On Hipp's compound Poisson approximations via concentration functions

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This paper is devoted to a refinement of Hipp's method in the compound Poisson approximation to the distribution of the sum of independent but not necessarily identically distributed random variables. Approximations by related Kornya–Presman signed measures are also considered. By using alternative proofs, we show that several constants in the upper bounds for the Kolmogorov and the stop-loss distances can be reduced. Concentration functions play an important role in Hipp's method. Therefore, we provide an improvement of the constant in Le Cam's bound for concentration functions of compound Poisson distributions. But we also follow Hipp's idea to estimate such concentration functions with the help of Kesten's concentration function bound for sums of independent random variables. In fact, under the assumption that the summands are identically distributed, we present a smaller constant in Kesten's bound, the proof of which is based on a slight sharpening of Le Cam's version of the Kolmogorov–Rogozin inequality.

Keywords: compound Poisson approximation; concentration functions; explicit constants; Hipp's method; individual risk model; Kolmogorov distance; Kornya–Presman signed measures; random sums; stop-loss distance; sums of independent random variables; upper bounds

1. Introduction

1.1. Motivation

The compound Poisson approximation of the distribution of the sum of independent but not necessarily identically distributed random variables has a long history. Such an approximation is reasonable when the summands are non-zero with small probabilities. In fact, in this case, the approximation error between the distributions involved is small. Though several upper bounds for different distances are nowadays available, there remain some difficult tasks. For instance, there is the problem of giving a good estimate for the constant which appears in the upper bound due to Zaĭtsev (1983, formula (6), p. 658); in fact, he improved the order of Le Cam's (1965, Theorem 3, p. 188; see also Le Cam 1986, Proposition 4, pp. 413–414) bound by using the so called 'method of triangular functions', which was invented by Arak and Zaĭtsev in the 1980s in order to find the optimal rate in Kolmogorov's (1956) second uniform limit theorem. For details of this method, see the monograph by Arak and Zaĭtsev (1988). Further developments can be found, for example, in Čekanavičius (2003) and his previous papers. Because of the complexity of this method,

constants are not explicitly specified; even if one followed the proofs by taking into account explicit constants, the final constant would be very large. In order to avoid this difficulty, Hipp (1985; 1986) invented his own method and proved some estimates, which are not easily comparable with the Zaĭtsev bound. As approximations, he considered not only the compound Poisson distribution but also finite signed measures, which can be derived from an expansion in the exponent. Apparently, such approximations were first considered by Kornya (1983) and Presman (1983), as a result of which we speak of Kornya–Presman signed measures. However, observe that the signed measures used by Kornya and Presman are slightly different (see also Hipp 1986). Further results in this direction were given, for example, by Kruopis (1986), Čekanavičius (1997), Barbour and Xia (1999) and Roos (2002). Note that Barbour *et al*. (1992a) and Barbour and Xia (1999; 2000) applied Stein's method but obtained some unwanted terms in their bounds, some of which could be removed by using Kerstan's approach (see Roos 2003). However, it should be mentioned that, in contrast to Kerstan's approach, Stein's method also works in the context of dependent variables. Roos (2003) gives a more detailed review of known results.

This paper is devoted to a refinement of Hipp's (1985; 1986) method. Motivated by the need for explicit approximation results for the individual aggregate claims distribution within the context of risk theory, Hipp has used concentration functions in his estimates of the Kolmogorov and stop-loss distances. The aim of the present paper is to present alternative proofs and smaller constants in the bounds. Additionally, we provide an improvement of the constant in Le Cam's (1986, remark on p. 408) bound for concentration functions of compound Poisson distributions. But we also follow Hipp's idea to give an estimate of such concentration functions with the help of Kesten's (1969) concentration function bound for sums of independent random variables. In fact, under the assumption that the summands are identically distributed, we present a smaller constant in Kesten's bound, the proof of which is based on a slight sharpening of Le Cam's (1986, Theorem 2, p. 411) version of the Kolmogorov–Rogozin inequality; see Kolmogorov (1958) and Rogozin (1961, Theorem 1, p. 95). For the theory of concentration functions, the reader is referred to Hengartner and Theodorescu (1973), Petrov (1975; 1995), and Arak and Zaĭtsev (1988). Note that, in the literature, many contributions on upper bounds of concentration functions can be found. But only a small number of them deal with explicit constants; see, for instance, Salikhov (1996) and Nagaev and Khodzhibagyan (1996). Though it is often possible to derive bounds for concentration functions for sums of independent but not necessarily identically distributed random variables, in this paper we only need to consider identically distributed summands.

1.2. Notation

1.2.1. Concentration functions

The concentration functions $\mathrm{Conc}(Q; \cdot)$, $\mathrm{Conc}^-(Q; \cdot) : [0, \infty) \to [0, 1]$ of a probability measure Q on $\mathbb{R}$ are defined by

$$\mathrm{Conc}(Q; t) = \sup_{x \in \mathbb{R}} Q([x, x + t]),$$

$$\mathrm{Conc}^-(Q; t) = \sup_{x \in \mathbb{R}} Q((x, x + t]), \qquad t \in [0, \infty).$$

We have listed some basic properties of concentration functions in the Appendix.

1.2.2. Stop-loss transforms

For a finite signed measure Q on $\mathbb{R}$, let $|Q|$ denote the total variation measure and $F_Q = Q((-\infty, \cdot])$ the distribution function of Q. The stop-loss transform π_Q of Q at a point $t \in \mathbb{R}$ is defined by

$$\pi_Q(t) = \int_{\mathbb{R}} (x - t)_+ \, \mathrm{d}Q(x).$$

Here and throughout this paper, $x_+ = x \vee 0$, $x \vee y = \max\{x, y\}$, and $x \wedge y = \min\{x, y\}$ for $x, y \in \mathbb{R}$. Whenever we deal with a stop-loss transform π_Q, to ensure that π_Q has finite values, we assume that $\int_{\mathbb{R}} |x| \, \mathrm{d}|Q|(x) < \infty$. For a real-valued random variable X with distribution $\mathcal{L}(X)$ and distribution function $F_X = F_{\mathcal{L}(X)}$, we set $\overline{F}_X = 1 - F_X$; if $\mathrm{E}(X)$ is finite, the stop-loss transform $\pi_X = \pi_{\mathcal{L}(X)}$ of X is finite and satisfies, for $t \in \mathbb{R}$,

$$\pi_X(t) = \mathrm{E}(X - t)_+ = \int_t^\infty \overline{F}_X(x) \, \mathrm{d}x = \mathrm{E}(X_+) - \int_0^t \overline{F}_X(x) \, \mathrm{d}x,$$

where, as usual, '$\int_x^y = -\int_y^x$' for $x, y \in \mathbb{R}$. Similar formulae for π_Q are possible when Q is a finite signed measure. Note that, in the context of stop-loss reinsurance, a risk X (i.e. a non-negative random variable) is divided between the ceding company and the reinsurer in such a way that the reinsurer has to pay the excess $(X - t)_+$ over an agreed retention $t > 0$, whereas the ceding company has to pay the remaining amount $X \wedge t$; hence $\pi_X(t)$ denotes the expected claim of the reinsurer.

1.2.3. Distances

As measures of accuracy, we consider the following distances

$$d_{\mathrm{KM}}(Q_1, Q_2) = \sup_{x \in \mathbb{R}} |F_{Q_1}(x) - F_{Q_2}(x)| \qquad \text{(Kolmogorov metric)},$$

$$d_{\mathrm{SL}}(Q_1, Q_2) = \sup_{t \in \mathbb{R}} |\pi_{Q_1}(t) - \pi_{Q_2}(t)| \qquad \text{(stop-loss metric)},$$

between two finite signed measures Q_1 and Q_2 on $\mathbb{R}$. For two real-valued random variables X and Y, we write

$$d_{\mathrm{KM}}(X, Y) = d_{\mathrm{KM}}(\mathcal{L}(X), \mathcal{L}(Y)) \quad \text{and} \quad d_{\mathrm{SL}}(X, Y) = d_{\mathrm{SL}}(\mathcal{L}(X), \mathcal{L}(Y)).$$

Sometimes it will be necessary to consider also the Fortet–Mourier metric

$$d_{\mathrm{FM}}(X,\ Y) = d_{\mathrm{FM}}(\mathcal{L}(X),\ \mathcal{L}(Y)) = \int_{\mathbb{R}} |F_X(x) - F_Y(x)| \, \mathrm{d}x$$

between X and Y, and an ℓ_1 version of the stop-loss metric

$$\tilde{d}_{\mathrm{SL}}(M,\ N) = \tilde{d}_{\mathrm{SL}}(\mathcal{L}(M),\ \mathcal{L}(N)) = \sum_{n=0}^{\infty} |\pi_M(n) - \pi_N(n)|,$$

between random variables M and N concentrated on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

1.2.4. Exponentials

In what follows, we need exponentials of finite signed measures. If Q denotes a finite signed measure on $\mathbb{R}$, then we set

$$\exp(Q) = \sum_{j=0}^{\infty} \frac{1}{j!} \, Q^{*j},$$

where, for $j \in \mathbb{N} = \{1, 2, \ldots\}$, Q^{*j} denotes the j-fold convolution of Q with itself and $Q^{*0} = \varepsilon_0$ is the Dirac measure at point 0. Note that $\exp(Q)$ is a finite signed measure. It is well known that, for finite signed measures Q_1 and Q_2, we have $\exp(Q_1) * \exp(Q_2) = \exp(Q_1 + Q_2)$; see, for example, Hipp (1985; 1986) and Hipp and Michel (1990, Chapter 4). This and other similar facts regarding finite signed measures can easily be proved with the help of the Hahn–Jordan decomposition and characteristic functions. For a probability distribution Q on $\mathbb{R}$ and parameter $t \in [0, \infty)$, we define the compound Poisson distribution by

$$\mathrm{CPo}(t,\ Q) = \exp(t(Q - \varepsilon_0)) = \sum_{j=0}^{\infty} \mathrm{po}(j,\ t) \, Q^{*j},$$

where $\mathrm{po}(j,\ t) = \mathrm{e}^{-t}\, t^j / j!$.

