analytic set X in an open set Ω of $\mathbb{R}^n$ can be written as the union of a unique irredundant locally finite family of irreducible global analytic sets X_i with $X = \bigcup_i X_i$.

Examples 2.5. (a) $f(x, y, z) = (x^2 + y^2)^2 z^2 + x^6 + y^6$ defines a special analytic function whose real zero set is $\{x = 0, y = 0\}$, which is irreducible.

(b) $f(x, y, z) = x^4 y^2 + y^4 z^2 + z^4 x^2 - 3x^2 y^2 z^2$ (Motzkin's polynomial) defines a special analytic function whose real zero set is $\{x^4 y^2 = y^4 z^2 = z^4 x^2\}$, that is,

$$\begin{aligned}\{x = 0, y = 0\} \cup \{x = 0, z = 0\} \cup \{y = 0, z = 0\}\\ \cup \{y = \pm x, z = \pm x\} \cup \{y = \pm x, z = \mp x\},\end{aligned}$$

which is reducible.

To introduce more exotic examples we need the following result:

Lemma 2.6. *The homogeneous polynomial* $F(x,y,z) = (z+ax)^2(z+by)^2 + z^4 + c^2x^2y^2 \in \mathbb{R}[x,y,z]$ *is irreducible in* $\mathbb{C}[x,y,z]$ *for all* $a,b,c \in \mathbb{R}$ *such that* $a,b,c \neq 0$, $c^2 \neq a^2b^2$.

Proof. First, note that the zero set of F in $\mathbb{R}^3$ is the union of the two lines $x=0, z=0$ and $y=0, z=0$. Next, we write

$$F = Ax^2 + 2azB^2x + z^2C,$$

where

$$\begin{aligned} A &= a^2z^2 + 2a^2byz + (a^2b^2+c^2)y^2 \\ &= (az+aby+\sqrt{-1}\,cy)(az+aby-\sqrt{-1}\,cy), \\ B &= z+by, \\ C &= (z+by)^2+z^2 = (z+by+\sqrt{-1}\,z)(z+by-\sqrt{-1}\,z). \end{aligned}$$

Let us show now that if $a,b,c,a^2b^2-c^2 \neq 0$, then F is irreducible. Suppose that F is reducible.

First, since $\gcd(A, zB, z^2C) = 1$ (because $\gcd(B,C) = 1$ and z does not divide A) we have that F cannot be written as $F = G_1G_2$, where $G_1 \in \mathbb{C}[x,y,z]$ is a polynomial of degree 0 with respect to x. Next, we see that F cannot be written as the product of two linear real factors with respect to x, namely,

$$F = (\alpha_1x+\beta_1)(\alpha_2x+\beta_2)$$

where $\alpha_i, \beta_i \in \mathbb{R}[y,z]$. If this were the case, the set $\{\alpha_1x+\beta_1 = 0\} \cap \mathbb{R}^3$, which has dimension 2, would be a subset of $\{F=0\} \cap \mathbb{R}^3$, which has dimension 1, a contradiction.

Thus, if F is reducible, it has two conjugated roots in $\mathbb{C}(y,z)$, namely,

$$\frac{-azB^2 \pm z\sqrt{a^2B^4-AC}}{A}.$$

Hence, $\sqrt{a^2B^4-AC} \in \sqrt{-1}\mathbb{R}(y,z)$ and, in fact, since $\mathbb{R}[y,z]$ is a normal domain, we have that $\sqrt{AC-a^2B^4} \in \mathbb{R}[y,z]$. Therefore, $AC - a^2B^4 = H^2$, where $H \in \mathbb{R}[y,z]$ is a quadratic form. Thus, $AC = a^2B^4 + H^2$ and looking at the factors of A and C we essentially have the following possibilities:

(i) C divides $aB^2 + iH$. Since $C \in \mathbb{R}[y,z]$, we have that C divides B^2, a contradiction.

(ii) $E_1 = (az + aby + \sqrt{-1}\,cy)(z + by + \sqrt{-1}\,z)$ divides $aB^2 + iH$, that is, $aB^2 + iH = \eta E_1$ for some $\eta \in \mathbb{C}$. Then, there exists $\lambda, \mu \in \mathbb{R}$ such that

$$\begin{aligned}\lambda(az^2 + (2ab - c)yz + b^2ay^2) + \mu(az^2 + (ab + c)yz + cby^2)\\ = aB^2 = a(z + by)^2 = az^2 + 2abyz + ab^2y.\end{aligned}$$

Thus, $\lambda + \mu = 1$, $\lambda(2ab - c) + \mu(ab + c) = 2ab$ and $\lambda ab^2 + \mu cb = ab^2$. Therefore,

$$\lambda = 1 - \mu, \quad \mu(2c - ab) = c, \quad b\mu(c - ab) = 0.$$

Hence, b, c being nonzero, we deduce that $c = ab$, a contradiction.

(iii) $E_2 = (az + aby + \sqrt{-1}\,cy)(z + by - \sqrt{-1}\,z)$ divides $aB^2 + iH$. Proceeding as in the previous case, we deduce that $c = -ab$, a contradiction.

Whence, we conclude that F is irreducible in $\mathbb{C}[x, y, z]$. □

Example 2.7. *Let $H : \mathbb{C} \to \mathbb{C}$ be an invariant holomorphic function such that $H^{-1}(0) = \{k \in \mathbb{Z} : k \geq 0\}$ and H has a zero of order one at each point of its zero set. Such a function exists by the Weierstrass Factorization Theorem. Let $a_k = H'(k) \neq 0$ for all integer $k \geq 0$ and let $M > 0$ be a real number such that $M^2 \neq \frac{1}{a_k^2 a_\ell^2}$ for all couple of integers $k, \ell \geq 0$. Let*

$$F(x, y, z) = (z + \sin(\pi x))^2(z + \sin(\pi y))^2 + z^4 + M^2H(x)^2H(y)^2,$$

which is an invariant holomorphic function on $\mathbb{C}^3$. Let us show that $f = F|_{\mathbb{R}^3}$ is a special analytic function whose real zero set is the net

$$S = \bigcup_{k\geq 0}\{x = k, z = 0\} \cup \bigcup_{\ell \geq 0}\{y = \ell, z = 0\}$$

which has infinitely many irreducible components.

Proof. Indeed, a straightforward computation shows that $\{F = 0\} \cap \mathbb{R}^n = S$. To show that f is a special analytic function we have to check that the restrictions of F to small enough invariant neighbourhoods U of $\mathbb{R}^n$ in $\mathbb{C}^n$ cannot be written as the product of two holomorphic functions $G_1, G_2 : U \to \mathbb{C}$ such that $G_i^{-1}(0) \cap \mathbb{R}^n \neq \emptyset$.

First, we show some crucial properties of F to prove the irreducibility of f.

(a) For each pair of non-negative integers $k, \ell \geq 0$ consider the point $p_{k,\ell} = (k, \ell, 0)$. Then the function germs $F_{p_{k,\ell}}$ are irreducible in the ring $\mathcal{O}_{\mathbb{C}^3, p_{k,\ell}}$ for all k, ℓ. This is because their initial forms at the points $p_{k,\ell}$ are irreducible by Lemma 2.6.

(b) For each integer $k \geq 0$ and each $\lambda \in \mathbb{R} \setminus \mathbb{Z}$ consider the point $p_{k,\lambda} = (k, \lambda, 0)$. Then, for all k, λ as before the function germs $F_{p_{k,\lambda}}$ are the product in $\mathcal{O}_{\mathbb{C}^3, p_{k,\lambda}}$ of two irreducible factors of order 1 that vanish at the line germ $x = k, z = 0$.

Indeed, after translating the point $p_{k,\lambda}$ to the origin, the initial form of $F_{p_{k,\lambda}}$ is $(z+ax)^2b^2 + c^2x^2$ for some real number $a, b, c > 0$. Thus, by classification of singularities, $F_{p_{k,\lambda}}$ is analytically equivalent either to $x^2 + z^2$ or to a polynomial of the type $x^2 + z^2 + \varepsilon y^k$ where $k \geq 3$ and $\varepsilon = \pm 1$. Since the zero set of $F_{p_{k,\lambda}}$ is the line germ $x = k, z = 0$, we conclude that $F_{p_{k,\lambda}}$ is analytically equivalent to $x^2 + z^2$. Hence

$$F_{p_{k,\lambda}} = F_1^2 + F_2^2 = (F_1 + \sqrt{-1}F_2)(F_1 - \sqrt{-1}F_2),$$

where the factors $F_1 + \sqrt{-1}F_2$, $F_1 - \sqrt{-1}F_2$ have order 1, are irreducible and vanish at the line germ $x = k, z = 0$.

Suppose now that there exist an open invariant neighbourhood U of $\mathbb{R}^n$ in $\mathbb{C}^n$ and two holomorphic functions $G_1, G_2 \colon U \to \mathbb{C}$ such that $G_i^{-1}(0) \cap \mathbb{R}^n \neq \emptyset$ and $F = G_1 G_2$. Then, for each $p \in S = F^{-1}(0) \cap \mathbb{R}^n$ we have that $F_p = G_{1,p} G_{2,p}$. Since the line $x = 0, z = 0$ is irreducible and it is contained in $F = 0$, we may assume that it is also contained in $G_1 = 0$. As the germ F_p is irreducible for the points $p = p_{0,\ell} = (0, \ell, 0)$, we have that all the lines $y = \ell, z = 0$ are contained in $G_1 = 0$. Furthermore, since the germ F_p is irreducible for the points $p = p_{k,0} = (k, 0, 0)$, we have that all the lines $x = k, z = 0$ are contained in $G_1 = 0$. Hence, S is a subset of $G_1 = 0$. Again, since the germ F_p is irreducible for the points $p = p_{k,\ell} = (k, \ell, 0)$, we have that no line of S can be contained in $G_2 = 0$. Hence, the lines being irreducible, we deduce that $G_2^{-1}(0) \cap \mathbb{R}^n$ has to be a discrete set contained in S but which does not intersect the set $\{(k, \ell, 0) : k, \ell \geq 0\}$.

Next, we take $p \in G_2^{-1}(0) \cap \mathbb{R}^n$; we may assume $p = (k, \lambda, 0)$ for certain integer $k \geq 0$ and certain $\lambda \in \mathbb{R}$ which is not a non negative integer. Since $F_p = G_{1,p} G_{2,p}$ is a product of two irreducible factors of order 1 that vanish at the line germ $x = k, z = 0$, we conclude that $G_{2,p}$ must vanish at the line germ $x = k, z = 0$, a contradiction.

