3. *On Hirota's Difference Equations*

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§1. The aim of this note is to exploit the operator approach [1] to soliton equations in studying the following non linear difference equation proposed by Hirota [2]:

$$(1.1)\qquad \begin{aligned}&\alpha f(\lambda+1,\mu,\nu)f(\lambda-1,\mu,\nu)+\beta f(\lambda,\mu+1,\nu)f(\lambda,\mu-1,\nu)\\&\quad+\gamma f(\lambda,\mu,\nu+1)f(\lambda,\mu,\nu-1)=0,\end{aligned}$$

where α, β and γ are constants satisfying $\alpha+\beta+\gamma=0$.

Hirota [2] found the difference Lax pair for (1.1), proved the existence of three soliton solutions and gave an ample list of non linear differential and/or difference equations obtained by taking suitable limits of (1.1). Among them is the KP (Kadomtsev-Petviashvili) equation which is written in Hirota's form as follows:

$$(1.2)\qquad (D_1^4+3D_2^2-4D_1D_3)\tau\cdot\tau=0.$$

He also remarked a significant coincidence of the phase shift term in soliton solutions of the equations (1.1) and (1.2).

Here I shall give an explicit transformation which connects the hierarchy of the KP equation and that of Hirota's difference equation.

One of the striking discoveries of Mikio and Yasuko Sato [3] on the former was that it admits the characters of the general linear group as its solutions with

$$(1.3)\qquad x_j=\text{trace}\,\frac{X^j}{j},\quad X\in GL(N)\quad (j=1,2,3,\cdots)$$

as the continuum variables. The transformation tells us that the latter, in a slightly modified form, admits them also as its solutions with the multiplicities of the eigenvalues of X as the discrete variables.

The transformation gives us also an operator solution to Hirota's difference equation in the sense of [1]. It reduces to operator solutions to equations in Hirota's list in the limit. Here I discuss briefly those for the two dimensional Toda lattice.

Finally I show that a similar consideration for the BKP hierarchy [1] leads us to the following discrete version:

$$(1.4)\qquad \begin{aligned}&\alpha f(\lambda+1,\mu,\nu)f(\lambda-1,\mu,\nu)+\beta f(\lambda,\mu+1,\nu)f(\lambda,\mu-1,\nu)\\&\quad+\gamma f(\lambda,\mu,\nu+1)f(\lambda,\mu,\nu-1)+\delta f(\lambda+1,\mu+1,\nu+1)\\&\qquad\times f(\lambda-1,\mu-1,\nu-1)=0,\end{aligned}$$

where α, β, γ and δ are constants satisfying $\alpha+\beta+\gamma+\delta=0$.

§2. Let us recall the operator solution to the KP hierarchy.

Let ψ_m and ψ_m^* $(m \in Z)$ be free fermions satisfying $[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0$ and $[\psi_m, \psi_n^*]_+ = \delta_{mn}$. We use their Fourier transforms $\psi(k) = \sum_{n\in Z} \psi_n k^n$ and $\psi^*(k) = \sum_{n \in Z} \psi_n^* k^{-n}$. Introducing an infinite set of variables $x=(x_1, x_2, x_3, \cdots)$ we set

$$\rho(x) = \iint \frac{dp}{2\pi i p} \frac{dq}{2\pi i q} \frac{p/q}{1-p/q} e^{-\xi(x,p)+\xi(x,q)} \psi(p)\psi^*(q) \tag{2.1}$$

where $\xi(x, k) = \sum_{j=1}^{\infty} x_j k^j$. It was shown in [1] that if g is an element of the Clifford group

$$\tau(x) = \langle : e^{\rho(x)} : g \rangle \tag{2.2}$$

satisfies the bilinear equation of the KP hierarchy.

Now we introduce two other infinite sets of variables $l=(l_1, l_2, l_3, \cdots)$ and $a=(a_1, a_2, a_3, \cdots)$. We relate them to x by

$$x_j = \sum_{i=1}^{\infty} l_i \frac{a_i^j}{j}. \tag{2.3}$$

Then the quadratic expression (2.1) is transformed into the following.

$$\rho(l\,; a) = \iint \frac{dp}{2\pi i p} \frac{dq}{2\pi i q} \frac{p/q}{1-p/q} \prod_{j=1}^{\infty} \left(\frac{1-a_j p}{1-a_j q}\right)^{l_j} \psi(p)\psi^*(q). \tag{2.4}$$

By using Wick's theorem we can show

Theorem 1. *Let g be an element of the Clifford group and set*

$$\tau(l\,; a) = \langle : e^{\rho(l;a)} : g \rangle. \tag{2.5}$$

We pick up a triple (l, m, n) among l_i's and the corresponding one (a, b, c) among a_i's. Then $\tau(l, m, n)$ $(\underset{\text{det}}{=} \tau(l\,; a))$ satisfies the following bilinear difference equation.

$$\begin{aligned} &a(b-c)\tau(l+1, m, n)\tau(l, m+1, n+1) \\ &+ b(c-a)\tau(l, m+1, n)\tau(l+1, m, n+1) \\ &+ c(a-b)\tau(l, m, n+1)\tau(l+1, m+1, n) = 0. \end{aligned} \tag{2.6}$$

In other words, by (2.3) we can transform solutions to the KP hierarchy into those to (2.6). Firstly, we have the following N soliton solution to (2.7) as was expected by Hirota [2].