2. Results

2.1. Hipp-type results

In the following proposition, we are concerned with the approximation by a compound Poisson distribution.

Proposition 1. *Let* $n \in \mathbb{N}$ *and* $X_1, \ldots, X_n$ *be non-negative and independent random variables. Set* $S_n = \sum_{i=1}^{n} X_i$ *and, for all* $i \in \{1, \ldots, n\}$,

$$p_i = P(X_i > 0), \qquad Q_i = P(X_i \in \cdot \mid X_i > 0),$$

$$\mu_i = \int x \, dQ_i(x), \qquad \mu_i^{(2)} = \int x^2 \, dQ_i(x),$$

$$c_i = \begin{cases} 1, & \text{if } Q_i \text{ is a Dirac measure,} \\ 2, & \text{otherwise,} \end{cases} \qquad c_i' = \frac{c_i - 1}{4}.$$

Let

$$\lambda = \sum_{i=1}^{n} p_i, \qquad Q = \frac{1}{\lambda} \sum_{i=1}^{n} p_i Q_i, \qquad H = \mathrm{CPo}(\lambda, Q),$$

$$\tilde{\lambda} = \frac{1}{2} \sum_{i=1}^{n} p_i(1 - p_i), \qquad \tilde{Q} = \frac{1}{\tilde{\lambda}} \sum_{i=1}^{n} \frac{p_i(1 - p_i)}{2} Q_i, \qquad \tilde{H} = \mathrm{CPo}(\tilde{\lambda}, \tilde{Q}).$$

If, for all i, $p_i < 1$, *then*

$$d_{\mathrm{KM}}(\mathcal{L}(S_n), H) \leqslant \frac{\pi^2}{8} \sum_{i=1}^{n} \frac{c_i \, p_i^2}{1 - p_i} \, \mathrm{Conc}^-(\tilde{H}; \mu_i), \tag{1}$$

$$d_{\mathrm{SL}}(\mathcal{L}(S_n), H) \leqslant \frac{\pi^2}{4} \sum_{i=1}^{n} \frac{p_i^2}{1 - p_i} \left(\mu_i + c_i' \left(\mu_i + \frac{\mu_i^{(2)}}{\mu_i} \right) \right) \mathrm{Conc}^-(\tilde{H}; \mu_i). \tag{2}$$

Remark 1. (a) Hipp and Michel (1990), p. 51) give an inequality essentially the same as (1), and their proof can be used to establish (1); see also Hipp (1985). The bound (2) is slightly sharper than the one in Hipp and Michel (1990, p. 54). In fact, for non-degenerate probability distribution Q_i, their bound contains the term $\mu_i + \mu_i^{(2)}/(2\mu_i)$ instead of the smaller value

$$\mu_i + \frac{1}{4} \left(\mu_i + \frac{\mu_i^{(2)}}{\mu_i} \right).$$

(b) It is well known that, under the assumptions of Proposition 1, the distribution $\mathcal{L}(S_n)$ is smaller than or equal to H in the stop-loss order, that is, for all $t \in \mathbb{R}$, we have $\pi_{S_n}(t) \leqslant \pi_H(t)$; see Hipp and Michel (1990, p. 43). This may be helpful when dealing with the stop-loss distance.

(c) As pointed out by Hipp (1985), in order to obtain higher accuracy, the concentration functions in the upper bounds of Proposition 1 should be evaluated rather than estimated. Indeed, in many applications, where $\tilde{Q}$ is an arithmetic probability distribution with $\tilde{Q}(\{h, 2h, 3h, \ldots\}) = 1$ and $h \in (0, \infty)$, $\mathrm{Conc}^-(\tilde{H}; \mu_i)$ can be evaluated by using Panjer's (1981) recursive algorithm. Nevertheless, we provide some general upper bounds for concentration functions in Section 2.3.

(d) Note that Zaĭtsev (1983, formula (6), p. 658) has shown that

$$d_{\mathrm{KM}}(\mathcal{L}(S_n),\ H) \leqslant c \max_{1 \leqslant i \leqslant n} p_i,$$

where c denotes an absolute constant.

Remark 2. The situation in Proposition 1 can be interpreted within risk theory: let us consider the individual model with a portfolio of $n \in \mathbb{N}$ independent policies, producing the non-negative individual claim amounts $X_1, \ldots, X_n$. Each X_i can be written as a random sum $X_i = \sum_{k=1}^{M_i} U_{i,k}$. Here, for i fixed and $k \in \mathbb{N}$, the $U_{i,k}$ are positive, independent and identically distributed random variables and M_i is a Bernoulli random variable independent of the $U_{i,k}$ with $P(M_i = 1) = 1 - P(M_i = 0) = p_i$. The p_i represents the probability that risk i produces a positive claim, and so we can assume that p_i is small. Further, $\mathcal{L}(U_{i,1}) = Q_i$ is the conditional distribution of the claim in risk i, given that a positive claim occurs in risk i. The aggregate claim in the individual model is defined by the sum S_n of all X_i. Frequently the distribution $\mathcal{L}(S_n)$ of S_n is quite involved and should be approximated by a simpler distribution. Due to the smallness of the p_i, an approximation by a compound Poisson distribution $\mathrm{CPo}(\lambda, Q)$ is particularly favourable. Note that we obtain this distribution by Poissonization: if, in the sum $X_i = \sum_{k=1}^{M_i} U_{i,k}$, we replace M_i with an independent Poisson distributed random variable N_i with the same mean as M_i, then we obtain random variables $Y_i = \sum_{k=1}^{N_i} U_{i,k}$. Now, $\mathcal{L}(\sum_{i=1}^{n} Y_i) = \mathrm{CPo}(\lambda, Q)$. From this, we see that Corollary 1 below is applicable. The distribution $\mathrm{CPo}(\lambda, Q)$ can also be obtained as the aggregate claims distribution $\sum_{j=1}^{\tilde{M}} V_j$ of a suitable collective model: here only the claims V_j and their total number $\tilde{M}$ are modelled. In the present context, the number of claims $\tilde{M}$ has a Poisson $\mathrm{Po}(\lambda)$ distribution with mean λ and the claims are independent (also of the number of claims) and identically distributed random variables with distribution Q.

The next proposition deals with the approximation by Kornya–Presman signed measures H_K (see below).

Proposition 2. *Let the assumptions of Proposition 1 be valid. Further, set $K \in \mathbb{N}$,*

$$H_K = \exp\left(\sum_{i=1}^{n} \sum_{k=1}^{K} \frac{(-1)^{k+1}}{k} p_i^k (Q_i - \varepsilon_0)^{*k}\right),$$

$$\tau(i,\ K) = \frac{(2p_i)^{K+1}}{(K+1)(1-2p_i)} \qquad \text{for } i \in \{1, \ldots, n\},$$

$$\delta = \sum_{i=1}^{n} (e^{\tau(i,K)} - 1), \qquad \eta = \frac{\pi^2}{8(1-\delta)},$$

where we assume that, for all i, $p_i < \frac{1}{2}$ and $\delta < 1$. Then

$$d_{\mathrm{KM}}(\mathcal{L}(S_n), H_K) \leqslant \eta \sum_{i=1}^{n} c_i (\mathrm{e}^{\tau(i,K)} - 1) \operatorname{Conc}^-(\tilde{H}; \mu_i), \tag{3}$$

$$d_{\mathrm{SL}}(\mathcal{L}(S_n), H_K) \leqslant \eta \sum_{i=1}^{n} (\mathrm{e}^{\tau(i,K)} - 1)\left(\mu_i + c_i'\left(\mu_i + \frac{\mu_i^{(2)}}{\mu_i}\right)\right) \operatorname{Conc}^-(\tilde{H}; \mu_i). \tag{4}$$

Remark 3. Inequality (3) is better than that of Hipp and Michel (1990, p. 82); see also Hipp (1986). In fact, their bound contains the values $\mathrm{e}^{2\tau(i,K)}$ and $\operatorname{Conc}^-(\tilde{H}; (K+1)\mu_i)$ instead of the better ones $\mathrm{e}^{\tau(i,K)}$ and $\operatorname{Conc}^-(\tilde{H}; \mu_i)$. For $d_{\mathrm{SL}}(\mathcal{L}(S_n), H_K)$, we found no comparable bounds in the literature; therefore (4) seems to be new. However, in Hipp (1986, formula (10)), a non-uniform inequality for the difference of the stop-loss transforms of $\mathcal{L}(S_n)$ and the signed measures originally used by Kornya (1983) was presented. Note that, as mentioned above, these signed measures differ slightly from the H_K of the present paper.