Thus, f is a special analytic function. □

2.8 Decomposition of real analytic functions. Now we proceed to prove the decomposition result Lemma 1.1 announced in the introduction. We first recall a well-known result to get rid of the squares.

Lemma 2.9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positive semidefinite analytic function. Then we can factorize $f = f_0^2 f'$, where f_0, f' are analytic functions on $\mathbb{R}^n$ such that f' is squares free in $\mathcal{O}(\mathbb{R}^n)$ and its zero set has codimension ≥ 2.*

Proof. Firstly, at each zero x of f, we write $f_x = \zeta_x^2 \eta_x \in \mathcal{O}_{\mathbb{R}^n,x}$, η_x without multiple factors; this factorization is unique up to units. The germ $\{\eta_x = 0\}$ has codimension ≥ 2, because otherwise some irreducible factor ξ_x of η_x would be real, and f_x would change sign at x.

Now, the ζ_x's generate a locally principal coherent sheaf $\mathcal{J} \subset \mathcal{O}_{\mathbb{R}^n}$. Since $H^1(\mathbb{R}^n, \mathbb{Z}_2) = 0$, $\mathcal{J}$ is globally generated by a global analytic function $f_0\colon \mathbb{R}^n \to \mathbb{R}$. An easy computation shows that f_0^2 divides f, and we have $f = f_0^2 f'$. Each germ f'_x coincides with η_x up to a unit, hence its zero set has codimension ≥ 2, f'_x does not change sign and it is squares free. □

Now we are ready to prove Lemma 1.1:

Proof of Lemma 1.1. By Lemma 2.9 there exist analytic functions $f_0, f'\colon \mathbb{R}^n \to \mathbb{R}$ such that f' is squares free, $\dim\{f' = 0\} \leq n-2$ and $f = f_0^2 f'$. Hence, all the special factors of f' divide it with multiplicity one. By 2.3, there exist:

- An open invariant Stein neighborhood $\mathcal{U}$ of $\mathbb{R}^n$ in $\mathbb{C}^n$ such that $\mathbb{R}^n$ is a deformation retract of $\mathcal{U}$,
- A holomorphic extension F' of f' to $\mathcal{U}$, and
- Holomorphic functions $H_j\colon \mathcal{U} \to \mathbb{C}$, $j \in J$, such that $\{S_j = H_j^{-1}(0)_{\mathbb{R}^n}\}_{j \in J}$ are the (complex) irreducible components of the germ $F^{-1}(0)_{\mathbb{R}^n}$ and H_j generates the ideal of $H_j^{-1}(0)$. Furthermore, if S_j is invariant we may assume that H_j is also invariant, hence $h_j = H_j|_{\mathbb{R}^n}$ defines an analytic function on $\mathbb{R}^n$.

Let $J_1 = \{j \in J : \dim(S_j \cap \mathbb{R}^n) = 0,\ S_j = \sigma(S_j)\}$, $J_2 = \{j \in J \setminus J_1 : S_j \neq \sigma(S_j)\}$ and $J_3 = J \setminus (J_1 \cup J_2)$. Consider the bijection $\hat{\sigma}\colon J_2 \to J_2$ defined by $S_{\hat{\sigma}(i)} = \sigma(S_i)$. This bijection defines on J_2 (together with the identity) an equivalence relation. For each equivalence class α we choose a representative $j \in \alpha$ and consider the set $J'_2 \subset J_2$ of such representatives. We have $J'_2 \cap \hat{\sigma}(J'_2) = \emptyset$ and $J'_2 \cup \hat{\sigma}(J'_2) = J_2$.

Next, let $D_1 = \bigcup_{j \in J_1} S_j \cap \mathbb{R}^n$, which is a discrete set. Thus, we can define the following sheaf

$$\mathcal{J}_x = \begin{cases} \prod_{j \in J_1, x \in S_j} h_j \cdot \mathcal{O}_{\mathbb{R}^n,x} & \text{if } x \in D_1, \\ \mathcal{O}_{\mathbb{R}^n,x} & \text{if } x \notin D_1. \end{cases}$$

This sheaf $\mathcal{J}$ is a locally principal coherent ideal sheaf whose zero set is D_1. Since the group $H^1(\mathbb{R}^n, \mathbb{Z}_2) = 0$, all locally principal sheaves are principal, and $\mathcal{J}$ has a global generator f_1. Since D_1 has dimension 0 we may assume that f_1 is positive semidefinite on $\mathbb{R}^n$. By the definition of $\mathcal{J}$ we have that f_1 divides f'.

Let $Z = \bigcup_{j \in J'_2} H_j^{-1}(0)$ which is an analytic subset of $\mathcal{U}$. Consider the coherent sheaf of ideals $\mathcal{I}$ defined on $\mathcal{U}$ by

$$\mathcal{I}_x = \begin{cases} \prod_{j \in J'_2, x \in S_j} H_j\, \mathcal{O}_{\mathbb{C}^n,x} & \text{if } x \in Z, \\ \mathcal{O}_{\mathbb{C}^n,x} & \text{if } x \notin Z. \end{cases}$$

As before (in 2.3), the locally principal coherent sheaf $\mathcal{J}$ is globally principal, say generated by a holomorphic function $\Gamma\colon \mathcal{U} \to \mathbb{C}$, whose zero set is $\Gamma^{-1}(0) = Z$. Moreover, by the definition of $\mathcal{I}$ we have that Γ divides F', and since F' is invariant, also the holomorphic function $\overline{\Gamma \circ \sigma}$ divides F'. By the properties of the set J_2' we have that Γ and $\overline{\Gamma \circ \sigma}$ do not have irreducible common factors in the ring $\mathcal{H}(\mathcal{U})$ whose zero set intersect $\mathbb{R}^n$. Hence, their product $F_2 = \Gamma \cdot \overline{\Gamma \circ \sigma}$ divides F' in a perhaps smaller neighbourhood of $\mathbb{R}^n$ in $\mathbb{C}^n$. Moreover, F_2 is an invariant holomorphic function such that $f_2 = F_2|_{\mathbb{R}^n}$ is a sum of two squares of analytic functions on $\mathbb{R}^n$.

Next, since f_1, f_2 are positive semidefinite analytic functions that divide f and which do not have common irreducible factors, we conclude that $f_3 = f'/(f_1 f_2)$ is a positive semidefinite analytic function on $\mathbb{R}^n$. Moreover, a straightforward computation shows that the irreducible complex factors of f_3 are H_j, $j \in J_3$, which are all special, as wanted. □

Remark 2.10 Since f_1 and f_2 are finite sums of squares of meromorphic function on $\mathbb{R}^n$, to prove that f is a finite or convergent sum of squares of meromorphic functions on $\mathbb{R}^n$ it is enough to check that for f_3. That is, we may always assume that all the complex irreducible factors of f are special and divide f with multiplicity one.

3. Examples

In this section we construct two examples of positive semidefinite analytic functions on $\mathbb{R}^3$ which are infinite sums of squares of meromorphic functions on $\mathbb{R}^3$, but for which we have not been able to decide whether or not they are finite sums of squares of meromorphic functions. We also produce an example of a positive semidefinite analytic function to which we cannot apply Theorem 1.6 but to which we can apply Corollary 1.7; hence, it is a finite sum of squares. Let us begin with such example.

Example 3.1. Let $f_0\colon \mathbb{R}^3 \to \mathbb{R}$ be the analytic function given by

$$f_0(x, y, z) = (z + x)^2(z + y)^2 + z^4 + 4x^2y^2,$$

which by Lemma 2.6 is a special analytic function. Note that f is a sum of three squares of analytic functions.

For each integer $\ell \geq 1$, let $f_\ell(x, y, z) = f_0(x - q_\ell, y - (\ell - q_\ell), z)$ where $q_\ell = \left[\frac{\ell}{2}\right]$. The zero set of f_ℓ is

$$X_\ell = \{x = q_\ell, z = 0\} \cup \{y = \ell - q_\ell, z = 0\}.$$

Since the family $\{X_\ell\}_\ell$ is locally finite, the set $X = \bigcup_\ell X_\ell$ is closed in $\mathbb{R}^3$. Hence, the presheaf

$$\mathcal{J}_x = \begin{cases} \prod_{\ell, x \in X_\ell} f_\ell \cdot \mathcal{O}(\mathbb{R}^3)_x & \text{if } x \in X, \\ \mathcal{O}(\mathbb{R}^3)_x & \text{if } x \notin X \end{cases}$$

is a subsheaf of the structure sheaf $\mathcal{O}_{\mathbb{R}^3}$. $\mathcal{J}$ is a locally principal coherent sheaf of ideals whose zero set is X. Since $H^1(\mathbb{R}^3, \mathbb{Z}_2) = 0$, all locally principal sheaves are principal, and $\mathcal{J}$ has a global generator f. Note that since X has codimension ≥ 2 we may assume that f is positive semidefinite on $\mathbb{R}^3$. Clearly, the special factors of f are the functions f_ℓ, $\ell \geq 1$.

Next, note that

$$X_\ell \cap X_{\ell+1} = \begin{cases} \{x = q_\ell, z = 0\} & \text{if } \ell \text{ is even}, \\ \{y = \ell - q_\ell, z = 0\} & \text{if } \ell \text{ is odd}, \end{cases}$$

which is not a discrete set for all $\ell \geq 1$. Thus, we cannot apply Theorem 1.6 to f. However, since $X_i \cap X_j$ is discrete if $i \equiv j \mod 2$, we can apply Corollary 1.7 with the partition $\{A_1, A_2\}$, where A_1 is the set of the non negative odd numbers and A_2 the set of the non negative even ones. Thus, we conclude that *f is a sum of 2^5 squares*.

Now let us construct the examples we have announced in the introduction which are infinite sums of squares but to which we cannot apply Corollary 1.7.

Examples 3.2. (a) Let $f_0 \colon \mathbb{R}^3 \to \mathbb{R}$ be the special analytic function described in Example 2.7, which is a sum of three squares in $\mathcal{O}(\mathbb{R}^3)$. For each integer $\ell \geq 1$ consider the analytic function $f_\ell(x, y, z) = f_0(x - \ell, y - \ell, z)$, whose zero set is

$$X_\ell = \bigcup_{k \geq \ell} \{x = k, z = 0\} \cup \bigcup_{k \geq \ell} \{y = k, z = 0\}$$

Since the family $\{X_\ell\}_\ell$ is locally finite, the set $X = \bigcup_\ell X_\ell$ is closed in $\mathbb{R}^3$, and as in Example 3.1 there exists a positive semidefinite analytic function $f \colon \mathbb{R}^n \to \mathbb{R}$ whose special factors are the functions f_ℓ, $\ell \geq 1$ (and each one divides f with multiplicity one). Thus, by Theorem 1.4, *f is a convergent sum of squares of analytic functions on $\mathbb{R}^3$*.

