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On Holditch's theorem

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Abstract. In this paper we present a certain modification of the Holditch construction. This construction allows to consider a geometric family of pairs of ring domains. It is proved that the ratio of areas of ring domains of each pair belonging to this family is constant. Problems on extremal chords of constant length sliding around a given oval with both endpoints on it are also considered.

Mathematics Subject Classification. Primary 53A04, Secondary 52A38.

Keywords. Crofton-type integral formula, Holditch's theorem, Ring domains, Sliding chord.

1. Introduction

Hamnet Holditch, president of Caius College in Cambridge, published in [13] a remarkable theorem. Let C be a convex curve, and a chord h of length $a+b$ be divided into parts of lengths a and b by a point A. Let $C_{a,b}$ denote a curve traced out by the point A when the chord h slides around with both endpoints on C. Holditch proved that the area of a ring domain bounded by C and $C_{a,b}$ is equal to πab, see Fig 1.

Arne Broman proved in [4] and [3] a much more general theorem and gave some kinematic applications. Further applications to mechanics were given in [11,12], and [14]. Some additional remarks on Holditch's theorem can be found also in [1,8,9], and [18], and recent related investigations are given in [15,16], and [10].

In this paper we modify the Holditch construction in which one ring domain is considered. In our modification we deal with a family of pairs of ring domains and obtain a natural geometric generalization. As an application we derive some Crofton-type formula for a ring domain.

We denote by $\mathcal{C}^*$ the family of all closed strictly convex curves of class C^1. Let $C \in \mathcal{C}^*$ and let p denote a fixed support function of C. The parametric representation of the curve C has the form

$$z(t) = p(t)\, e^{it} + \dot{p}(t)\, ie^{it} \qquad \text{for} \quad t \in [0, 2\pi], \tag{1.1}$$

where the dot denotes differentiation with respect to t, see [2] and [17]. We denote by $\mathcal{C}$ a subfamily of $\mathcal{C}^*$ defined as follows: a curve $C \in \mathcal{C}^*$ belongs to $\mathcal{C}$ if and only if the function $R = p + \ddot{p}$ satisfies the inequality

$$R = p + \ddot{p} > 0. \tag{1.2}$$

Note that the function R is the curvature radius of C if the curve is of class C^2.

We fix $\alpha \in (0, \pi)$, and we denote by $z_\alpha(t)$ the intersection point of the tangent lines at $z(t)$ and $z(t+\alpha)$. A curve $C_\alpha\colon t \to z_\alpha(t)$ is called an α-*isoptic*, see Fig. 2.

We will use the notations introduced in [6] and [7], namely

$$z_\alpha(t) = z(t) + \lambda(t, \alpha)\, ie^{it} = z(t+\alpha) + \mu(t, \alpha)\, ie^{i(t+\alpha)}, \tag{1.3}$$

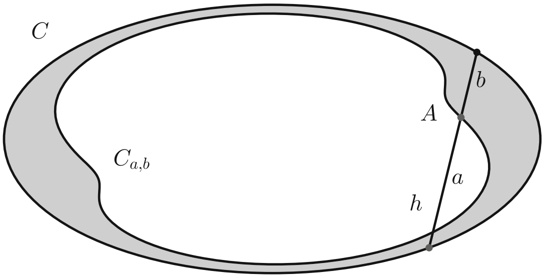

FIGURE 1 Illustration of Holditch's theorem

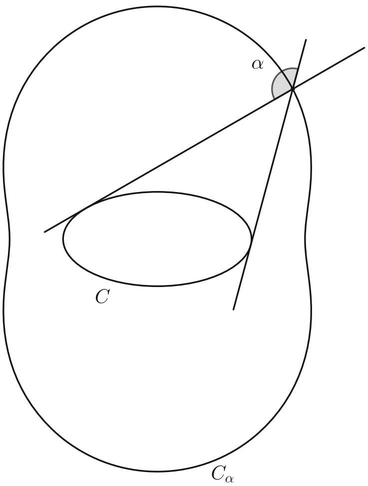

FIGURE 2 An α-isoptic of the curve C

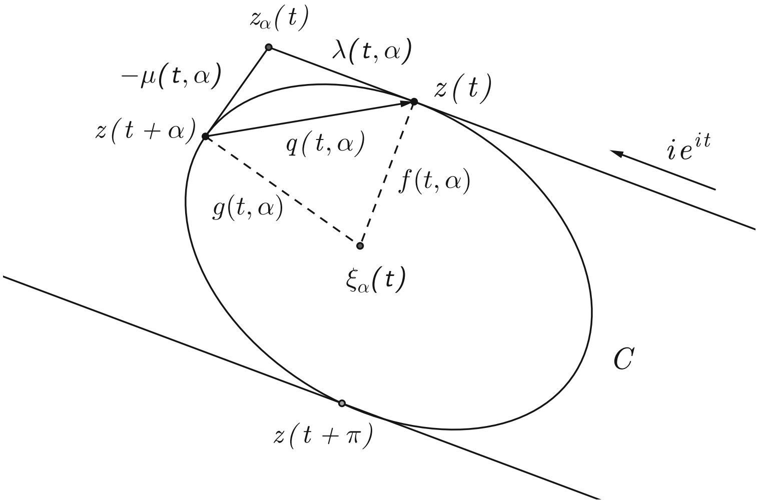

FIGURE 3 The functions f and g

where

$$\lambda(t,\alpha)\sin\alpha = p(t+\alpha) - p(t)\cos\alpha - \dot{p}(t)\sin\alpha, \tag{1.4}$$

$$\mu(t,\alpha)\sin\alpha = p(t+\alpha)\cos\alpha - \dot{p}(t+\alpha)\sin\alpha - p(t). \tag{1.5}$$