Remark 4. The idea behind the use of the finite signed measure H_K is the following: using the log series and characteristic functions, it is easy to show that, for $i \in \{1, \ldots, n\}$,

$$\mathcal{L}(X_i) = \varepsilon_0 + p_i(Q_i - \varepsilon_0) = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} p_i^k (Q_i - \varepsilon_0)^{*k}\right).$$

Note that, since $p_i < \frac{1}{2}$, the infinite sum in the exponent converges with respect to the total variation norm and forms a finite signed measure. We obtain

$$\mathcal{L}(S_n) = \exp\left(\sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} p_i^k (Q_i - \varepsilon_0)^{*k}\right);$$

see also Hipp and Michel (1990, Chapter 4). Therefore, we should expect H_K to be a good approximation of $\mathcal{L}(S_n)$ if K is large. In fact, from Proposition 2, it follows that $d_{\mathrm{KM}}(\mathcal{L}(S_n), H_K)$ and $d_{\mathrm{SL}}(\mathcal{L}(S_n), H_K)$ tend to zero as $K \to \infty$, if $p_i < \frac{1}{2}$ and if the respective moments of Q_i are finite for all i.

2.2. Prerequisites for Proposition 1

The following theorem seems to be new.

Theorem 1. *Let* $X_1, X_2, X_3, \ldots$ *be non-negative, independent and identically distributed random variables. For* $n \in \mathbb{Z}_+$, *set* $S_n = \sum_{i=1}^{n} X_i$. *Let* M *and* N *be* $\mathbb{Z}_+$*-valued random variables with the same finite expectation. Let* Y *denote a random variable in* $\mathbb{R}$. *We assume that all* $Y, M, N, X_1, X_2, \ldots$ *are independent.*

(a) *Then we have*

$$d_{\mathrm{KM}}(S_M + Y, \ S_N + Y) \leqslant \tfrac{1}{2} d_{\mathrm{FM}}(M, \ N) \, d_{\mathrm{KM}}(Y, \ X_1 + Y).$$

(b) *If* $E(X_1) < \infty$, *then*

$$d_{SL}(S_M + Y, S_N + Y) \leq \tilde{d}_{SL}(M, N) E[(X_1 \wedge X_2) \operatorname{Conc}^-(\mathcal{L}(Y); X_1 + X_2)].$$

Note that the upper bounds in Theorem 1 are small when $\mathcal{L}(M) \approx \mathcal{L}(N)$, when $\mathcal{L}(X_1) \approx \varepsilon_0$, or when $\mathcal{L}(Y)$ has a small concentration. Theorem 1 and the telescopic sum decomposition enable us to give results concerning the approximation of sums of independent but not necessarily identically distributed random variables.

Corollary 1. *Let* $n \in \mathbb{N}$ *and* $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ *be independent random variables. For each* $i \in \{1, \ldots, n\}$, X_i *and* Y_i *are given by random sums of the form* $X_i = \sum_{k=1}^{M_i} U_{i,k}$ *and* $Y_i = \sum_{k=1}^{N_i} U_{i,k}$, *where, for* i *fixed, the* $U_{i,1}, U_{i,2}, U_{i,3}, \ldots$ *are non-negative, independent and identically distributed random variables and the* M_i *and* N_i *are random variables in* $\mathbb{Z}_+$ *with* $E(M_i) = E(N_i) < \infty$. *We assume that all* $M_i, N_i, U_{i,k}$ *are independent. Set* $S_n = \sum_{i=1}^{n} X_i$, $T_n = \sum_{i=1}^{n} Y_i$ *and, for* $i \in \{1, \ldots, n\}$, $Z_i = \sum_{j=1}^{i-1} X_j + \sum_{j=i+1}^{n} Y_j$.

(a) *Then we have*

$$d_{KM}(S_n, T_n) \leq \frac{1}{2} \sum_{i=1}^{n} d_{FM}(M_i, N_i) \, d_{KM}(Z_i, U_{i,1} + Z_i).$$

(b) *If* $E(U_{i,1}) < \infty$ *for all* $i \in \{1, \ldots, n\}$, *then*

$$d_{SL}(S_n, T_n) \leq \sum_{i=1}^{n} \tilde{d}_{SL}(M_i, N_i) E[(U_{i,1} \wedge U_{i,2}) \operatorname{Conc}^-(\mathcal{L}(Z_i); U_{i,1} + U_{i,2})].$$

Proof. The assertion easily follows from Theorem 1 in conjunction with the well-known telescopic sum decomposition

$$\mathcal{L}(S_n) - \mathcal{L}(T_n) = \sum_{i=1}^{n} (\mathcal{L}(X_i + Z_i) - \mathcal{L}(Y_i + Z_i)),$$

which, in turn, can be shown via induction over n. □

Corollary 1 is used in the proof of Proposition 1.

2.3. Concentration function bounds

The following proposition is devoted to Le Cam-type bounds for the concentration functions of a compound Poisson distribution. The absolute constant $(2e)^{-1/2}$ in the bounds is the best possible.

Proposition 3. *Let* Q *be a probability distribution on* $\mathbb{R}$. *Then, for* $t \in (0, \infty)$ *and* $s \in [0, \infty)$,

$$\mathrm{Conc}(\mathrm{CPo}(t, Q); s) \leqslant \frac{1}{\sqrt{2etf(s, Q)}}, \tag{5}$$

$$\mathrm{Conc}^-(\mathrm{CPo}(t, Q); s) \leqslant \frac{1}{\sqrt{2etg(s, Q)}}, \tag{6}$$

where

$$f(s, Q) = \max\{Q((-\infty, -s)), Q((s, \infty))\},$$

$$g(s, Q) = \max\{Q((-\infty, -s]), Q([s, \infty))\}.$$

In (5) and (6), equalities hold when $s \in (0, 1)$, $t = \frac{1}{2}$, and $Q = \varepsilon_1$ is the Dirac measure at point 1 such that $\mathrm{CPo}(t, Q) = \mathrm{Po}(\frac{1}{2})$.

Remark 5. From Le Cam's (1986, remark on p. 408) more general inequality for the concentration function of an infinitely divisible probability distribution, it follows that, under the assumptions of Proposition 3,

$$\mathrm{Conc}(\mathrm{CPo}(t, Q); s) \leqslant \left(\frac{2\pi}{t\, Q(\{x : |x| > s\})}\right)^{1/2} \tag{7}$$

(see also Le Cam 1965, Proposition 5, p. 183; Arak and Zaĭtsev 1988, Theorems 2.5 and 2.6, p. 46). Since $f(s, Q) \geqslant 2^{-1}Q(\{x : |x| > s\})$, it follows from (5) that the constant $\sqrt{2\pi} \approx 2.51$ in (7) can be replaced with $e^{-1/2} \approx 0.61$.

The bound (6) can be used to estimate the concentration functions in the upper bounds in Propositions 1 and 2. However, other bounds can be derived with the help of a Kesten-type inequality for the concentration function of the sum of independent and identically distributed random variables:

Proposition 4. *Let $S_n = \sum_{i=1}^{n} X_i$ be the sum of $n \in \mathbb{N}$ independent and identically distributed random variables $X_1, \ldots, X_n$. Then, for $t \in [0, \infty)$,*

$$\mathrm{Conc}(\mathcal{L}(S_n); t) \leqslant 6.33 \frac{\mathrm{Conc}(\mathcal{L}(X_1); t)}{\sqrt{(n+1)(1 - \mathrm{Conc}(\mathcal{L}(X_1); t))}}. \tag{8}$$

This inequality remains valid if Conc *is everywhere replaced by* Conc$^-$.

Remark 6. From a more general result of Kesten (1969, Corollary 1, pp. 134–135), it follows that, under the assumptions of Proposition 4,

$$\mathrm{Conc}(\mathcal{L}(S_n); t) \leqslant 4\sqrt{2}(1 + 9c)\frac{\mathrm{Conc}(\mathcal{L}(X_1); t)}{\sqrt{n(1 - \mathrm{Conc}(\mathcal{L}(X_1); t))}}. \tag{9}$$

Here, c is an absolute constant satisfying the classical Kolmogorov–Rogozin inequality (see Kolmogorov 1958; Rogozin 1961, Theorem 1, p. 95), which states that, under the same assumptions,

$$\mathrm{Conc}(\mathcal{L}(S_n);\, t) \leqslant \frac{c}{\sqrt{n(1 - \mathrm{Conc}(\mathcal{L}(X_1);\, t))}}. \tag{10}$$

Since $c \leqslant 1$ (see Remark 8 below), the leading constant in (9) is bounded from above by $40\sqrt{2} \leqslant 56.6$, which is considerably larger than our 6.33. A further advantage of (8) over (9) is the factor $(n + 1)^{-1/2}$ instead of $n^{-1/2}$.

Corollary 2. *Under the assumptions of Proposition 3, we have*

$$\mathrm{Conc}(\mathrm{CPo}(t,\, Q);\, s) \leqslant \mathrm{e}^{-t} + \frac{6.33\, \mathrm{Conc}(Q;\, s)(1 - \mathrm{e}^{-t})}{\sqrt{t(1 - \mathrm{Conc}(Q;\, s))}}. \tag{11}$$

This inequality remains valid if Conc *is everywhere replaced by* Conc⁻.