Moreover, we claim: *There exist an open invariant neighbourhood U of $\mathbb{R}^3$ in $\mathbb{C}^3$, a holomorphic extension F of f to U, and a sequence of invariant holomorphic functions $\{G_k\}_k$ on U which converges uniformly to F in the compact sets of U.*

Indeed, let U_0 be an open invariant neighbourhood of $\mathbb{R}^3$ in $\mathbb{C}^3$ to which we can extend holomorphically f. We denote such extension by F. Let $\{K_k\}_{k \geq 1}$ be an exhaustion by compact sets of U_0, that is, $U_0 = \bigcup_{k\geq 1} K_k$ and $K_k \subset \operatorname{Int}(K_{k+1})$ for all $k \geq 1$. We may assume that $(1,1,0) \in K_1$. For each $k \geq 1$ we set

$$J_k = \{\ell \geq 1 : F_\ell^{-1}(0) \cap K_k \neq \emptyset\}.$$

We have that $1 \in J_k$ and $J_k \subset J_{k+1}$ for all $k \geq 1$. We write

$$H_k = \prod_{\ell \in J_k} F_k,$$

which is a finite product of sums of 4 squares of analytic functions; hence, H_k is a sum of 4 squares itself. For each $k \geq 1$, the analytic function $\Lambda_k = \frac{F}{H_k}$ does not vanish on the compact set K_k. Let

$$\varepsilon_k = \frac{1}{2k} \cdot \frac{1}{\sup_{K_k} |H_k| + 1} \cdot \frac{\inf_{K_k} |\Lambda_k|}{\inf_{K_k} |\Lambda_k| + 1} > 0$$

and $B_k = \sqrt{\Lambda_k + \varepsilon_k}$ for each $k \geq 1$. As one can check, $\Lambda_k + \varepsilon_k$ does not vanish on $K_k \cup \mathbb{R}^n$. Hence, B_k is holomorphic on an open set $U_k \subset \mathbb{C}^n$ which contains $K_k \cup \mathbb{R}^3$.

Using that $\{K_k\}_k$ is an exhaustion of U_0, one can verify that there exists an open neighbourhood U of $\mathbb{R}^n$ in $\mathbb{C}^n$ contained in $\bigcap_{k\geq 1} U_k$. The functions

$$G_k = B_k^2 H_k$$

are sum of four squares of invariant holomorphic functions on U. Moreover, a straightforward computation shows that the sequence $\{G_k\}_k$ converges to F uniformly in the compact sets of U.

However, we do not know whether or not f is a *finite* sum of squares of meromorphic functions on $\mathbb{R}^3$. Note that the lines $\{x = \ell, z = 0\}$ and $\{y = \ell, z = 0\}$ belong exactly to the zero set of $f_1, \dots, f_\ell$ for all $\ell \geq 1$. Hence, we cannot apply Corollary 1.7 to this example.

(b) The description of the following example requires an initial preparation. Consider the following distribution of the natural numbers into an infinite array

$$\begin{array}{ccccccc}
\swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \\
1 & 2 & 4 & 7 & 11 & 16 & \cdots \\
3 & 5 & 8 & 12 & 17 & \cdots & \\
6 & 9 & 13 & 18 & \cdots & & \\
10 & 14 & 19 & \cdots & & & \\
15 & 20 & \cdots & & & & \\
21 & \cdots & & & & & \\
\vdots & \ddots & & & & &
\end{array}$$

and the finite sets $S_k = \{a_k + 1, \dots, a_k + k\}$ where $a_k = \frac{1}{2}k(k-1)$, $k \geq 1$. The set S_k corresponds to the oblique line $\{\alpha_{1k}, \alpha_{2,k-2}, \dots, \alpha_{k,1}\}$ of the previous array.

For each $k \geq 1$ we also construct (inductively) a set

$$T_k = \left\{b_{kj} : k - \left[\tfrac{k}{2}\right] \leq j \leq k-1\right\}$$

such that $b_{kj} \in S_j \setminus \bigcup_{\ell=1}^{k-1} T_\ell$. We take $T_1 = \emptyset$ and $T_2 = \{1\}$. By the definition of the sets T_k, for any given j, we have $S_j \cap T_k \neq \emptyset$ if and only if $k - \left[\frac{k}{2}\right] \leq j \leq k-1$. Thus, $j+1 \leq k \leq 2j$, and this means that S_j intersects exactly j of the T_k's, which are $T_{j+1}, \dots, T_{2j}$. Since the set S_j has j different elements then the T_k's can be constructed with the desired conditions. We denote $C_k = S_k \cup T_k$.

Next, for each $k \geq 1$ we consider the holomorphic function

$$F_k = (\sin(\pi x) + z)^2(\sin(\pi y) + z)^2 + z^4 + M^2(x-k)^2 \prod_{\ell \in C_k} (y-\ell)^2,$$

where $M > 0$ is a positive real number such that $M^2 \cdot \prod_{\ell \in C_k, \ell \neq j}(j-\ell)^2 \neq 1$ for all $j \in C_k$. We have that the real analytic function $f_k = F_k|_{\mathbb{R}^n}$ is a special analytic function whose real zero set is

$$X_k = \{x = k, z = 0\} \cup \bigcup_{\ell \in C_k} \{y = \ell, z = 0\}.$$

One can check, proceeding similarly to Example 2.7, that f is a special factor.

Once again, the family $\{X_k\}_k$ is locally finite; hence, the set $X = \bigcup_k X_k$ is closed in $\mathbb{R}^3$ and there exists a positive semidefinite analytic function $f\colon \mathbb{R}^n \to \mathbb{R}$ whose special factors are the functions f_k, $k \geq 1$. Hence, by Theorem 1.4, *f is a convergent sum of squares of analytic functions on $\mathbb{R}^3$.*

Proceeding as in the previous example (a), one can produce a sequence $\{g_\ell\}_{\ell \geq 1}$ of sums of four squares of analytic functions on $\mathbb{R}^3$ which converges uniformly to f (in the sense described in the introduction).

However, we have do not know whether or not f is a finite sum of squares of meromorphic functions on $\mathbb{R}^3$. Note that the lines $\{y = \ell, z = 0\}$ belong exactly to the zero set of two f_k's, and the lines $\{x = \ell, z = 0\}$ belong exactly to the zero set of f_ℓ. Moreover, for all $k \geq 1$ the zero set X_k has finitely many irreducible components.

Let us explain why we cannot apply Corollary 1.7 to this example. For, we have to check that there does not exist a finite partition $\mathcal{P} = \{A_1, \dots, A_r\}$ of $\mathbb{N}$ such that for each $j = 1, \dots, r$ and each pair $k, \ell \in A_j$ the intersection $X_k \cap X_\ell$ is a discrete set. Let $\ell \geq r$ be an integer. By the definition of the sets X_k, it follows that X_ℓ shares an irreducible component of dimension 1 with $X_{\ell+1}, \dots, X_{2\ell}$. This means that the integers $\ell, \ell+1, \dots, 2\ell$ should belong to different elements of the partition $\mathcal{P}$. But this is impossible because the partition has $r < \ell + 1$ elements.

A natural question is if it is possible to go a little bit further determining whether or not there is a positive semidefinite analytic function $f\colon \mathbb{R}^3 \to \mathbb{R}$ to which we cannot apply Corollary 1.7 but which satisfy the following conditions:

(i) The function f has infinitely many special factors whose zero sets have all finitely many irreducible components.

(ii) The special factors of f are all sums of p squares in $\mathcal{O}(\mathbb{R}^3)$ for certain integer $p \geq 1$.

(iii) f is the uniform limit of a convergent sequence of analytic functions which are sum of q squares in $\mathcal{O}(\mathbb{R}^3)$ for certain integer $q \geq 1$.

(iv) There exists and integer $r \geq 1$ such that the number of irreducible components of a special factor of f is $\leq r$.

(v) Each irreducible component of the zero set of f belongs to at most the zero sets of s of the special factors of f for the same fixed integer $s \geq 1$.

Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a positive semidefinite analytic function satisfying the conditions (i) to (v) above. Let $\{f_k\}_{k\geq 1}$ be the special factors of f, $X_k = f_k^{-1}(0)$ and $\{Y_\ell\}_{\ell\geq 1}$ the irreducible components of $f^{-1}(0) = \bigcup_{k\geq 1} X_k$. Consider the set $S = S_f = \{(k,\ell) : Y_\ell \subset X_k\} \subset \mathbb{N}^2$ and the projections $\pi_i : S \to \mathbb{N},\ (x_1,x_2) \to x_i$ for $i = 1,2$. The set S has the following properties:

(1) $\pi_i(S) = \mathbb{N}$ for $i = 1,2$.

(2) The fibers $\pi_1^{-1}(k)$ and $\pi_2^{-1}(\ell)$ have respectively less than or equal to s and r points for all $k,\ell \in \mathbb{N}$.

Conversely, for each set $S \subset \mathbb{N}^2$ satisfying the properties (1) and (2) above there exists a positive semidefinite analytic function $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that $S_f = S$. To check that, it is enough to proceed as in the example 3.2 (b).

Thus, the existence of an analytic function $f\colon \mathbb{R}^3 \to \mathbb{R}$ satisfying (i) to (v) above to which we cannot apply Corollary 1.7 is equivalent to the existence of a set $S \subset \mathbb{N}^2$ satisfying the conditions (1) and (2) above and the following one:

(3) There is no finite partition $\mathcal{P} = \{A_1,\dots,A_m\}$ of $\mathbb{N}$ such that for all $j = 1,\dots,m$ and all $\alpha,\beta \in A_j$, we have that $\pi_2(\pi_1^{-1}(\alpha)) \cap \pi_2(\pi_1^{-1}(\beta)) = \emptyset$.

As far as we known, after consulting several specialists in the matter, this is an open problem which seems to be difficult.

4. Proofs of the main results

The purpose of this section is to prove Theorem 1.4 and Theorem 1.6 announced in the introduction. Before that we need some preliminary results.

Proposition 4.1. *Let $U \subset \mathbb{C}^n$ be an open set and let $F_k, G_k\colon U \to \mathbb{C}$ be invariant holomorphic functions such that the series $\sum_{k\geq 1} F_k^2$, $\sum_{k\geq 1} G_k^2$ converge in the sense of Definition* 1.2 (ii)*, that is,* $\sum_{k\geq 1} \sup_K |F_k^2|$, $\sum_{k\geq 1} \sup_K |G_k^2| < +\infty$ *for all compact sets $K \subset U$. Then the series $\sum_{j,k\geq 1} F_j^2 G_k^2$ converges on U to $\sum_{j\geq 1} F_j^2 \cdot \sum_{k\geq 1} G_k^2$ in the sense of Definition* 1.2 (ii).