Moreover, if

$$b(t,\alpha) = p(t+\alpha)\sin\alpha + \dot{p}(t+\alpha)\cos\alpha - \dot{p}(t), \tag{1.6}$$

$$B(t,\alpha) = p(t) - p(t+\alpha)\cos\alpha + \dot{p}(t+\alpha)\sin\alpha, \tag{1.7}$$

then we have

$$z(t) - z(t+\alpha) = B(t,\alpha)e^{it} - b(t,\alpha)ie^{it}, \tag{1.8}$$

$$\lambda(t,\alpha) = b(t,\alpha) - B(t,\alpha)\cot\alpha, \tag{1.9}$$

$$\mu(t,\alpha)\sin\alpha = -B(t,\alpha), \tag{1.10}$$

see [7]. We denote by $\xi_\alpha(t)$ the intersection point of the normal lines at $z(t)$ and $z(t+\alpha)$. We have

$$\xi_\alpha(t) = z(t) - f(t,\alpha)e^{it} = z(t+\alpha) - g(t,\alpha)e^{i(t+\alpha)}, \tag{1.11}$$

see Fig. 3.

Simple calculations lead us to the formulas

$$f(t,\alpha)\sin\alpha = p(t)\sin\alpha + \dot{p}(t+\alpha) - \dot{p}(t)\cos\alpha, \tag{1.12}$$

$$g(t,\alpha)\sin\alpha = p(t+\alpha)\sin\alpha + \dot{p}(t+\alpha)\cos\alpha - \dot{p}(t). \tag{1.13}$$

Comparing (1.6) and (1.13), we get

$$g(t,\alpha)\sin\alpha = b(t,\alpha). \tag{1.14}$$

Moreover, it is easy to verify that

$$f(t,\alpha) = B(t,\alpha) + b(t,\alpha)\cot\alpha. \tag{1.15}$$

2. Extremal chords

Let us fix a curve $C \in \mathcal{C}$. We consider lengths of chords joining a fixed point $z(t)$ and $z(t+\alpha)$ for $\alpha \in (0, 2\pi)$. Let

$$H_t(\alpha) = |z(t) - z(t+\alpha)| \qquad \text{for} \quad \alpha \in (0, 2\pi). \tag{2.1}$$

We denote by $\langle -, - \rangle$ the Euclidean scalar product. Differentiating the function H_t given by the formula (2.1) and making use of (1.8) and (1.15), we obtain

$$\begin{aligned} H_t'(\alpha) H_t(\alpha) &= \left\langle \frac{\partial}{\partial \alpha}(z(t) - z(t+\alpha)),\ z(t) - z(t+\alpha) \right\rangle \\ &= \left\langle -R(t+\alpha) i e^{i(t+\alpha)},\ B(t,\alpha) e^{it} - b(t,\alpha) i e^{it} \right\rangle \\ &= R(t+\alpha)(B(t,\alpha)\sin\alpha + b(t,\alpha)\cos\alpha) \\ &= R(t+\alpha) f(t,\alpha)\sin\alpha \end{aligned}$$

and

$$H_t'(\alpha) = \frac{R(t+\alpha)}{H_t(\alpha)} f(t,\alpha)\sin\alpha. \tag{2.2}$$

The above formula implies immediately the following statement.

If the chord joining a fixed point $z(t)$ and a point $z(t+\alpha)$ for some $\alpha \in (0, 2\pi)$ has maximal length, then the normal line at $z(t+\alpha)$ intersects C at $z(t)$.

Let

$$ww(C) = \min_{t \in [0,2\pi]} \max_{\alpha \in [0,2\pi]} H(t,\alpha). \tag{2.3}$$

Now we consider the particular but important case of ellipses.