Remark 7. (a) The bound (11) can be much better than (5) if t is large and if Q has a small concentration function.

(b) Bening *et al.* (1997, Theorem 8, pp. 370–371) have shown that, under the assumptions of Proposition 3,

$$\mathrm{Conc}(\mathrm{CPo}(t,\, Q);\, s) \leqslant c(\epsilon,\, \delta)\, \frac{s + 1}{\sqrt{t}}, \tag{12}$$

where

$$c(\epsilon,\, \delta) = \left(\frac{96}{95}\right)^2 \max\left\{1,\, \frac{1}{\delta}\right\} \sqrt{\frac{\pi}{\epsilon}}$$

and $\epsilon,\, \delta > 0$ are defined in such a way that the characteristic function $\varphi_Q(x) = \int \mathrm{e}^{ixy}\, dQ(y)$ of Q satisfies

$$|\varphi_Q(x)| \leqslant 1 - \epsilon x^2 \qquad \text{whenever } |x| \leqslant \delta.$$

In fact, for each non-degenerate probability distribution Q, there exist positive numbers ϵ and δ with such a property (see Petrov 1975, Theorem 1.2.2, p. 11). It is easily shown that, under the present assumptions, $\max\{1,\, \delta^{-1}\}\epsilon^{-1/2} \geqslant 1$. Therefore the bound in (12) is often worse than the one in (11).

The proof of Proposition 4 is based on a refinement of Le Cam's version of the Kolmogorov–Rogozin inequality for the concentration function of the sum of independent random variables.

Proposition 5. *Under the assumptions of Proposition 4, we have*

$$\mathrm{Conc}(\mathcal{L}(S_n);\, t) \leqslant \left(\frac{1 - [\mathrm{Conc}(\mathcal{L}(X_1);\, t)]^{n+1}}{(n + 1)(1 - \mathrm{Conc}(\mathcal{L}(X_1);\, t))}\right)^{1/2}.$$

This inequality remains valid if Conc *is everywhere replaced by* Conc⁻.

Remark 8. From the more general Theorem 2 in Le Cam (1986, p. 411), it follows that, in

the Kolmogorov–Rogozin inequality (see (10)), one can choose $c = 1$. But his inequality is slightly better. In fact, under the assumptions of Proposition 4, his result implies that

$$\text{Conc}(\mathcal{L}(S_n);\ t) \leqslant \left(\frac{1 - \exp(-n(1 - \text{Conc}(\mathcal{L}(X_1);\ t)))}{n(1 - \text{Conc}(\mathcal{L}(X_1);\ t))} \right)^{1/2}.$$

However, it is easily shown that this bound is always larger than or equal to the one of Proposition 5.

3. Remaining proofs

In what follows, we use the forward difference operator $\Delta b : \mathbb{Z}_+ \to \mathbb{Z}_+$ of a sequence $b : \mathbb{Z}_+ \to \mathbb{Z}_+$, which is defined by $\Delta b_n = b_n - b_{n+1}$ for $n \in \mathbb{Z}_+$. Powers of Δ are understood in the sense of composition, that is, we have $\Delta^k b = \Delta(\Delta^{k-1} b)$ for $k \in \{2, 3, \ldots\}$. Sometimes we use the following version of Abel's summation formula.

Lemma 1. *For $n \in \mathbb{Z}_+$, let a_n, $b_n \in \mathbb{R}$, $A_n = \sum_{i=0}^{n} a_i$. If $\sum_{n=0}^{\infty} |a_n| < \infty$ and $\sum_{n=0}^{\infty} |b_n| < \infty$ then*

$$\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} a_n \sum_{m=n}^{\infty} \Delta b_m = \sum_{m=0}^{\infty} A_m \Delta b_m.$$

For the proof of Theorem 1, we need the following lemma.

Lemma 2. *Let the assumptions of Theorem 1 be valid and set, for $y \in \mathbb{R}$ $b_n(y) = F_{S_n}(y)$ and, for $n \in \mathbb{Z}_+$, $A_n = F_M(n) - F_N(n)$.*

(a) *For $y \in \mathbb{R}$, we have*

$$F_{S_M}(y) - F_{S_N}(y) = \sum_{n=0}^{\infty} A_n \Delta b_n(y).$$

(b) *If $\mathrm{E}(X_1) < \infty$, then, for $t \in \mathbb{R}$,*

$$\pi_{S_M}(t) - \pi_{S_N}(t) = \sum_{n=1}^{\infty} (\pi_M(n) - \pi_N(n)) \int_0^t \Delta^2 b_{n-1}(y) \, \mathrm{d}y.$$

Proof. We may assume that, for all i, $X_i \neq 0$ with positive probability. Assertion (a) follows with the help of Abel's summation formula. Here, we have used the fact that $\sum_{n=0}^{\infty} F_{S_n}(y)$ is a renewal function which is bounded on intervals of finite lengths (see, for example, Feller 1971, p. 359).

We now prove (b). We may assume that $t \in [0, \infty)$. Then

$$\pi_{S_M}(t) - \pi_{S_N}(t) = \int_t^{\infty} (F_{S_N}(y) - F_{S_M}(y)) \, \mathrm{d}y = \int_0^t (F_{S_M}(y) - F_{S_N}(y)) \, \mathrm{d}y,$$

where we have used the fact that, under the present assumptions, $E(S_M) = E(S_N) < \infty$. Application of Abel's summation formula to assertion (a) gives

$$F_{S_M}(y) - F_{S_N}(y) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} A_m \right) \Delta^2 b_n(y)$$

$$= \sum_{n=1}^{\infty} [\pi_M(n) - \pi_N(n)] \Delta^2 b_{n-1}(y)$$

for $y \in \mathbb{R}$, where we have taken into account that $\sum_{n=0}^{\infty} |A_n| \leqslant E(M + N) < \infty$ and that, for $k \in \mathbb{Z}_+$,

$$\sum_{n=0}^{k-1} A_n = - \sum_{n=k}^{\infty} A_n = \pi_M(k) - \pi_N(k).$$

To complete the proof of (b), we use Fubini's theorem, which is permitted since

$$\int_0^t \sum_{n=1}^{\infty} |(\pi_M(n) - \pi_N(n)) \Delta^2 b_{n-1}(y)| \, dy \leqslant 4 E(M + N) \int_0^t \sum_{n=0}^{\infty} F_{S_n}(y) \, dy < \infty.$$

$\square$

Proof of Theorem 1. Let $b_n(y)$ and A_n be defined as in Lemma 2. Using Lemma 2(a), we obtain, for all $c, x \in \mathbb{R}$,

$$P(S_M + Y \leqslant x) - P(S_N + Y \leqslant x) = E[F_{S_M}(x - Y) - F_{S_N}(x - Y)]$$

$$= E\left[\sum_{n=0}^{\infty} A_n \, \Delta b_n(x - Y) \right]$$

$$= \sum_{n=0}^{\infty} A_n (E[\Delta b_n(x - Y)] - c),$$

where the latter equality follows from Fubini's theorem and $E(M) = E(N) < \infty$, that is, $\sum_{n=0}^{\infty} A_n = 0$. It follows that

$$d_{KM}(S_M + Y, \ S_N + Y) \leqslant d_{FM}(M, \ N) \sup_{n \in \mathbb{Z}_+} \sup_{x \in \mathbb{R}} |E[\Delta b_n(x - Y)] - c|.$$

Since the X_i are non-negative, we have, for all $n \in \mathbb{Z}_+$ and $x \in \mathbb{R}$, $\Delta b_n(x - Y) \geqslant 0$, giving

$$0 \leqslant E[\Delta b_n(x - Y)]$$

$$= P(S_n + Y \leqslant x) - P(S_{n+1} + Y \leqslant x)$$

$$= E[F_Y(x - S_n) - F_{X_1 + Y}(x - S_n)]$$

$$\leqslant d_{KM}(Y, \ X_1 + Y).$$

Hence, if we set $c = 2^{-1} d_{KM}(Y, \ X_1 + Y)$, we obtain

$$|E[\Delta b_n(x - Y)] - c| \leqslant \tfrac{1}{2} d_{KM}(Y, \ X_1 + Y).$$

Assertion (a) immediately follows.