The proof of the proposition follows straightforwardly from the following result whose proof is a standard exercise of the theory of convergent series that we do not include here.

Lemma 4.2. *Let $\{a_i\}_{i\in\mathbb{N}}, \{b_i\}_{i\in\mathbb{N}}$ be two sequences of complex numbers such that the series $\sum_{i\geq 1} a_i$, $\sum_{i\geq 1} b_i$ converge to complex numbers a, b and the series $\sum_{i\geq 1} |a_i|$, $\sum_{i\geq 1} |b_i|$ converge to non-negative real numbers a^*, b^*. Then the series $\sum_{i,j\geq 1} a_i b_j$ converges to ab, that is, for all $\varepsilon > 0$ there exists a finite subset $I_\varepsilon \subset \mathbb{N}^2$ such that if $I \subset \mathbb{N}^2$ is a finite subset that contains I_ε then $\left|\sum_{(i,j)\in I} a_i b_j - ab\right| < \varepsilon$. Moreover, the series $\sum_{i,j\geq 1} |a_i b_j|$ converges to $a^* b^*$.*

Lemma 4.3. *Let $f, f'\colon \mathbb{R}^n \to \mathbb{R}$ be two positive semidefinite analytic functions such that $f^{-1}(0) = f'^{-1}(0) = S$. Suppose that there exists a discrete set $D \subset S$ such that the meromorphic function f/f' is analytic on $\mathbb{R}^n$ off the discrete set D. Then, there exist analytic functions $h_1, h_2\colon \mathbb{R}^n \to \mathbb{R}$ such that $h_i^{-1}(0) \subset D$, h_2 is a sum of $2^n + n$ squares and $h_1^2 f = h_2 f'$.*

Proof. Indeed, consider the coherent sheaf $(f' : f)\mathcal{O}_{\mathbb{R}^n}$. This sheaf is generated in a neighborhood of each $y \in D$ by finitely many sections $\delta_1, \dots, \delta_{r_y} \in \mathcal{O}(\mathbb{R}^n)$. By the standard sum of squares trick, $f_y/f'_y = \eta_y/\delta_y$ for $\delta = \sum_k \delta_k^2$ and some $\eta_y \in \mathcal{O}(\mathbb{R}^n_y)$. Furthermore, y is an isolated zero of δ. For that, suppose that there is $x \neq y$ arbitrarily close to y with $\delta(x) = 0$. Then, all δ_k's vanish at x, and since the ideal $(f'_x : f_x)$ is generated by them, it contains no unit. This means that f_x/f'_x is not analytic, a contradiction.

The ideals $\mathcal{I}_y = (\delta_y)$, $y \in D$, glue to define a locally principal coherent sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_{\mathbb{R}^n}$, whose zero set is D. Since $H^1(\mathbb{R}^n, \mathbb{Z}_2) = 0$, all locally principal sheaves are principal, and $\mathcal{I}$ has a global generator Δ. This means that each germ Δ_y/δ_y is a unit for all $y \in D$. This Δ is a non-negative analytic function on $\mathbb{R}^n$ whose zero set is D, and $f'' = \Delta^2 f/f'$ is analytic. Moreover f'' is strictly positive on $\mathbb{R}^n \setminus D$. Thus, by [BKS] there exists an analytic function $\Lambda\colon \mathbb{R}^n \to \mathbb{R}$ such that $\Lambda^{-1}(0) \subset f''^{-1}(0) = D$ and $\Lambda^2 f''$ is a sum of $2^n + n$ squares. Hence,

$$(\Lambda\Delta)^2 f = (\Lambda^2 f'') f'$$

and taking $h_1 = \Lambda\Delta$ and $h_2 = \Lambda^2 f''$, we are done. □

Lemma 4.4. *Let $U \subset \mathbb{C}^n$ be an open set and let $F_k \colon U \to \mathbb{R}$, $k \geq 1$, be invariant holomorphic functions such that the sum $\sum_{k\geq 1} F_k$ converges in the sense of Definition* 1.2 (ii). *Let $Y \subset U$ be a closed set and let $G \colon U \to \mathbb{R}$ be a real analytic function. If G_x divides $F_{k,x}$ for all $k \geq 1$ and all $x \in Y$, then G divides $F = \sum_{k\geq 1} F_k$ in $\mathcal{O}(W)$, where W is a small enough neighbourhood of Y in U.*

Proof. It is enough to check that for all $x \in Y$, G_x divides F_x. Fix $x \in Y$, and let $G_x = \prod_{i=1}^r G_{i,x}^{d_i}$ be the decomposition of the germ G_x into irreducible complex factors. We take a compact neighborhood $W^x \subset U$ of x such that:

- $G_1, \dots, G_r$ are holomorphic on W^x, and
- $G_i^{-1}(0) \cap W^x$ is an irreducible complex analytic set in W^x for all $i = 1, \dots, r$.

It is enough to see that $G_{i,x}^{d_i}$ divides F_x for $i = 1, \dots, r$. By hypothesis, for each $k \geq 1$ the germ $G_{i,x}^{d_i}$ divides $F_{k,x}$, and an easy computation shows that $G_{i,x}$ divides all derivatives $D^\alpha F_{k,x}$ of degree $|\alpha| < d_i$. That means that $D^\alpha F_k$ vanishes on the intersection of $G_i^{-1}(0)$ with a small neighborhood of x (depending on k). Thus, since $G_i^{-1}(0) \cap W^x$ is irreducible, $D^\alpha F_k$ vanishes on $G_i^{-1}(0) \cap W^x$. As this holds for each $k \geq 1$ and $D^\alpha F|_{W^x} = \sum_{k\geq 1} D^\alpha F_k$, we have that $D^\alpha F|_{W^x}$ vanishes on $G_k^{-1}(0) \cap W^x$. Whence, $G_{k,x}$ divides all derivatives $D^\alpha F_x$ with $|\alpha| < d_k$. This concludes the proof up to the lemma that follows. □

Lemma 4.5. *Let $G, F \in \mathbb{C}\{z\} = \mathbb{C}\{z_1, \dots, z_n\}$ be analytic germs such that G is irreducible and let d be a positive integer. Suppose that G divides all derivatives $D^\alpha F$ of degree $|\alpha| < d$. Then, G^d divides F.*

Proof. We proceed by induction on d. If $d = 1$ the result is clear. Suppose the result true for d and that G divides $D^\alpha F = \frac{\partial^{|\alpha|} F}{\partial z^\alpha}$ for $|\alpha| < d+1$. By induction, G^d divides $F, \frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_n}$. In particular, there exists $H \in \mathbb{C}\{z_1, \dots, z_n\}$ such that $F = G^d H$. Hence,

$$\frac{\partial F}{\partial z_i} = dG^{d-1}\frac{\partial G}{\partial z_i} H + G^d \frac{\partial H}{\partial z_i}.$$

Since G^d divides all derivatives $\frac{\partial F}{\partial z_i}$, we see that G divides all products $\frac{\partial G}{\partial z_i} H$. But G is irreducible and cannot divide all its derivatives $\frac{\partial G}{\partial z_i}$, hence G divides H. Thus, G^{d+1} divides F, as wanted. □

Next, we proceed to prove Theorem 1.4.

Proof of Theorem 1.4. We will split the proof into several steps.

Step 1: *Preparation.* By the decomposition result Lemma 1.1 we may assume that all the complex irreducible factors $\{h_j\}_{j\in J}$ of f are special and divide f with multiplicity one. As we have seen in 2.3 we may assume that there exist holomorphic extensions H_j of h_j to a Stein neighborhood $\mathcal{U}_0$ of $\mathbb{R}^n$ in $\mathbb{C}^n$.

We write $X_j = H_j^{-1}(0) \cap \mathbb{R}^n$ for all $j \in J$. By hypothesis and by Proposition 1.3, for each j there are invariant holomorphic functions $G_j, B_{jk} \colon U_j \to \mathbb{C}$, defined on an open neighborhood $U_j \subset \mathcal{U}_0$ of X_j in $\mathbb{C}^n$, such that $G_j^2 H_j|_{U_j} = \sum_k B_{jk}^2$ (the series converging in the strong sense of Definition 1.2 (ii)) and $G_j^{-1}(0) \cap \mathbb{R}^n \subset X_j$. We write $T_j = G_j^{-1}(0) \subset U_j$ and shrinking U_j if necessary, we may assume that the family $\{T_j\}_{j \in J}$ is locally finite in $\mathbb{C}^n$. Consider the open set in $\mathbb{C}^n$:

$$U = \Big(\mathbb{C}^n \setminus \bigcup_{j \geq 1} \mathrm{Cl}_{\mathbb{C}^n}(T_j)\Big) \cup \bigcup_{j \geq 1} \Big(U_j \setminus \bigcup_{k \neq j} (\mathrm{Cl}_{\mathbb{C}^n}(T_k) \setminus U_k)\Big).$$

A straightforward computation shows that $T_j \cap U$ is closed in U for all $j \in J$. Let $\mathcal{U} \subset U \cap \mathcal{U}_0$ be an open invariant Stein neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ such that $\mathbb{R}^n$ is a deformation retract of $\mathcal{U}$ ([Ca]). Denote again by T_j the intersections of T_j with $\mathcal{U}$ which are analytic (complex) subsets of $\mathcal{U}$. We denote $V_j = U_j \cap \mathcal{U}$ and keep F for the restriction of F to $\mathcal{U}$, and G_j, B_{jk} for those of G_j, B_{jk} to V_j. It holds:

- $T_j \subset V_j$, and
- all T_j's are closed analytic subsets of $\mathcal{U}$, as well as their union $T = \bigcup_j T_j$.

Step 2: *Extension of denominators.* Fix $j \in J$ and consider the coherent sheaf of ideals $\mathcal{J}$ defined on $\mathcal{U}$ by

$$\mathcal{J}_x = \begin{cases} G_j \cdot \mathcal{O}_{\mathbb{C}^n,x} & \text{if } x \in T_j, \\ \mathcal{O}_{\mathbb{C}^n,x} & \text{if } x \notin T_j. \end{cases}$$

As it has been done before, the locally principal coherent sheaf $\mathcal{J}$ is globally principal, say generated by a holomorphic function $\Gamma_j \colon \mathcal{U} \to \mathbb{C}$, whose zero set is $\Gamma_j^{-1}(0) = T_j$. In a small enough neighborhood of T_j we have that $\Gamma_j = G_j v_j$ where v_j is a holomorphic unit. Hence, in that neighborhood $\overline{\Gamma_j \circ \sigma} = G_j \cdot \overline{v_j \circ \sigma}$, and therefore $E_j = \overline{\Gamma_j \circ \sigma}/\Gamma_j$ is a unit in $\mathcal{H}(\mathcal{U})$. Moreover, one can check that $E_j \cdot \overline{E_j \circ \sigma} = 1$. Let $\Delta_j \colon \mathcal{U} \to \mathbb{C}$ be a holomorphic function such that $\Delta_j^2 = E_j$ and $\Delta_j \cdot \overline{\Delta_j \circ \sigma} = 1$. A straightforward computation, already done in 2.3, shows that $G'_j = \Gamma_j \Delta_j$ is an invariant holomorphic function that generates $\mathcal{J}$.