$$\begin{aligned} \tau(x) = 1 &+ \sum_{1\le i\le N} c_i \sum_{j=1}^{\infty} \left(\frac{1-a_j q_i}{1-a_j p_i}\right)^{l_j} \\ &+ \sum_{1 \le i \le i' \le N} c_i c_{i'} c_{ii'} \prod_{j=1}^{\infty} \left(\frac{1-a_j q_i}{1-a_j p_i}\right)^{l_j} \left(\frac{1-a_j q_{i'}}{1-a_j p_{i'}}\right)^{l_j} + \cdots, \end{aligned} \tag{2.7}$$

where $c_{ii'} = (p_i - p_{i'})(q_i - q_{i'})/(p_i - q_{i'})(q_i - p_{i'})$. Secondly, taking l to be integers, we have

Theorem 2. *Let Y be a Young diagram. Let (a, b, c) be three distinct eigenvalues of an element X in $GL(N)$ with a sufficiently large integer N, and let (l, m, n) be their multiplicities. The character corresponding to Y solves the equation* (2.6) *with respect to the variation of (l, m, n).*

The correspondence between (1.1) and (2.7) is given by

$$\lambda = m+n+1, \quad \mu = n+l+1, \quad \nu = l+m+1, \tag{2.8}$$

(2.9) $$\alpha=a(b-c),\quad \beta=b(c-a),\quad \gamma=c(a-b)$$

and

(2.10) $$\tau(l,m,n)=f(m+n,n+l,l+m).$$

§ 3. Let us consider the following limit $\varepsilon\to 0$ in (2.7).

(3.1) $$\begin{gathered} a=\infty,\quad b=\varepsilon,\quad c=\varepsilon^{-1}, \\ l+n-1=s,\quad m\varepsilon=x,\quad n\varepsilon=y. \end{gathered}$$

The resulting equation is called the two dimensional Toda equation [2]:

(3.2) $$\begin{aligned} &D_xD_yf(s,x,y)\cdot f(s,x,y) \\ &\quad=f(s,x,y)^2-f(s-1,x,y)f(s+1,x,y). \end{aligned}$$

By taking $a_j=0$ ($a_j\neq a,b,c$) the quadratic expression (2.5) is scaled to

(3.3) $$\rho(s,x,y)=\iint\frac{dp}{2\pi ip}\frac{dq}{2\pi iq}\frac{(p/q)^{s+1}}{1-p/q}\frac{e^{qx+q^{-1}y}}{e^{px+p^{-1}y}}\psi(p)\psi^*(q).$$

Thus we have

Theorem 3. *If g is an element of the Clifford group, then*

(3.4) $$\tau(s,x,y)=\langle : e^{\rho(s,x,y)} : g\rangle$$

solves the equation (3.2).

§ 4. Now we consider the BKP hierarchy [1]. Let ϕ_n ($n\in \boldsymbol{Z}$) be neutral free fermions satisfying $[\phi_m,\phi_n]_+=(-)^m\delta_{m,-n}$. We use the Fourier transform $\phi(k)=\sum_{n\in \boldsymbol{Z}}\phi_nk^n$. In this section we set $x=(x_1,x_3,x_5,\cdots)$,

(4.1) $$\xi(x,k)=x_1k+x_3k^3+x_5k^5+\cdots$$

and

(4.2) $$\rho_B(x)=\iint\frac{dp}{2\pi ip}\frac{dq}{2\pi iq}\frac{q/p}{1+q/p}e^{-\xi(x,p)-\xi(x,q)}\phi(p)\phi(q).$$

Then, if g is an element of the Clifford group

(4.3) $$\tau_B(x)=\langle : e^{\rho_B(x)} : g\rangle$$

solves the BKP hierarchy.

We set

(4.4) $$x_j=2\sum_{i=1}^{\infty}l_i\frac{a_i^j}{j}.$$

Then $\rho_B(x)$ reduces to

(4.5) $$\begin{aligned} &\rho_B(l,a) \\ &=\iint\frac{dp}{2\pi ip}\frac{dq}{2\pi iq}\frac{q/p}{1+q/p}\prod_{j=1}^{\infty}\left(\frac{1-a_jp}{1+a_jp}\right)^{l_j}\left(\frac{1-a_jq}{1+a_jq}\right)^{l_j}\phi(p)\phi(q). \end{aligned}$$

Then we have

Theorem 4. *Let g be an element of the Clifford group. We set*

(4.6) $$\tau_B(l,a)=\langle : e^{\rho_B(l,a)} : g\rangle.$$

We pick up a triple (l,m,n) among l_j's and the corresponding one (a,b,c) among a_j's. Then $\tau(l,m,n)$ ($\underset{\det}{=}\tau_B(l,a)$) solves the following bilinear difference equation.

(4.7) $$\begin{aligned} &(a+b)(a+c)(b-c)\tau(l+1,m,n)\tau(l,m+1,n+1) \\ &+(b+c)(b+a)(c-a)\tau(l,m+1,n)\tau(l+1,m,n+1) \end{aligned}$$

$$+(c+a)(c+b)(a-b)\tau(l,m,n+1)\tau(l+1,m+1,n)$$
$$+(a-b)(b-c)(c-a)\tau(l+1,m+1,n+1)\tau(l,m,n)=0.$$

We also note that the BKP equation

$$(D_1^6-5D_1^3D_3-5D_3^2+9D_1D_5)\tau\cdot\tau=0 \tag{4.8}$$

is recovered in a suitable limit of (4.7).

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References

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