We now prove (b). For $t \in \mathbb{R}$, we have

$$\pi_{S_M + Y}(t) - \pi_{S_N + Y}(t) = E[(S_M + Y - t)_+ - (S_N + Y - t)_+]$$

$$= E[\pi_{S_M}(t - Y) - \pi_{S_N}(t - Y)]$$

$$= E\left[\sum_{n=1}^{\infty}(\pi_M(n) - \pi_N(n))\int_0^{t-Y} \Delta^2 b_{n-1}(y)\,\mathrm{d}y\right],$$

where we have used Lemma 2(b). Since the X_n are non-negative, independent and identically distributed with finite mean, $\pi_{S_n}(y)$ is a convex sequence in $n \in \mathbb{Z}_+$ for $y \in \mathbb{R}$ fixed, that is, $\Delta^2 \pi_{S_n}(y) \geqslant 0$ for all $n \in \mathbb{Z}_+$ (see, for example, Müller and Stoyan 2002, p. 160). In fact, this follows from the equalities

$$\Delta^2 \pi_{S_n}(y) = \pi_{S_{n+2}}(y) - 2\pi_{S_{n+1}}(y) + \pi_{S_n}(y)$$

$$= E[(S_n + X_{n+1} + X_{n+2} - y)_+$$

$$- (S_n + X_{n+1} - y)_+ - (S_n + X_{n+2} - y)_+ + (S_n - y)_+],$$

and the obvious fact that, for all $\alpha, \beta, \gamma \geqslant 0$,

$$(\alpha + \beta - y)_+ + (\alpha + \gamma - y)_+ \leqslant (\alpha + \beta + \gamma - y)_+ + (\alpha - y)_+.$$

From this we see that, for $n \in \mathbb{N}$, $\int_0^{t-Y} \Delta^2 b_{n-1}(y)\,\mathrm{d}y \geqslant 0$, and therefore we arrive at

$$d_{SL}(S_M + Y, \ S_N + Y) \leqslant \tilde{d}_{SL}(M, \ N)\sup_{n \in \mathbb{N}}\sup_{t \in \mathbb{R}} E\left[\int_0^{t-Y} \Delta^2 b_{n-1}(y)\,\mathrm{d}y\right].$$

For all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, we have, by conditioning on the values of S_{n-1}, X_n and X_{n+1},

$$E\left[\int_0^{t-Y} \Delta^2 b_{n-1}(y)\,\mathrm{d}y\right] = E\left[\int_0^{t-Y} [P(S_{n-1} \leqslant y) - P(S_n \leqslant y)\right.$$

$$\left. - P(S_{n-1} + X_{n+1} \leqslant y) + P(S_{n+1} \leqslant y)]\,\mathrm{d}y\right]$$

$$= \int E[Z_{a,b,c}(t - Y)]\,\mathrm{d}\mathcal{L}((S_{n-1}, X_n, X_{n+1}))(a, b, c),$$

where, for $x \in \mathbb{R}$,

$$Z_{a,b,c}(x) = \int_0^x (\mathbf{1}_{(-\infty,\,y]}(a) - \mathbf{1}_{(-\infty,\,y]}(a+b)$$

$$- \mathbf{1}_{(-\infty,\,y]}(a+c) + \mathbf{1}_{(-\infty,\,y]}(a+b+c))\,dy$$

$$= \begin{cases} x - a, & \text{if} \quad a < x \leqslant a + (b \wedge c), \\[2mm] b \wedge c, & \text{if} \quad a + (b \wedge c) < x \leqslant a + (b \vee c), \\[2mm] a + b + c - x, & \text{if} \quad a + (b \vee c) < x \leqslant a + b + c, \\[2mm] 0, & \text{otherwise} \end{cases}$$

$$\leqslant (b \wedge c)\,\mathbf{1}_{(a,\,a+b+c]}(x).$$

Here, for a set A, $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ otherwise. This leads to

$$E\left[\int_0^{t-Y} \Delta^2 b_{n-1}(y)\,dy\right]$$

$$\leqslant \int (b \wedge c)\,P(t - Y \in (a,\ a+b+c])\,d\mathcal{L}((S_{n-1},\ X_n,\ X_{n+1}))(a,b,c)$$

$$\leqslant E[(X_n \wedge X_{n+1})\,\mathrm{Conc}^-(\mathcal{L}(Y);\ X_n + X_{n+1})].$$

Assertion (b) follows from the inequalities above. $\square$

The proof of Proposition 1 requires the following three lemmas.

Lemma 3. *Let the assumptions of Proposition 1 be valid. Further, let $Y_1, \ldots, Y_n$ be random variables with distributions $\mathcal{L}(Y_j) = \mathrm{CPo}(p_j, Q_j)$ for $j \in \{1, \ldots, n\}$. We assume that all $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are independent. For $i \in \{1, \ldots, n\}$ fixed, set $Z_i = \sum_{j=1}^{i} X_j + \sum_{j=i+1}^{n} Y_j$ and $Z_i' = Z_i - X_i$. Then, for all $t \in [0, \infty)$,*

$$\mathrm{Conc}(\mathcal{L}(Z_i');\ t) \leqslant \frac{1}{1 - p_i}\,\mathrm{Conc}(\mathcal{L}(Z_i);\ t), \qquad \mathrm{Conc}(\mathcal{L}(Z_i);\ t) \leqslant \frac{\pi^2}{4}\,\mathrm{Conc}(\tilde{H};\ t).$$

The above inequalities remain valid if Conc *is everywhere replaced by* Conc^-.

Proof. For the proof with respect to Conc^-, see Hipp and Michel (1990, pp. 52–53) or, for a preliminary version, Hipp (1985, p. 231), where the main argument is a suitable smoothing lemma for arbitrary probability measures. The proof is completed by using the continuity properties of concentration functions (see Lemma 9 below). $\square$

Lemma 4. (a) *Let X and Y be two real-valued random variables. If $\mu = E|X| < \infty$, then*

$$d_{\mathrm{KM}}(Y,\ X + Y) \leqslant c\,\mathrm{Conc}^-(\mathcal{L}(Y);\ \mu),$$

where $c = 2$. If X is almost surely constant, then we can set $c = 1$.

(b) *Let Y be a real-valued random variable and let X_1, X_2 be non-negative, independent and identically distributed random variables with $\mathrm{E}(X_1^2) < \infty$. Then*

$$\mathrm{E}[(X_1 \wedge X_2)\,\mathrm{Conc}^-(\mathcal{L}(Y); X_1 + X_2)]$$

$$\leqslant 2\left(\mu + c'\left(\mu + \frac{\mu^{(2)}}{\mu}\right)\right)\mathrm{Conc}^-(\mathcal{L}(Y); \mu),$$

where $\mu = \mathrm{E}(X_1)$, $\mathrm{E}(X_1^2) = \mu^{(2)}$ and $c' = \frac{1}{4}$. If X_1 is almost surely constant, then we can set $c' = 0$.

Proof. Assertion (a) was implicitly shown in Hipp and Michel (1990, p. 52); see also Hipp (1985, pp. 230–231). In fact, the argument is the following. We may assume that $\mu > 0$. For $y \in \mathbb{R}$, we have

$$|P(Y \leqslant y) - P(X + Y \leqslant y)| \leqslant \int_{\mathbb{R}} |P(Y \leqslant y) - P(Y \leqslant y - x)|\, \mathrm{d}\mathcal{L}(X)(x).$$

The integrand is equal to $P(Y \in I(x, y))$, where $I(x, y) = (y \wedge (y - x), \, y \vee (y - x)]$ is a half-open interval with length $|x|$. Dividing this interval into smaller ones, we see that

$$P(Y \in I(x, y)) \leqslant \left\lceil \frac{|x|}{\mu} \right\rceil \mathrm{Conc}^-(\mathcal{L}(Y); \mu),$$

where, for $x \in \mathbb{R}$, $\lceil x \rceil \in \mathbb{Z}$ is defined by $x \leqslant \lceil x \rceil < x + 1$. Therefore

$$d_{\mathrm{KM}}(Y, X + Y) \leqslant \mathrm{Conc}^-(\mathcal{L}(Y); \mu)\,\mathrm{E}\left\lceil \frac{|X|}{\mu} \right\rceil,$$

from which (a) follows.

Assertion (b) can be shown in the same way. Indeed, we have

$$\mathrm{E}[(X_1 \wedge X_2)\,\mathrm{Conc}^-(\mathcal{L}(Y); X_1 + X_2)]$$

$$\leqslant \mathrm{E}\left[(X_1 \wedge X_2)\left(1 + \frac{X_1 + X_2}{\mu}\right)\right]\mathrm{Conc}^-(\mathcal{L}(Y); \mu)$$

$$= \left(\mathrm{E}(X_1 \wedge X_2) + \frac{1}{\mu}\,\mathrm{E}(X_1 \wedge X_2)^2 + \mu\right)\mathrm{Conc}^-(\mathcal{L}(Y); \mu)$$

and $X_1 \wedge X_2 \leqslant (X_1 + X_2)/2$. $\qquad\square$

Let $\mathrm{Bi}(n, p)$ denote the binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0, 1]$.

Lemma 5. *For $p \in [0, 1]$,*

$$d_{\mathrm{FM}}(\mathrm{Bi}(1, p), \mathrm{Po}(p)) = 2(\mathrm{e}^{-p} - 1 + p) \leqslant p^2, \qquad \tilde{d}_{\mathrm{SL}}(\mathrm{Bi}(1, p), \mathrm{Po}(p)) = \frac{p^2}{2}.$$

Proof. The assertions are easily shown. See also Roos (2001, Proposition 1 and remark after Proposition 2).   □

Proof of Proposition 1. For $i \in \{1, \ldots, n\}$, let

$$P_i = \mathcal{L}(X_i), \qquad \hat{P}_i = \mathrm{CPo}(p_i, Q_i), \qquad M'_i = \left(\mathop{*}_{j=1}^{i-1} P_j \right) * \left(\mathop{*}_{j=i+1}^{n} \hat{P}_j \right).$$

Then, according to Corollary 1 and Lemmas 4 and 5,

$$d_{\mathrm{KM}}(\mathcal{L}(S_n), H) \leqslant \tfrac{1}{2} \sum_{i=1}^{n} d_{\mathrm{FM}}(\mathrm{Bi}(1, \ p_i), \ \mathrm{Po}(p_i)) \, d_{\mathrm{KM}}(M'_i, \, Q_i * M'_i)$$

$$\leqslant \tfrac{1}{2} \sum_{i=1}^{n} c_i \, p_i^2 \, \mathrm{Conc}^{-}(M'_i; \, \mu_i)$$

and, similarly,

$$d_{\mathrm{SL}}(\mathcal{L}(S_n), H) \leqslant \sum_{i=1}^{n} p_i^2 \left(\mu_i + c'_i \left(\mu_i + \frac{\mu_i^{(2)}}{\mu_i} \right) \right) \mathrm{Conc}^{-}(M'_i; \, \mu_i).$$