Consider also the real analytic function $g'_j = G'_j|_{\mathbb{R}^n}$. The zero set of G'_j is T_j and the zero set of g'_j is $T_j \cap \mathbb{R}^n \subset X_j \subset h_j^{-1}(0)$. Now, since G'_j generates $\mathcal{J}$, G_j generates $\mathcal{J}|_{V_j}$, and these functions are invariant, there exist an invariant holomorphic function $Q_j \colon V_j \to \mathbb{C}$ such that $G'_j|_{V_j} = Q_j G_j$. We deduce:

$$G_j'^2 H_j = Q_j^2 (G_j^2 H_j) = Q_j^2 \sum_k B_{jk}^2 = \sum_k C_{jk}^2,$$

where $C_{jk} = Q_j B_{jk}$, and the series $\sum_k C_{jk}^2$ satisfies the convergence condition Definition 1.2 (ii). Note moreover that the zero set of G_j is T_j.

Step 3: *Glueing of denominators.* After the preceding preparation, we glue the denominators G'_j. Consider the coherent sheaf of ideals $\mathcal{J}$ defined on $\mathcal{U}$ by

$$\mathcal{J}_x = \begin{cases} \prod_{j,x\in T_j} G'_j \cdot \mathcal{O}_{\mathbb{C}^n,x} & \text{if } x \in T_j, \\ \mathcal{O}_{\mathbb{C}^n,x} & \text{if } x \notin T. \end{cases}$$

As in the preceding arguments, the locally principal coherent sheaf $\mathcal{J}$ is globally principal, says generated by a holomorphic function $\Gamma\colon \mathcal{U} \to \mathbb{C}$, whose zero set is $\Gamma^{-1}(0) = T$. As in the previous step we can substitute Γ by an invariant holomorphic function $G\colon \mathcal{U} \to \mathbb{C}$ that generates $\mathcal{J}$. Consider the real analytic function $g = G|_{\mathbb{R}^n}$. The zero set of G is $T = \bigcup_j T_j$ and the zero set of g is $\bigcup_j T_j \cap \mathbb{R}^n \subset \bigcup_j X_j \subset f^{-1}(0)$. Moreover, by the construction of G and G'_j, we have that each G'_j divides G and for all $x \in T$

$$G \cdot \mathcal{O}_{\mathbb{C}^n,x} = \prod_{j,x\in T_j} G'_j \cdot \mathcal{O}_{\mathbb{C}^n,x}.$$

Step 4: *Globalization of sums of squares.* Here we find global sums of squares $\sum_k A_{jk}^2$ to replace the sums $\sum_k C_{jk}^2$, which are defined only on the V_j's, such that their restrictions to $\mathbb{R}^n$ vanish only at the corresponding X_j.

After shrinking $\mathcal{U}$ if necessary, we may assume that the connected component of $H_j^{-1}(0)$ that intersects $\mathbb{R}^n$ in X_j is contained in V_j (this can be done using an auxiliary open set U similar to the one constructed in Step 1). Up to shrinking V_j, we may assume that it is invariant and does not intersect other connected components of $({G'_j}^2 H_j)^{-1}(0)$ different to the one that intersects $\mathbb{R}^n$ in X_j. By Proposition 2.2, applied to $\Phi = ({G'_j}^2 H_j)^2$, $V = V_j$ and $C_k = C_{jk}$, there exist invariant holomorphic functions $A_{jk}\colon \mathcal{U} \to \mathbb{C}$, such that $\sum_k \sup_K |A_{jk}|^2 < +\infty$ for all compact sets $K \subset \mathcal{U}$, and $({G'_j}^2 H_j)^2$ divides $A_{jk} - C_{jk}$ on V_j.

On V_j we have:

$$\sum_k A_{jk}^2 - {G'_j}^2 H_j = \sum_k A_{jk}^2 - \sum_k C_{jk}^2 = \sum_k \left(A_{jk}^2 - C_{jk}^2\right),$$

and this series is convergent on compact sets, as $\sum_k A_{jk}^2$ and $\sum_k C_{jk}^2$ are so. By construction, $({G'_j}^2 H_j)^2$ divides on V_j each term $A_{jk}^2 - C_{jk}^2 = (A_{jk} + C_{jk})(A_{jk} - C_{jk})$, hence it divides their sum $\sum_k A_{jk}^2 - {G'_j}^2 H_j$. Thus, if we set $A_{j0} = {G'_j}^2 H_j$, there is a holomorphic function $\Psi_j\colon V_j \to \mathbb{C}$ such that on V_j we have:

$$\sum_{k\geq 0} A_{jk}^2 = {G'_j}^2 H_j + (1+\Psi_j)\left({G'_j}^2 H_j\right)^2 = u_j {G'_j}^2 H_j, \quad \text{where } u_j = 1 + (1+\Psi_j){G'_j}^2 H_j.$$

Clearly, u_j has no zeros in $({G'_j}^2 H_j)^{-1}(0) \cap V_j$, hence, u_j is a holomorphic unit in a perhaps smaller neighbourhood V_j of $T_j \cup H_j^{-1}(0)$. Moreover, the restriction of $\sum_{k\geq 0} A_{jk}^2$ to $\mathbb{R}^n$ vanishes only at X_j, because $A_{j0}^{-1}(0) \cap \mathbb{R}^n = X_j$.

Step 5a: *Glueing of sums of squares under the condition* (a). Here we paste all the sums of squares $\sum_k A_{jk}^2$ to get a single one, if the irreducible components $\{Y_i\}_{i\in I}$ of $X = \bigcup_{j\in J} X_j$ satisfy the condition (a) in the statement, that is, $Y_i \cap Y_k$ is a discrete set for $i \neq k$.

We may assume that $I = \mathbb{N}$, because if I is a finite set the result is a straightforward consequence of Proposition 4.1. Note that each X_j is a union of some of the Y_i's and that each Y_i is a subset of finitely many X_j's. This fact can be checked taking germs at any point of Y_i.

For each $i \in I$ we set $J_i = \{j \in J : Y_i \subset X_j\}$, which is a finite set. By Proposition 4.1, the function $\prod_{j\in J_i} \sum_k {A_{jk}}^2$ is a convergent sum of squares

$$\sum_\ell {A'_{i\ell}}^2 = \prod_{j\in J_i} \sum_k {A_{jk}}^2$$

on $\mathcal{U}$ in the sense of Definition 1.2 (ii). Note that for each $i \in I$ we have $Y_i \subset \bigcap_{j\in J_i} X_j \subset \bigcap_{j\in J_i} V_j$. Hence in $W_i = \bigcap_{j\in J_i} V_j$ we have

$$\sum_\ell {A'_{i\ell}}^2 = \prod_{j\in J_i} \sum_k {A_{jk}}^2 = \prod_{j\in J_i} u_j {G'_j}^2 H_j = u'_i F_i,$$

where $u'_i = \prod_{j\in J_i} u_j$ is a holomorphic unit on W_i and $F_i = \prod_{j\in J_i} {G'_j}^2 H_j$. Note that F_i divides $G^2 F$ for all $i \in I$.

For each $i \in I$ we choose a compact set K_i such that $K_1 \neq \emptyset$, $K_i \subset \operatorname{Int}(K_{i+1})$ and $\bigcup_{i\in I} K_i = \mathcal{U}$, that is, the family $\{K_i\}_{i\in I}$ is an exhaustion of $\mathcal{U}$ by compact sets. For each $i \in I$ set

$$\mu_i = \sup_{K_i} \left|\frac{G^2 F}{F_i}\right|^2 \sum_\ell \sup_{K_i} |{A'_{i\ell}}^2| \quad \text{and} \quad \gamma_i = \frac{1}{\sqrt{2^i \mu_i}}.$$

We have

$$\sum_\ell \sup_{K_i} \left|\gamma_i \frac{G^2 F}{F_i} A'_{i\ell}\right|^2 \leq \gamma_i^2 \sup_{K_i} \left|\frac{G^2 F}{F_i}\right|^2 \sum_\ell \sup_{K_i} |{A'_{i\ell}}^2| \leq \frac{1}{2^i}\,.$$

Now, let K be a compact subset of the open set $\mathcal{U}$. As $\mathcal{U} \subset \bigcup_{i\geq 1} \operatorname{Int}_{\mathbb{C}^n}(K_i)$, K is contained in some K_{i_0}, hence in all K_i for $i \geq i_0$, and so:

$$\begin{aligned}
&\sum_{i,\ell} \sup_K \left| \gamma_i \frac{G^2 F}{F_i} A'_{i\ell} \right|^2 \\
&\quad = \sum_{i=1}^{i_0-1} \sum_{\ell} \sup_K \left| \gamma_i \frac{G^2 F}{F_i} A'_{i\ell} \right|^2 + \sum_{i \geq i_0} \sum_{\ell} \sup_K \left| \gamma_i \frac{G^2 F}{F_i} A'_{i\ell} \right|^2 \\
&\quad \leq \sum_{i=1}^{i_0-1} \sup_K \left| \gamma_i \frac{G^2 F}{F_i} \right|^2 \sum_{\ell} \sup_K |{A'_{i\ell}}^2| + \sum_{i \geq i_0} \sum_{\ell} \sup_{K_i} \left| \gamma_i \frac{G^2 F}{F_i} A'_{i\ell} \right|^2 \\
&\quad \leq \sum_{i=1}^{i_0-1} \sup_K \left| \gamma_i \frac{G^2 F}{F_i} \right|^2 \sum_{\ell} \sup_K |{A'_{i\ell}}^2| + \sum_{i \geq i_0} \frac{1}{2^i} \\
&\quad \leq \sum_{i=1}^{i_0-1} \sup_K \left| \gamma_i \frac{G^2 F}{F_i} \right|^2 \sum_{\ell} \sup_K |{A'_{i\ell}}^2| + 1 < +\infty.
\end{aligned}$$

Consequently, the sum of squares

$$F' = G^4 F^2 + \sum_{i,\ell} \Big(\gamma_i \frac{G^2 F}{F_i} A'_{i\ell} \Big)^2$$

is convergent in the sense of Definition 1.2 (ii).