Lemma 3 gives

$$\mathrm{Conc}^{-}(M'_i; \, \mu_i) \leqslant \frac{\pi^2}{4(1 - p_i)} \, \mathrm{Conc}^{-}(\tilde{H}; \, \mu_i),$$

which completes the proof.   □

Proof of Proposition 2. For $i \in \{1, \ldots, n\}$, let

$$P_i = \mathcal{L}(X_i), \qquad R = \mathcal{L}(S_n),$$

$$R_K^{(i)} = \sum_{k=1}^{K} \frac{(-1)^{k+1}}{k} \, p_i^k (Q_i - \varepsilon_0)^{*k}, \qquad R^{(i)} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \, p_i^k (Q_i - \varepsilon_0)^{*k},$$

$$U^{(i)} = R_K^{(i)} - R^{(i)}, \qquad H_K^{(i)} = \exp(R_K^{(i)}), \qquad M''_i = \left(\mathop{*}_{j=1}^{i-1} P_j \right) * \left(\mathop{*}_{j=i+1}^{n} H_K^{(j)} \right).$$

Then the telescopic sum decomposition (cf. proof of Corollary 1) gives

$$d_{\mathrm{KM}}(R, H_K) \leqslant \sum_{i=1}^{n} \sup_{x \in \mathbb{R}} \left| [M''_i * (P_i - H_K^{(i)})]((-\infty, x]) \right| =: T.$$

Using the telescopic sum decomposition again, we obtain, for $i \in \{1, \ldots, n\}$,

$$R - P_i * M''_i = \left[\mathop{*}_{j=i+1}^{n} P_j - \mathop{*}_{j=i+1}^{n} H_K^{(j)} \right] * \left(\mathop{*}_{j=1}^{i} P_j \right) = \sum_{j=i+1}^{n} M''_j * (P_j - H_K^{(j)}).$$

Since $P_i = \exp(R^{(i)})$, this yields

$$M_i'' * (P_i - H_K^{(i)}) = M_i'' * P_i * (\varepsilon_0 - \exp(U^{(i)}))$$

$$= (R + M_i'' * P_i - R) * (\varepsilon_0 - \exp(U^{(i)}))$$

$$= \left(R - \sum_{j=i+1}^{n} M_j'' * (P_j - H_K^{(j)}) \right) * (\varepsilon_0 - \exp(U^{(i)})).$$

In view of Abel's summation formula, we see that the second convolution factor is equal to

$$\varepsilon_0 - \exp(U^{(i)}) = \sum_{r=0}^{\infty} a_r^{(i)} Q_i^{*r} = \sum_{r=0}^{\infty} A_r^{(i)} Q_i^{*r} * (\varepsilon_0 - Q_i), \tag{13}$$

where the coefficients $a_r^{(i)}$ are real-valued and $A_r^{(i)} = \sum_{m=0}^{r} a_m^{(i)}$ for $r \in \mathbb{Z}_+$. In fact, (13) is valid, since it can be shown that $B^{(i)} := \sum_{r=0}^{\infty} |A_r^{(i)}| < \infty$ (see below). Hence, for $i \in \{1, \ldots, n\}$,

$$\sup_{x \in \mathbb{R}} |[M_i'' * (P_i - H_K^{(i)})]((-\infty, x])| \leq B^{(i)}(d_{\mathrm{KM}}(R, Q_i * R) + 2T).$$

This implies

$$T \leq \sum_{i=1}^{n} B^{(i)}(d_{\mathrm{KM}}(R, Q_i * R) + 2T),$$

and therefore, letting $B = \sum_{i=1}^{n} B^{(i)}$,

$$(1 - 2B)T \leq \sum_{i=1}^{n} B^{(i)} d_{\mathrm{KM}}(R, Q_i * R).$$

Here, we have used the fact that, since $M_i'' * (P_i - H_K^{(i)})$ is a finite signed measure for all i, T must be finite. In order to give an estimate for B, we use, for a power series $g(z) = \sum_{i=0}^{\infty} g_i z^i$ with $|z| \leq 1$, the notation $\|g(z)\| = \sum_{i=0}^{\infty} |g_i|$. Further, we make use of the simple property that $\|g(z)\tilde{g}(z)\| \leq \|g(z)\| \|\tilde{g}(z)\|$, where $g(z)$ and $\tilde{g}(z)$ are two such power series. For $|z| \leq 1$, let

$$G_K^{(i)}(z) = - \sum_{k=K+1}^{\infty} \frac{(-1)^{k+1}}{k} p_i^k (z-1)^k.$$

Then it follows that, for all $i \in \{1, \ldots, n\}$,

$$B^{(i)} = \left\| \frac{1}{1-z}(1 - \exp(G_K^{(i)}(z))) \right\|$$

$$\leq \left\| \frac{G_K^{(i)}(z)}{z-1} \right\| \sum_{m=1}^{\infty} \frac{\|G_K^{(i)}(z)\|^{m-1}}{m!}$$

$$\leq \frac{1}{2}\left(\exp\left(\sum_{k=K+1}^{\infty} \frac{(2p_i)^k}{k} \right) - 1 \right)$$

$$\leq \frac{1}{2}(e^{\tau(i,K)} - 1),$$

giving $B \leq \delta/2$. Since $d_{\mathrm{KM}}(R, Q_i * R) \leq 4^{-1}\pi^2 c_i \mathrm{Conc}^-(\tilde{H}; \mu_i)$ (cf. Lemmas 4(a) and 3), we arrive at the first inequality.

The second assertion is shown in the same manner. Here, we may assume that, for all i, $\mu_i < \infty$ and $\mu_i^{(2)} < \infty$. Now

$$d_{\mathrm{SL}}(R, H_K) \leq \sum_{i=1}^{n} \sup_{x \in \mathbb{R}} \left| \int_x^{\infty} [M_i'' * (P_i - H_K^{(i)})]((y, \infty)) \, \mathrm{d}y \right| =: \tilde{T}.$$

By using Abel's summation formula, we have, for $i \in \{1, \ldots, n\}$,

$$\varepsilon_0 - \exp(U^{(i)}) = \sum_{r=0}^{\infty} A_r^{(i)} Q_i^{*r} * (\varepsilon_0 - Q_i) = \sum_{r=0}^{\infty} \tilde{A}_r^{(i)} Q_i^{*r} * (\varepsilon_0 - Q_i)^{*2},$$

with $\tilde{A}_r^{(i)} = \sum_{m=0}^{r} A_m^{(i)}$ for $r \in \mathbb{Z}_+$. This leads to

$$M_i'' * (P_i - H_K^{(i)}) = \left(R - \sum_{j=i+1}^{n} M_j'' * (P_j - H_K^{(j)}) \right) * \sum_{r=0}^{\infty} \tilde{A}_r^{(i)} Q_i^{*r} * (\varepsilon_0 - Q_i)^{*2}.$$

Hence

$$\sup_{x \in \mathbb{R}} \left| \int_x^{\infty} [M_i'' * (P_i - H_K^{(i)})]((y, \infty)) \, \mathrm{d}y \right|$$

$$\leq \sum_{r=0}^{\infty} |\tilde{A}_r^{(i)}| \left(\sup_{x \in \mathbb{R}} \left| \int_x^{\infty} [Q_i^{*r} * (\varepsilon_0 - Q_i)^{*2} * R]((y, \infty)) \, \mathrm{d}y \right| + 4\tilde{T} \right).$$

It is easy to see that, for $r \in \mathbb{Z}_+$,

$$\sup_{x \in \mathbb{R}} \left| \int_x^{\infty} [Q_i^{*r} * (\varepsilon_0 - Q_i)^{*2} * R]((y, \infty)) \, \mathrm{d}y \right| = \sup_{x \in \mathbb{R}} \left| \mathrm{E} \int_0^{x-S_n} \Delta^2 b_r^{(i)}(y) \, \mathrm{d}y \right|,$$

where $b_r^{(i)}(y) = Q_i^{*r}((-\infty, y])$. Proceeding as in the proof of Theorem 1(b), we obtain, together with Lemma 4(b), that

$$\tilde{T} \leqslant \sum_{i=1}^{n} \sum_{r=0}^{\infty} |\tilde{A}_r^{(i)}| \left(2 \left(\mu_i + c_i' \left(\mu_i + \frac{\mu_i^{(2)}}{\mu_i} \right) \right) \mathrm{Conc}^-(R; \mu_i) + 4\tilde{T} \right),$$

giving

$$(1 - 4\tilde{B})\tilde{T} \leqslant 2 \sum_{i=1}^{n} \tilde{B}^{(i)} \left(\mu_i + c_i' \left(\mu_i + \frac{\mu_i^{(2)}}{\mu_i} \right) \right) \mathrm{Conc}^-(R; \mu_i),$$

where $\tilde{B} = \sum_{i=1}^{n} \tilde{B}^{(i)}$ and $\tilde{B}^{(i)} = \sum_{r=0}^{\infty} |\tilde{A}_r^{(i)}|$. Here, we have used the fact that $\tilde{T} < \infty$, which can easily be shown by using the simple inequality

$$\sup_{t \in \mathbb{R}} |\pi_{Q_1' * Q_2'}(t)| \leqslant \sup_{t \in \mathbb{R}} |\pi_{Q_1'}(t)| \, |Q_2'|(\mathbb{R})$$

for two finite signed measures Q_1' and Q_2' on $\mathbb{R}$. Similarly to the above,

$$\tilde{B}^{(i)} = \left\| \frac{1}{(1-z)^2} (1 - \exp(G_K^{(i)}(z))) \right\| \leqslant \frac{1}{4} (\mathrm{e}^{\tau(i,K)} - 1),$$

and hence, we obtain $\tilde{B} \leqslant \delta/4$. The second assertion now follows. $\square$

For the proofs of Proposition 3, Lemma 6 below, and Proposition 5, we use a splitting technique due to Lévy (cf. Le Cam 1986, p. 412).