For a fixed $r \in I$, we claim: $F_r = \prod_{j \in J_r} {G'_j}^2 H_j$ *divides the convergent sum*

$$\sum_{\ell} \Big(\gamma_i \frac{G^2 F}{F_i} A'_{i\ell} \Big)^2 = \gamma_i^2 \left(\frac{G^2 F}{F_i} \right)^2 \sum_{\ell} {A'_{i\ell}}^2$$

in $\mathcal{H}(W_r)$ *for all* $i \in I$. Indeed, in $W_r = \bigcap_{j \in J_r} V_j$ we have

$$\begin{aligned}
&\gamma_i^2 \left(\frac{G^2 F}{F_i} \right)^2 \sum_{\ell} {A'_{i\ell}}^2 \\
&\quad = \gamma_i^2 \left(\frac{G^2 F}{F_i \prod_{j \in J_r \setminus J_i} {G'_j}^2 H_j} \right)^2 \cdot \prod_{j \in J_r \setminus J_i} {G'_j}^4 H_j^2 \cdot \prod_{j \in J_i} \sum_k A_{jk}^2 \\
&\quad = \gamma_i^2 \left(\frac{G^2 F}{F_r \prod_{j \in J_i \setminus J_r} {G'_j}^2 H_j} \right)^2 \cdot \prod_{j \in J_r \setminus J_i} {G'_j}^4 H_j^2 \cdot \prod_{j \in J_r \cap J_i} \sum_k A_{jk}^2 \cdot \prod_{j \in J_i \setminus J_r} \sum_k A_{jk}^2 =
\end{aligned}$$

$$= \gamma_i^2 \left(\frac{G^2 F}{F_r \prod_{j \in J_i \setminus J_r} {G'_j}^2 H_j} \right)^2 \cdot \prod_{j \in J_r \setminus J_i} {G'_j}^4 H_j^2 \cdot \prod_{j \in J_r \cap J_i} u_j {G'_j}^2 H_j \cdot \prod_{j \in J_i \setminus J_r} \sum_k A_{jk}^2$$

$$= \gamma_i^2 \left(\frac{G^2 F}{F_r \prod_{j \in J_i \setminus J_r} {G'_j}^2 H_j} \right)^2 \cdot F_r \cdot \prod_{j \in J_r \setminus J_i} {G'_j}^2 H_j \cdot \prod_{j \in J_r \cap J_i} u_j \cdot \prod_{j \in J_i \setminus J_r} \sum_k A_{jk}^2.$$

Thus, F_r divides $\sum_\ell \big(\gamma_i^2 \frac{G^2 F}{F_i} A'_{i\ell})\big)^2$ in $\mathcal{H}(W_r)$, as wanted.

Next, we denote by f', f_i the restrictions to $\mathbb{R}^n$ of F', F_i for all $i \in I$. We claim: *For each $r \in I$ we have $g_x^2 f_x \mathcal{O}_{\mathbb{R}^n,x} = f'_x \mathcal{O}_{\mathbb{R}^n,x}$ for all $x \in Y_r \setminus \bigcup_{i \neq r} Y_i$.*

Before showing this we summarize the following facts already proved:

(i) $G_x^2 F_x \mathcal{O}_{\mathbb{C}^n,x} = F_{r,x} \mathcal{O}_{\mathbb{C}^n,x}$ for all $x \in Y_r \setminus \bigcup_{i \neq r} Y_i$.

(ii) $\sum_\ell {A'_{r\ell}}^2 = u'_r F_r$ in W_r.

(iii) $\gamma_r^2 \frac{G^4 F^2}{F_r^2}$ does not vanish at $Y_r \setminus \bigcup_{i \neq r} Y_i$. Hence, by (ii), we have that

$$\sum_\ell \gamma_r^2 \frac{G^4 F^2}{F_r^2} {A'_{r\ell}}^2 = \gamma_r^2 \frac{G^4 F^2}{F_r^2} \sum_\ell {A'_{r\ell}}^2 = F_r u'_r \gamma_r^2 \frac{G^4 F^2}{F_r^2} = F_r \Phi_r$$

where Φ_r is an invariant holomorphic function on W_r whose restriction $\Phi_r|_{W_r \cap \mathbb{R}^n}$ is positive semidefinite and does not vanish at $Y_r \setminus \bigcup_{i \neq r} Y_i$.

(iv) For all $i \neq r$ we have that F_r divides $\sum_\ell \big(\gamma_i \frac{G^2 F}{F_i} A'_{i\ell}\big)^2$. Therefore, the holomorphic function

$$\sum_{i \neq r} \sum_\ell \left(\gamma_i \frac{G^2 F}{F_i} A'_{i\ell} \right)^2$$

is divisible by F_r in $\mathcal{O}(W_r)$. Moreover, the quotient Θ_r of such division is invariant and its restriction $\Theta_r|_{W_r \cap \mathbb{R}^n}$ is positive semidefinite on $W_r \cap \mathbb{R}^n$.

Now, we turn to prove our claim. Let $r \in I$ and $x \in Y_r \setminus \bigcup_{i \neq r} Y_i$. We have that

$$\begin{aligned} F'_x \mathcal{O}_{\mathbb{C}^n,x} &= \bigg(\sum_\ell \Big(\gamma_r^2 \frac{G^2 F}{F_r} A'_{r\ell,x} \Big)^2 + \sum_{i \neq r} \sum_\ell \Big(\gamma_i^2 \frac{G^2 F}{F_i} A'_{i\ell,x} \Big)^2 + G_x^4 F_x^2 \bigg) \mathcal{O}_{\mathbb{C}^n,x} \\ &= \big(F_{r,x} \Phi_{r,x} + F_{r,x} \Theta_{r,x} + G_x^4 F_x^2 \big) \mathcal{O}_{\mathbb{C}^n,x} \\ &= \big(F_{r,x} (\Phi_{r,x} + \Theta_{r,x}) + G_x^4 F_x^2 \big) \mathcal{O}_{\mathbb{C}^n,x} \\ &= G_x^2 F_x \mathcal{O}_{\mathbb{C}^n,x}, \end{aligned}$$

hence, our claim is true.

Thus, since ${f'}^{-1}(0) = (g^2 f)^{-1}(0) = \bigcup_{i \in I} Y_i$, the meromorphic function $g^2 f / f'$ is analytic on $\mathbb{R}^n$ off the discrete set $D = \bigcup_{i,r: i \neq r} Y_i \cap Y_r$. By Lemma 4.3, there exist analytic functions $\Delta_1, \Delta_2 \colon \mathbb{R}^n \to \mathbb{R}$ such that $\Delta_i^{-1}(0) \subset D$, Δ_2 is a finite sum of squares and $\Delta_1^2 g^2 f = \Delta_2 f'$.

Consequently, since moreover $(\Delta_1 g)^{-1}(0) \subset f^{-1}(0)$, f is a sum of squares of meromorphic functions on $\mathbb{R}^n$ with controlled bad set.

Step 5b: *Glueing of sums of squares under the condition* (b). In this step, we paste all the sums of squares $\sum_k A_{jk}^2$ to get a single one if the irreducible components $\{Y_i\}_{i\in I}$ of $X = \bigcup_{j\in J} X_j$ satisfy the condition (b) in the statement, that is, each Y_i is a compact set.

Again, we may assume that $I = \mathbb{N}$ because if I is finite the result follows straightforwardly from Proposition 4.1. Fix $i \in I$; since Y_i is a compact set and the family $\{X_j\}_{j\in J}$ is locally finite, we deduce that Y_i intersects finitely many X_j's. We define $J_i = \{j \in J : Y_i \cap X_j \neq \emptyset\}$ which is a finite set and

$$F_i = \prod_{j\in J_i} {G'_j}^2 H_j.$$

By Proposition 4.1, the function $\prod_{j\in J_i}\sum_k {A_{jk}}^2$ is a convergent sum of squares

$$\sum_\ell {A'_{i\ell}}^2 = \prod_{j\in J_i}\sum_k {A_{jk}}^2$$

on $\mathcal{U}$ in the sense of Definition 1.2 (ii). We claim: *If $x \in Y_i$, then* $\sum_\ell {A'_{i\ell,x}}^2 \mathcal{O}_{\mathbb{C}^n,x} = F_{i,x}\mathcal{O}_{\mathbb{C}^n,x}$. Indeed, let $J_x = \{j \in J : x \in X_j\}$ which is a subset of J_i. Recall that for each $j \in J$ the restriction of $\sum_k {A_{jk}}^2$ to $\mathbb{R}^n$ vanishes only at X_j (see Step 4). We have that

$$\begin{aligned} F_{i,x}\mathcal{O}_{\mathbb{C}^n,x} &= \prod_{j\in J_i} {G'_{j,x}}^2 H_{j,x}\mathcal{O}_{\mathbb{C}^n,x} = \prod_{j\in J_x} {G'_{j,x}}^2 H_{j,x}\mathcal{O}_{\mathbb{C}^n,x} \\ &= \prod_{j\in J_x}\sum_k {A_{jk,x}}^2\mathcal{O}_{\mathbb{C}^n,x} = \prod_{j\in J_i}\sum_k {A_{jk,x}}^2\mathcal{O}_{\mathbb{C}^n,x} = \sum_\ell {A'_{i\ell,x}}^2\mathcal{O}_{\mathbb{C}^n,x}. \end{aligned}$$

In the same way as in Step 5a, we can find real numbers $\gamma_i > 0$ such that the sum of squares

$$F' = G^4F^2 + \sum_{i,\ell}\Big(\gamma_i\,\frac{G^2F}{F_i}A'_{i\ell}\Big)^2$$

is convergent in the sense of Definition 1.2 (ii).