Proof of Proposition 3. Let $s \in [0, \infty)$. The proof is based on the decomposition

$$Q = c_1 Q_1 + c_2 Q_2 + c_3 Q_3,$$

where Q_1, Q_2 and Q_3 are probability measures concentrated on $(-\infty, -s)$, $[-s, s]$ and (s, ∞), respectively, and

$$c_1 = Q((-\infty, -s)), \qquad c_2 = Q([-s, s]), \qquad c_3 = Q((s, \infty)).$$

Then, for $t \in (0, \infty)$, $\mathrm{CPo}(t, Q) = *_{i=1}^{3} \mathrm{CPo}(tc_i, Q_i)$ and therefore, by Lemma 8(c) below,

$$\mathrm{Conc}(\mathrm{CPo}(t, Q); s) \leqslant \min\{\mathrm{Conc}(\mathrm{CPo}(tc_1, Q_1); s), \, \mathrm{Conc}(\mathrm{CPo}(tc_3, Q_3); s)\}.$$

We obtain

$$\mathrm{Conc}(\mathrm{CPo}(tc_1, Q_1); s) = \sup_{x \in \mathbb{R}} \left(\sum_{n=0}^{\infty} \mathrm{po}(n, \, tc_1) \, Q_1^{*n}([x, \, x+s]) \right)$$

$$\leqslant \left(\sup_{n \in \mathbb{Z}_+} \mathrm{po}(n, \, tc_1) \right) \sup_{x \in \mathbb{R}} \sum_{n=0}^{\infty} Q_1^{*n}([x, \, x+s]).$$

It is well known that, for $y \in (0, \infty)$,

$$\sup_{n \in \mathbb{Z}_+} \mathrm{po}(n, \, y) \leqslant \frac{1}{\sqrt{2\mathrm{e}y}}; \tag{14}$$

see, for example, Barbour *et al*. (1992b, p. 262) or Hipp and Michel (1990, pp. 46–47).

Further, for all $x \in \mathbb{R}$, it can be shown that $\sum_{n=0}^{\infty} Q_1^{*n}([x, \ x + s]) \leqslant 1$. Indeed, if $T_1, T_2, \ldots$ are independent and identically distributed random variables with $\mathcal{L}(T_1) = Q_1$, then we may assume that, for all $i \in \mathbb{N}$, $T_i < -s$ and therefore

$$\sum_{n=0}^{\infty} Q_1^{*n}([x, \ x + s]) = P\left(\bigcup_{n=0}^{\infty} \left\{\sum_{i=1}^{n} T_i \in [x, \ x + s]\right\}\right) \leqslant 1.$$

Hence

$$\mathrm{Conc}(\mathrm{CPo}(tc_1, Q_1); s) \leqslant \frac{1}{\sqrt{2etc_1}}.$$

Similarly, $\mathrm{Conc}(\mathrm{CPo}(tc_3, Q_3); s) \leqslant (2e \, tc_3)^{-1/2}$. Combining the estimates above, (5) is shown. Inequality (6) can be derived from (5) by using Lemma 9 below. Since $\mathrm{CPo}(t, \varepsilon_1) = \mathrm{Po}(t)$ for $t \in [0, \infty)$ and since, in (14), equality holds for $y = \frac{1}{2}$, we see that the remaining part of the assertion is true.   □

For the proof of Proposition 5, we need the following lemma, which is similar to Proposition 2 in Le Cam (1986, pp. 409–410). However, there are some differences. In contrast to Lemma 6 below, in Le Cam's Proposition 2 it was assumed that the summands X_i have symmetric but not necessarily identical distributions.

Lemma 6. *Let* $n \in \mathbb{N}$ *and* $X_1, \ldots, X_n$ *be independent and identically distributed random variables. Set* $S_n = \sum_{i=1}^{n} X_i$. *Let* $x \in \mathbb{R}$, $t > 0$ *be fixed. We assume that the* X_i *admit the decomposition* $\mathcal{L}(X_i) = \mathcal{L}(I_i Y_i + (1 - I_i) Z_i)$ *for* $i \in \{1, \ldots, n\}$, *where* $\{I_i\}$, $\{Y_i\}$ *and* $\{Z_i\}$ *are sets of identically distributed random variables with* $\mathcal{L}(I_i) = \mathrm{Bi}(1, \frac{1}{2})$, $P(Y_i \leqslant x) = P(Z_i \geqslant x + t) = 1$. *We assume that all* I_i, Y_i, Z_i *are independent. Then*

$$\mathrm{Conc}^-(\mathcal{L}(S_n); t) \leqslant \frac{1}{\sqrt{n+1}}.$$

Proof. Set $T_n = \sum_{i=1}^{n} I_i$ and, for $m \in \{0, \ldots, n\}$, $\tilde{Z}_m = \sum_{i=1}^{m} Y_i + \sum_{i=1}^{n-m} Z_i$. For $y \in \mathbb{R}$, we then have

$$P(S_n \in (y, \ y + t)) = \sum_{m=0}^{n} P(T_n = m) \, P(\tilde{Z}_m \in (y, \ y + t))$$

$$\leqslant \left(\sup_{m \in \mathbb{Z}_+} P(T_n = m)\right) \sum_{m=0}^{n} P(\tilde{Z}_m \in (y, \ y + t))$$

$$\leqslant \frac{1}{\sqrt{n+1}},$$

where we have used the fact that $\sup_{m \in \mathbb{Z}_+} P(T_n = m) \leqslant (1 + n)^{-1/2}$ (see, for example, Le Cam 1986, proof of Proposition 2, p. 410) and that, since $\tilde{Z}_m - \tilde{Z}_{m+1} \geqslant t$ almost surely for

$m \in \{0, \ldots, n-1\}$, the events $\{\tilde{Z}_m \in (y, \, y+t)\}$ for $m \in \{0, \ldots, n\}$ are pairwise disjoint. $\qquad \square$

Proposition 5 can be proved by adapting the proof of Theorem 2 in Le Cam (1986, p. 411); cf. Remark 8 of the present paper. In what follows, we give an alternative direct proof.

Proof of Proposition 5. Let $t \in (0, \infty)$ and let $x = x_t \in \mathbb{R}$ be a median of the distribution function $2^{-1}(F(y) + F(y+t))$ for $y \in \mathbb{R}$, where F is the distribution function of X_1. This means that

$$F(x-) + F((x+t)-) \leqslant 1 \leqslant F(x) + F(x+t),$$

where $F(x-) = \lim_{y \uparrow x} F(y)$. Therefore $a \in [0, 1]$ exists such that

$$q := F(x-) + a \, P(X_1 = x) = 1 - F((x+t)-) - a \, P(X_1 = x+t).$$

This leads to

$$1 - P(X_1 \in [x, \, x+t]) \leqslant 2q \leqslant 1 - P(X_1 \in (x, \, x+t)). \tag{15}$$

In particular, $q \leqslant \frac{1}{2}$. Let us assume that $q > 0$. For $y \in \mathbb{R}$, set

$$F_1(y) = \frac{F(y)}{q} \mathbf{1}_{(-\infty, \, x)}(y) + \mathbf{1}_{[x, \, \infty)}(y),$$

$$F_2(y) = \frac{F(y) - (1-q)}{q} \mathbf{1}_{[x+t, \, \infty)}(y),$$

$$F_3(y) = \frac{1}{2}(F_1(y) + F_2(y)),$$

$$F_4(y) = \begin{cases} \dfrac{F(y) - q}{1 - 2q} \mathbf{1}_{[x, \, x+t)}(y) + \mathbf{1}_{[x+t, \, \infty)}(y), & \text{if } q < \frac{1}{2}, \\[2ex] \mathbf{1}_{[x, \, \infty)}(y), & \text{if } q = \frac{1}{2}. \end{cases}$$

It is easy to verify that the $F_1, \ldots, F_4$ are distribution functions with

$$F = 2qF_3 + (1 - 2q)F_4.$$

Further, the distributions with distribution functions F_1, F_2, F_4 are concentrated on $(-\infty, \, x]$, $[x+t, \, \infty)$ and $[x, \, x+t]$, respectively. Let $\{Y_1, \ldots, Y_n\}$, $\{Z_1, \ldots, Z_n\}$ and $\{I_1, \ldots, I_n\}$ be families of identically distributed random variables with $F_{Y_i} = F_3$, $F_{Z_i} = F_4$ and $\mathcal{L}(I_i) = \text{Bi}(1, 2q)$ for $i \in \{1, \ldots, n\}$, where we assume that all Y_i, Z_i, I_i for $i \in \{1, \ldots, n\}$ are independent. Then S_n is equal in distribution to

$$\sum_{i=1}^{n} [I_i Y_i + (1 - I_i) Z_i].$$

Set $T_n = \sum_{i=1}^{n} I_i$ and, for $m \in \{0, \ldots, n\}$, $R_m = \sum_{i=1}^{m} Y_i$. For $y \in \mathbb{R}$, we now obtain

$$P(S_n \in (y, \ y + t)) = \sum_{m=0}^{n} P(T_n = m) \, P\left(R_m + \sum_{i=1}^{n-m} Z_i \in (y, \ y + t)\right)$$

$$\leqslant \sum_{m=0}^{n} P(T_n = m) \, \mathrm{Conc}^-(\mathcal{L}(R_m); \, t),$$

where we have used Lemma 8 below. From Lemma 6, we see that $\mathrm{Conc}^-(\mathcal{L}(R_m); \, t) \leqslant (m+1)^{-1/2}$. Therefore, using Jensen's inequality, the equality

$$P(T_n + 1 = m) = \frac{m}{(n+1)2q} \, P(T_{n+1} = m)$$

for $m \in \{0, \ldots, n+1\}$, and (15), we derive

$$\mathrm{Conc}^-(\mathcal{L}(S_n); \, t) \leqslant \mathrm{E}\, \frac{1}{\sqrt{T_n + 1}} \leqslant \left(\mathrm{E}\, \frac{1}{T_n + 1}\right)^{1/2} = \left(\frac{1 - (1 - 2q)^{n+1}}{(n+1)\, 2q}\right)^{1/2}$$

$$\leqslant \left(\frac{1 - [\mathrm{Conc}(\mathcal{L}(X_1); \, t)]^{n+1}}{(n+1)(1 - \mathrm{Conc}(\mathcal{L}(X_1); \, t))}\right)^{1/2},$$

where $t \in (0, \ \infty)$. In the case $q = 0$, the upper bound we have just proved can be set to be one by a continuity argument, since here we have $P(X_1 \in [x, \ x + t]) = 1$ and therefore $\mathrm{Conc}(\mathcal{L}(X_1); \, t) = 1$. The proof is completed by using Lemma 9 below. $\qquad\square$

For the proof of Proposition 4, we need the following lemma.

Lemma 7. *Let the assumptions of Proposition 4 be valid. For $s, t \in (0, \infty)$ and $\alpha = 1 - \mathrm{Conc}(\mathcal{L}(X_1); \, s)$, we have*

$$\mathrm{Conc}^-(\mathcal{L}(S_n); \, t) \leqslant \mathrm{Conc}^-(\mathcal{L}(X_1); \, t) \sum_{m=0}^{n-1} \mathrm{Conc}^-(\mathcal{L}(S_m); \, s + t)\alpha^{n-1-m}.$$

Proof. Let $x \in \mathbb{R}$ be arbitrary and set $I = (x, \ x + t]$. According to Lemma 9(d) below, $y \in \mathbb{R}$ exists such that $\alpha = P(X_1 \notin J)$, where we define $J = [y, \ y + s]$. Then we have (cf. Petrov 1995, p. 70)

$$P(S_n \in I) = \int_{I-J} P(X_1 + x \in I, \ X_1 \in J) \, \mathrm{d}\mathcal{L}(S_{n-1})(x)$$

$$+ \alpha \int_{-\infty}^{\infty} P(S_{n-1} + x \in I) \, \mathrm{d}\mathcal{L}(X_1 | X_1 \notin J)(x),$$

where $I - J = \{z_1 - z_2 \,|\, z_1 \in I, \ z_2 \in J\} = (x - y - s, \ x - y + t]$. This yields

$$\mathrm{Conc}^-(\mathcal{L}(S_n); \, t) \leqslant \mathrm{Conc}^-(\mathcal{L}(X_1); \, t)\mathrm{Conc}^-(\mathcal{L}(S_{n-1}); \, s + t)$$

$$+ \alpha \mathrm{Conc}^-(\mathcal{L}(S_{n-1}); \, t).$$

The assertion now follows by induction over n. ☐

Proof of Proposition 4. According to Lemma 9 below, it suffices to show the assertion for Conc^-. Let $t \in (0, \infty)$. Let us first assume that $\text{Conc}^-(\mathcal{L}(X_1); t) \leqslant \beta$, where $\beta \in (0, 1)$. Then, by Lemma 9 below, $s \in [t, \infty)$ exists such that

$$\text{Conc}^-(\mathcal{L}(X_1); s) \leqslant \beta \leqslant \text{Conc}(\mathcal{L}(X_1); s).$$

In particular, we have $\alpha := 1 - \text{Conc}(\mathcal{L}(X_1); s) \leqslant 1 - \beta$. Using Lemma 7, Proposition 5 and the simple inequality $\text{Conc}^-(\mathcal{L}(X_1); s + t) \leqslant 2\beta$, we obtain

$$\text{Conc}^-(\mathcal{L}(S_n); t) \leqslant \text{Conc}^-(\mathcal{L}(X_1); t) \sum_{m=0}^{n-1} \frac{(1-\beta)^{n-1-m}}{\sqrt{(m+1)(1-2\beta)}}.$$

Set $\beta = 0.3322$. Then simple calculus shows that

$$\sqrt{n+1} \sum_{m=0}^{n-1} \frac{(1-\beta)^{n-1-m}}{\sqrt{(m+1)(1-2\beta)}} \leqslant 6.33, \tag{16}$$

giving the assertion in the present case. In fact, it is not difficult to prove that, if we denote the left-hand side of (16) by f_n, then $f_n \leqslant f_{n+1}$ for $n \leqslant 6$ and $f_n \geqslant f_{n+1}$ for $n \geqslant 7$. Therefore, $\sup_{n \in \mathbb{N}} f_n = f_7 = 6.329 \ldots$. If $\text{Conc}^-(\mathcal{L}(X_1); t) \geqslant \beta$, then the assertion follows easily from Proposition 5. In fact,

$$\text{Conc}^-(\mathcal{L}(S_n); t) \leqslant \frac{1}{\sqrt{(m+1)(1 - \text{Conc}^-(\mathcal{L}(X_1); t))}}$$

$$\leqslant \frac{\text{Conc}^-(\mathcal{L}(X_1); t)}{\beta\sqrt{(m+1)(1 - \text{Conc}^-(\mathcal{L}(X_1); t))}},$$

where $1/\beta \leqslant 3.1$. ☐

Proof of Corollary 2. The assertion follows from

$$\text{Conc}(\text{CPo}(t, Q) ; s) \leqslant \sum_{n=0}^{\infty} \text{po}(n, t) \, \text{Conc}(Q^{*n} ; s)$$

$$\leqslant e^{-t} + \sum_{n=1}^{\infty} \text{po}(n, t) \frac{6.33 \, \text{Conc}(Q; s)}{\sqrt{(n+1)(1 - \text{Conc}(Q; s))}},$$

the simple inequality $\sum_{n=1}^{\infty} \text{po}(n, t)/\sqrt{n+1} \leqslant (1 - e^{-t})/\sqrt{t}$ and Lemma 9 below. ☐

Appendix: Concentration functions

For the proof of the following lemmas, see Hengartner and Theodorescu (1973).

Lemma 8 (Basic properties of concentration functions). *Let* t, $s \in [0, \infty)$ *and* X *and* Y *be independent real-valued random variables. Then:*

    (a) $\mathrm{Conc}(\mathcal{L}(X); s) \leqslant \mathrm{Conc}(\mathcal{L}(X); s + t)$.
    (b) $\mathrm{Conc}(\mathcal{L}(X); s + t) \leqslant \mathrm{Conc}(\mathcal{L}(X); s) + \mathrm{Conc}(\mathcal{L}(X); t)$.
    (c) $\mathrm{Conc}(\mathcal{L}(X + Y); s) \leqslant \min\{\mathrm{Conc}(\mathcal{L}(X); s), \; \mathrm{Conc}(\mathcal{L}(Y); s)\}$.
    (d) *Assertions* (a)–(c) *also hold if* Conc *is everywhere replaced by* Conc^-.

Lemma 9 (Continuity properties of concentration functions). *Let* $s \in (0, \infty)$, $t \in [0, \infty)$, *and* Q *be a probability distribution on* $\mathbb{R}$. *Then:*

    (a) $\mathrm{Conc}^-(Q; t) = \sup_{x \in \mathbb{R}} Q((x, \; x + t)) = \sup_{x \in \mathbb{R}} Q([x, \; x + t))$.
    (b) $\mathrm{Conc}(Q; \cdot)$ *is continuous from the right;* $\mathrm{Conc}^-(Q; \cdot)$ *is continuous from the left.*
    (c) $\mathrm{Conc}(Q; s-) = \mathrm{Conc}^-(Q; s)$ *and* $\mathrm{Conc}^-(Q; t+) = \mathrm{Conc}(Q; t)$.
    (d) *There exists an* $x_t \in \mathbb{R}$ *such that* $\mathrm{Conc}(Q; t) = Q([x_t, x_t + t])$.

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