For a fixed $r \in I$, we claim: *F_r divides* $\sum_\ell \big(\gamma_i \frac{G^2F}{F_i}A'_{i\ell}\big)^2 = \gamma_i^2\big(\frac{G^2F}{F_i}\big)^2\sum_\ell {A'_{i\ell}}^2$ *for all $i \in I$, in a small enough neighbourhood of Y_r in $\mathcal{U}$.* Indeed, if $x \in Y_r$ and

$J_x = \{j \in J : x \in X_j\}$, we have

$$\begin{aligned}
&\gamma_i^2 \left(\frac{G_x^2 F_x}{F_{i,x}}\right)^2 \sum_\ell \left(A'_{i\ell,x}\right)^2 \mathcal{O}_{\mathbb{C}^n,x} \\
&\quad = \left(\frac{\prod_{j\in J_x} {G'_{j,x}}^2 H_{j,x}}{\prod_{j\in J_x\cap J_i} {G'_{j,x}}^2 H_{j,x}}\right)^2 \prod_{j\in J_i}\sum_k A_{jk,x}^2 \mathcal{O}_{\mathbb{C}^n,x} \\
&\quad = \prod_{j\in J_x\setminus J_i} {G'_{j,x}}^4 H_{j,x}^2 \prod_{j\in J_i\cap J_x}\sum_k A_{jk,x}^2 \mathcal{O}_{\mathbb{C}^n,x} \\
&\quad = \prod_{j\in J_x\setminus J_i} {G'_{j,x}}^4 H_{j,x}^2 \prod_{j\in J_i\cap J_x} {G'_{j,x}}^2 H_{j,x} \mathcal{O}_{\mathbb{C}^n,x} \\
&\quad = \prod_{j\in J_x} {G'_{j,x}}^2 H_{j,x} \prod_{j\in J_x\setminus J_i} {G'_{j,x}}^2 H_{j,x} \mathcal{O}_{\mathbb{C}^n,x} = F_{r,x} \prod_{j\in J_x\setminus J_i} {G'_{j,x}}^2 H_{j,x} \mathcal{O}_{\mathbb{C}^n,x}.
\end{aligned}$$

The last equality is a straightforward consequence of the fact that $J_x \subset J_r$. Thus, we deduce that $F_r = \prod_{j\in J_r} {G'_j}^2 H_j$ divides $\sum_\ell \left(\gamma_i^2 \frac{G^2F}{F_i} A'_{i\ell}\right)^2$ in a small enough neighbourhood of Y_r in $\mathcal{U}$, as wanted.

Next, we denote by f', f_i the restrictions to $\mathbb{R}^n$ of F', F_i for all $i \in I$. We claim: *For each $r \in I$ we have $g_x^2 f_x \mathcal{O}_{\mathbb{R}^n,x} = f'_x \mathcal{O}_{\mathbb{R}^n,x}$ for all $x \in Y_r$.*

Before showing this we summarize the following facts already known:

(i) $G_x^2 F_x \mathcal{O}_{\mathbb{C}^n,x} = F_{r,x}\mathcal{O}_{\mathbb{C}^n,x}$ for all $x \in Y_r$.

(ii) $\sum_\ell {A'_{r\ell,x}}^2 \mathcal{O}_{\mathbb{C}^n,x} = F_r \mathcal{O}_{\mathbb{C}^n,x}$ for all $x \in Y_r$.

(iii) $\gamma_r^2 \frac{G^4F^2}{F_r^2}$ does not vanish at Y_r. Hence, by (ii), we have that

$$\sum_\ell \gamma_r^2 \frac{G^4F^2}{F_r^2} {A'_{r\ell}}^2 \mathcal{O}_{\mathbb{C}^n,x} = F_r \mathcal{O}_{\mathbb{C}^n,x}.$$

for all $x \in Y_r$. Thus, there exist an open neighbourhood W_r of Y_r in $\mathcal{U}$ and an invariant holomorphic function Φ_r on W_r whose restriction $\Phi_r|_{W_r\cap\mathbb{R}^n}$ is positive semidefinite, such that

$$\sum_\ell \gamma_r^2 \frac{G^4F^2}{F_r^2} {A'_{r\ell}}^2 = F_r \Phi_r,$$

and it does not vanish at Y_r.

(iv) For all $i \neq r$ we have that F_r divides $\sum_\ell \left(\gamma_i \frac{G^2F}{F_i} A'_{i\ell}\right)^2$ in a small neighbourhood of Y_r in $\mathcal{U}$. Therefore, by 4.4, the holomorphic function

$$\sum_{i\neq r}\sum_\ell \left(\gamma_i^2 \frac{G^2F}{F_i} A'_{i\ell}\right)^2$$

is divisible by F_r in $\mathcal{H}(W'_r)$ where W'_r is a small enough neighbourhood of Y_r. Moreover, the quotient Θ_r is invariant and its restriction $\Theta_r|_{W'_r \cap \mathbb{R}^n}$ is positive semidefinite on $W'_r \cap \mathbb{R}^n$.

Now, we turn to prove our claim. Indeed, let $r \in I$ and $x \in Y_r$. We have that

$$\begin{aligned}
F'_x \mathcal{O}_{\mathbb{C}^n,x} &= \Big(\sum_\ell \Big(\gamma_r^2 \frac{G^2 F}{F_r} A'_{r\ell,x} \Big)^2 + \sum_{i \neq r} \sum_\ell \Big(\gamma_i^2 \frac{G^2 F}{F_i} A'_{i\ell,x} \Big)^2 + G_x^4 F_x^2 \Big) \mathcal{O}_{\mathbb{C}^n,x} \\
&= \big(F_{r,x} \Phi_{r,x} + F_{r,x} \Theta_{r,x} + G_x^4 F_x^2 \big) \mathcal{O}_{\mathbb{C}^n,x} \\
&= (F_{r,x} \big(\Phi_{r,x} + \Theta_{r,x}) + G_x^4 F_x^2 \big) \mathcal{O}_{\mathbb{C}^n,x} \\
&= G_x^2 F_x \mathcal{O}_{\mathbb{C}^n,x}.
\end{aligned}$$

Thus, since $f'^{-1}(0) = (g^2 f)^{-1}(0) = \bigcup_{i \in I} Y_i$, the meromorphic function $g^2 f / f'$ is analytic and positive semidefinite on $\mathbb{R}^n$. Consequently, there exists a positive semidefinite analytic unit $u \colon \mathbb{R}^n \to \mathbb{R}$ such that $g^2 f = f' u^2$, and therefore f is an infinite sum of squares of meromorphic functions. □

Remark 4.6 Note the following:

(1) In the step 5a of the proof of Theorem 1.4 we have only used the fact that $Y_i \cap Y_k$ is a discrete set for $i \neq k$ to apply, in a crucial way, Lemma 4.3 at the end of such step. However, it seems difficult to get similar results to Lemma 4.3 for a more general situations, because if $n \geq 3$ the special irreducible factors could appear whenever the dimension of the zero set of a positive semidefinite analytic function is > 0. Recall that if the zero set of a special factors is not compact we do not know a priori if it is a sum of squares of meromorphic functions.

(2) In the step 5b of the proof of Theorem 1.4 we only have used that the analytic sets Y_i are compact to have that: *each Y_i intersects only finitely many of the X_j's.* □

Before proving Theorem 1.6 we need a preliminary additional result whose proof is similar to the one of Theorem 1.4. However, its particular delicate technical details strongly suggest to reproduce the full proof and not only to give a patch for the one of Theorem 1.4.

Proposition 4.7. *Let $q \geq 1$ be an integer and let $\{f_j \colon \mathbb{R}^n \to \mathbb{R}\}_{j \in J}$ be a family of positive semidefinite analytic functions such that*

(a) *$f_j^{-1}(0) \cap f_k^{-1}(0)$ is a discrete set if $j \neq k$,*

(b) *$\{f_j^{-1}(0)\}_{j \in J}$ is a locally finite family in $\mathbb{R}^n$, and*

(c) *f_j is a sum of q squares with controlled bad set at $f_j^{-1}(0)$ for all $j \in J$.*

Then there exist analytic functions $g_1, g_2, f, f', f'' \colon \mathbb{R}^n \to \mathbb{R}$ such that

(i) $f_x\mathcal{O}_{\mathbb{R}^n,x} = \prod_{j,x\in f_j^{-1}(0)} f_{j,x}\mathcal{O}_{\mathbb{R}^n,x}$ *for all* $x \in f^{-1}(0) = \bigcup_{j\in J} f_j^{-1}(0)$,

(ii) $g_i^{-1}(0) \subset f^{-1}(0)$ *for* $i = 1, 2$,

(iii) f' *is a sum of* q *squares of analytic functions on* $\mathbb{R}^n$,

(iv) f'' *is a sum of* $2^n + n$ *squares of analytic functions on* $\mathbb{R}^n$ *and its zero set, which is contained in* $f^{-1}(0)$*, is discrete, and*

(v) $g_1^2 f = f''(f' + g_2^4 f^2)$.

Proof. We will split the proof into several steps.

Step 1: *Preparation.* First, we write $X_j = f_j^{-1}(0)$ for all $j \in J$. In the same way as the Step 1 of the proof of Theorem 1.4, there exist:

- an open invariant Stein neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ such that $\mathbb{R}^n$ is a deformation retract of $\mathcal{U}$ ([Ca]),
- open neighbourhoods V_j's of the X_j's in $\mathcal{U}$, and
- invariant holomorphic functions $G_j, F_j, B_{jk} \colon V_j \to \mathbb{C}$, $1 \le k \le q$, such that $G_j^2 F_j = \sum_k B_{jk}^2$, $F_j = f_j|_{\mathbb{R}^n\cap V_j}$ and $Y_j = G_j^{-1}(0) \cap \mathbb{R}^n \subset X_j$, which satisfy the following properties:
  - $T_j = G_j^{-1}(0) \subset V_j$, $S_j = F_j^{-1}(0) \subset V_j$, and
  - all T_j's, S_j's are closed analytic subsets of $\mathcal{U}$, as well as their unions $S = \bigcup_j S_j$, $T = \bigcup_j T_j$.

Step 2: *Extension of denominators and the positive semidefinite analytic functions.* Proceeding as in Step 2 of the proof of Theorem 1.4, for each $j \in J$ one can construct invariant holomorphic functions $G'_j \colon \mathcal{U} \to \mathbb{C}$ such that ${G'_j}^{-1}(0) = T_j$ and

$$G_j'^2 F_j = \sum_k C_{jk}^2,$$

where the series $\sum_k C_{jk}^2$ verifies the convergence condition (ii) on the open set V_j in Definition 1.2.

In a similar way, one can get invariant holomorphic functions F'_j on $\mathcal{U}$ that extend the functions f_j to $\mathcal{U}$ after multiplying f_j by the square of a suitable strictly positive analytic function on $\mathbb{R}^n$. We denote again by F_j the functions F'_j and by f_j their restrictions to $\mathbb{R}^n$. We also denote by G_j the functions G'_j.

Step 3: *Glueing of denominators and the positive semidefinite analytic functions.* Proceeding as in Step 3 of the proof of Theorem 1.4, one can construct an invariant holomorphic function $G \colon \mathcal{U} \to \mathbb{C}$ such that $G^{-1}(0) = \bigcup_j T_j$ and

$$G \cdot \mathcal{O}_{\mathbb{C}^n,x} = \prod_{j,x\in T_j} G_j \cdot \mathcal{O}_{\mathbb{C}^n,x}.$$

Moreover, if g denotes the real analytic function $g = G|_{\mathbb{R}^n}$, we have that $g^{-1}(0) = \bigcup_j T_j \cap \mathbb{R}^n \subset \bigcup_j X_j \subset f^{-1}(0)$.

Analogously, it can be obtained an invariant holomorphic function F on $\mathcal{U}$ such that $f = F|_{\mathbb{R}^n}$ is a positive semidefinite analytic function on $\mathbb{R}^n$ and

$$F \cdot \mathcal{O}_{\mathbb{C}^n,x} = \prod_{j,x\in S_j} F_j \cdot \mathcal{O}_{\mathbb{C}^n,x} \quad \text{for all } x \in S.$$

Step 4: *Globalization of sums of squares.* Here we find global sums of squares $\sum_{k=1}^{q} A_{jk}^2$ to replace the sums $\sum_{k=1} C_{jk}^2$, which are defined only on the V_j's.

Up to shrinking V_j, we may assume that V_j is invariant and does not intersect other connected components of $(G_j{}^2 F_j)^{-1}(0)$ different to the one that intersects $\mathbb{R}^n$ in X_j. By Proposition 2.2, applied to $\Phi = (G_j{}^2 F_j)^2$, $V = V_j$ and $C_k = C_{jk}$, there exist invariant holomorphic functions $A_{jk}\colon \mathcal{U} \to \mathbb{C}$, $1 \leq k \leq q$ such that $(G_j{}^2 F_j)^2$ divides $A_{jk} - C_{jk}$ in $\mathcal{H}(V_j)$.

In V_j we have:

$$\sum_{k=1}^{q} A_{jk}^2 - G_j{}^2 F_j = \sum_{k=1}^{q} A_{jk}^2 - \sum_{k=1} C_{jk}^2 = \sum_{k=1}^{q} \big(A_{jk}^2 - C_{jk}^2\big).$$

By construction, $(G_j{}^2 F_j)^2$ divides in $\mathcal{H}(V_j)$ each term $A_{jk}^2 - C_{jk}^2 = (A_{jk}+C_{jk})(A_{jk}-C_{jk})$, hence it divides their sum $\sum_{k=1}^{q} A_{jk}^2 - G_j{}^2 F_j$. Thus there is a holomorphic function $\Psi_j\colon V_j \to \mathbb{C}$ such that on V_j we have

$$\sum_{k=1}^{q} A_{jk}^2 = G_j{}^2 F_j + \Psi_j (G_j{}^2 F_j)^2 = u_j G_j{}^2 F_j, \quad \text{where } u_j = 1 + \Psi_j G_j{}^2 F_j.$$

Clearly, u_j has no zeros in $(G_j{}^2 F_j)^{-1}(0) \cap V_j$, hence, u_j is a holomorphic unit in a perhaps smaller neighbourhood V_j of $F_j^{-1}(0) \cup T_j$.

Step 5: *Glueing of sums of squares.* Here we paste all the sums of squares $\sum_k A_{jk}^2$ to get a single one. We may assume that $J = \mathbb{N}$ since the case where J is a finite set is similar but easier.

Let $\{K_j\}_{j\in J}$ be an exhaustion of $\mathcal{U}$ by compact sets (indexed using the set J). Define, for each $j \in J$:

$$M_j = \max_{1\leq i\leq q} \left\{ \sup_{K_j} \left| \frac{G^2 F}{G_j^2 F_j} \right|^2 |A_{ji}| \right\}.$$

and $\gamma_j = \frac{1}{2^j M_j}$. On the compact set K_j we have

$$\left| \gamma_j \left(\frac{G^2 F}{G_j^2 F_j} \right)^2 A_{jk} \right| \leq \frac{1}{2^j}\,.$$

Then, each infinite sum of holomorphic functions on $\mathcal{U}$

$$A_k = \sum_j \gamma_j \Big(\frac{G^2 F}{G_j^2 F_j}\Big)^2 A_{jk}, \quad k = 1, \dots, q,$$

is a well-defined holomorphic function on $\mathcal{U}$. Next, on V_j one writes

$$A_k = \gamma_j \Big(\frac{G^2 F}{G_j^2 F_j}\Big)^2 A_{jk} + \sum_{\ell \neq j} \gamma_\ell \Big(\frac{G^2 F}{G_\ell^2 F_\ell}\Big)^2 A_{\ell k} = \gamma_j \Lambda_j^2 A_{jk} + \Delta_{jk} G_j^4 F_j^2$$

where $\Lambda_j = \frac{G^2 F}{G_j^2 F_j}$ and Δ_{jk} are holomorphic functions on V_j. This is because $G_j^2 F_j$ divides $\frac{G^2 F}{G_\ell^2 F_\ell}$ for $j \neq \ell$. Hence, on V_j

$$F' = \sum_{k=1}^q A_k^2 = (\gamma_j \Lambda_j^2)^2 \sum_{k=1}^q A_{jk}^2 + \Delta_j G_j^4 F_j^2 = (\gamma_j^2 \Lambda_j^4 u_j + \Delta_j G_j^2 F_j) G_j^2 F_j,$$

where $\Delta_j = \sum_{k=1}^q (2\gamma_j \Lambda_j^2 A_{jk} \Delta_{jk} + \Delta_{jk}^2 G_j^4 F_j^2)$.

Thus, if $x \in X_j \setminus \bigcup_{\ell \neq j} X_\ell$ for some $j \in J$ we deduce that

$$f'_x \mathcal{O}_{\mathbb{R}^n,x} = g_{j,x}^2 f_{j,x} \mathcal{O}_{\mathbb{R}^n,x} = g_x^2 f_x \mathcal{O}_{\mathbb{R}^n,x}.$$

Next, we consider $f' + g^4 f^2$ which is a sum of $q+1$ squares of analytic functions on $\mathbb{R}^n$ and satisfies the same properties as f' for the germs at the points of $X_j \setminus \bigcup_{\ell \neq j} X_\ell$ for all $j \in J$. Since $(f' + g^4 f^2)^{-1}(0) = (g^2 f)^{-1}(0) = \bigcup_{j \in J} X_j$, the meromorphic function $g^2 f/(f' + g^4 f^2)$ is analytic on $\mathbb{R}^n$ off the discrete set $D = \bigcup_{j,\ell : j \neq \ell} X_j \cap X_\ell$. By Lemma 4.3, there exists analytic functions $\Delta_1, \Delta_2 \colon \mathbb{R}^n \to \mathbb{R}$ such that $\Delta_i^{-1}(0) \subset D$, Δ_2 is a sum of $2^n + n$ squares and $\Delta_1^2 g^2 f = \Delta_2 (f' + g^4 f^2)$.

Finally, we write $f'' = \Delta_2$, $g_1 = \Delta_1 g$ and $g_2 = g$, and taking account of the fact that $(\Delta_1 g)^{-1}(0) \subset f^{-1}(0)$ we are done. $\square$

Now, we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. First, by Lemma 1.1 there exist analytic functions $f_0, f_1, f_2, f_3 \colon \mathbb{R}^n \to \mathbb{R}$ such that f_1, f_2, f_3 are positive semidefinite, $f = f_0^2 f_1 f_2 f_3$ and

- $f_1^{-1}(0)$ is a discrete set,
- f_2 is a sum of two squares of analytic functions on $\mathbb{R}^n$, and
- All the irreducible complex factors of f_3 are special and divide f_3 with multiplicity 1. In fact, the special factors of f are the same that the ones of f_3.

Next we claim: *There exist analytic functions* $g_1, g_2, f_3', f_3'' \colon \mathbb{R}^n \to \mathbb{R}$ *such that*

(i) $g_i^{-1}(0) \subset f_3^{-1}(0)$

(ii) f_3' *is a sum of* 2^{n+r-1} *squares of analytic functions on* $\mathbb{R}^n$,

(iii) f_3'' *is positive semidefinite and its zero set, which is a subset of* $f_3^{-1}(0)$, *is discrete, and*

(iv) $g_1^2 f_3 = f_3''(f_3' + g_2^4 f_3^2)$.

We begin with some preparation. We say that two elements $j_1, j_2 \in J$ are equivalent $j_1 \sim j_2$ if and only if $X_{j_1} = X_{j_2}$. The previous relation gives and equivalence relation in J. Consider the quotient set $A = J/\sim$. For each $\alpha \in A$ we set $X_\alpha = X_j$ for any $j \in \alpha$. The set X_α is well defined because if $j_1, j_2 \in \alpha$, then $X_{j_1} = X_{j_2}$. Since the family $\{X_j\}_{j \in J}$ is locally finite, each $\alpha \in A$ is a finite set.

At this point, we recall that the general Pfister's theory says that if K is a field of zero characteristic and $a, b \in K$ are sum of 2^d squares in K then ab is also a sum of 2^d squares in K (see [Pf], [L, XI.1.9]).

Let $f_{3,\alpha} = \prod_{j \in \alpha} h_j$, which is a sum of 2^r squares of meromorphic functions on a neighborhood of X_α because each h_j's is a sum of 2^r squares of meromorphic functions on a neighborhood of X_α. By Proposition 1.3, we have that $f_{3,\alpha}$ is a sum of 2^{r+n-1} squares of meromorphic functions on a perhaps smaller neighborhood of X_α with controlled bad set. Now, the claim follows straightforwardly from Proposition 4.7.

Next, we have that

$$g_1^2 f = g_1^2 f_0^2 f_1 f_2 f_3 = f_0^2 f_2 f_1 g_1^2 f_3 = (f_0^2 f_2)(f_1 f_3'')(f_3' + g_2^4 f_3^2)$$

where

- $f_0^2 f_2$ is a sum of $2 \leq 2^{n+r}$ squares of analytic functions on $\mathbb{R}^n$,
- $f_1 f_3''$ is positive semidefinite and its zero set is discrete, hence by [BKS] a sum of $2^n + n \leq 2^{n+r}$ squares of meromorphic functions on $\mathbb{R}^n$ with controlled bad set, and
- $(f_3' + g_2^4 f_3^2)$ is a sum of $2^{n+r-1} + 1 \leq 2^{n+r}$ squares of analytic functions on $\mathbb{R}^n$.

Thus, $g_1^2 f$ is a finite product of sums of 2^{n+r} squares of meromorphic functions on $\mathbb{R}^n$; hence, it is a sum of 2^{n+r} squares of meromorphic functions on $\mathbb{R}^n$. □

Remark 4.8 We cannot, however, guarantee that for such expression of f, as a sum of 2^{n+r} squares of meromorphic functions on $\mathbb{R}^n$, the bad set is controlled. To control the bad set we should apply again Proposition 1.3, which produces a new controlled increase in the number of squares.

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Received September 1, 2005

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