Proposition 2.1. *Let us fix an ellipse E, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$. Then the maximal length of chords is given by the formula*

$$\begin{cases} 3^{\frac{3}{2}} \dfrac{a^2 b^2}{(a^2+b^2)^{\frac{3}{2}}}, & \text{if } a > b\sqrt{2}, \\ 2b, & \text{if } a \le b\sqrt{2}. \end{cases} \tag{2.4}$$

Proof. For a given point $P(r,s)$ of E we consider the normal line to E at P. This normal line intersects E at the second point $\tilde{P}(\tilde{r}, \tilde{s})$, where

$$(\tilde{r}, \tilde{s}) = \left(r \frac{a^6 s^2 - b^6 r^2 - 2a^4 b^2 s^2}{b^6 r^2 + a^6 s^2}, s \frac{b^6 r^2 - 2a^2 b^4 r^2 - a^6 s^2}{b^6 r^2 + a^6 s^2} \right). \tag{2.5}$$

Hence the distance between the points P and $\tilde{P}$ is $d(P, \tilde{P}) = 2(b^4 r^2 + a^4 s^2)^{\frac{3}{2}}$ $(b^6 r^2 + a^6 s^2)^{-1}$. Since $a^2 s^2 = a^2 b^2 - b^2 r^2$, it suffices to find the minimum of the function

$$d(r) = 2b\frac{(a^4-(a^2-b^2)r^2)^{\frac{3}{2}}}{a^6-(a^4-b^4)r^2} \qquad \text{for} \qquad r \in (-a,a).$$

□

Example. Let us fix an ellipse E, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Each chord of the length m which slides around the curve E with endpoints on E determines some curve E_m. We find the equation of the curve E_m. For this aim we solve the system of equations

$$\begin{cases} b^2x^2+a^2y^2-a^2b^2=0,\\ b^2ux+a^2vy-a^2b^2=0, \end{cases}$$

where the second equation represents the line dual to an exterior point (u,v) with respect to E. Then its intersection points with E are

$$(r,s) = \left(a^2\frac{b^2u+v\sqrt{a^2v^2+b^2u^2-a^2b^2}}{a^2v^2+b^2u^2}, b^2\frac{a^2v-u\sqrt{a^2v^2+b^2u^2-a^2b^2}}{a^2v^2+b^2u^2}\right),$$

$$(\tilde r,\tilde s) = \left(a^2\frac{b^2u-v\sqrt{a^2v^2+b^2u^2-a^2b^2}}{a^2v^2+b^2u^2}, b^2\frac{a^2v+u\sqrt{a^2v^2+b^2u^2-a^2b^2}}{a^2v^2+b^2u^2}\right).$$

Since $m^2=(r-\tilde r)^2+(s-\tilde s)^2$, we get

$$4(b^2x^2+a^2y^2-a^2b^2)(b^4x^2+a^4y^2)-m^2(b^2x^2+a^2y^2)^2=0, \tag{2.6}$$

where we substituted x,y instead of u,v, respectively.

3. Sliding a chord around a curve

We denote by $\mathcal{N}$ a family of functions $\nu\colon [0,+\infty)\to\mathbb{R}$ of the class $C^1(0,+\infty)$ satisfying the following conditions:

$$\dot\nu > 0 \qquad \text{for} \qquad t\in(0,+\infty), \tag{3.1}$$

$$t<\nu(t)<t+\pi \qquad \text{for} \qquad t\in[0,+\infty), \tag{3.2}$$

$$\nu(t+2\pi)=\nu(t)+2\pi \qquad \text{for} \qquad t\in[0,+\infty). \tag{3.3}$$

Let C be a curve $t\to z(t)$ given by (1.1), and $\nu\in\mathcal{N}$ be a function. We associate with C a vector field Q along the curve C, defined as follows:

$$Q(t)=z(t)-z(\nu(t)). \tag{3.4}$$

In view of (1.8) we have

$$Q(t)=B(t,\nu(t)-t)e^{it}-b(t,\nu(t)-t)ie^{it}. \tag{3.5}$$

Differentiating (3.5) and using the formulas

$$\begin{cases} \frac{\partial b}{\partial\alpha}=R(t+\alpha)\cos\alpha, & \frac{\partial b}{\partial t}=B(t,\alpha)+R(t+\alpha)\cos\alpha-R(t),\\ \frac{\partial B}{\partial\alpha}=R(t+\alpha)\sin\alpha, & \frac{\partial B}{\partial t}=-b(t,\alpha)+R(t+\alpha)\sin\alpha \end{cases} \tag{3.6}$$

given in [7], we obtain

$$\dot{Q} = \dot{\nu}R(\nu)\sin(\nu - t)e^{it} + (R - \dot{\nu}R(\nu)\cos(\nu - t))ie^{it}. \tag{3.7}$$

We note that

$$\begin{aligned}\langle Q, \dot{Q}\rangle &= \dot{\nu}BR(\nu)\cdot\sin(\nu - t) - (R - \dot{\nu}R(\nu)\cdot\cos(\nu - t))\, b\\ &= \dot{\nu}R(\nu)\,(B\sin(\nu - t) + b\cos(\nu - t)) - bR\\ &= (\dot{\nu}R(\nu)\cdot f(t, \nu - t) - Rg(t, \nu - t))\sin(\nu - t).\end{aligned}$$

Hence we have equivalence of the following conditions:

$$\langle Q, \dot{Q}\rangle = 0, \tag{3.8}$$

$$\dot{\nu} = \frac{R(t)\cdot g(t, \nu - t)}{R(\nu)\cdot f(t, \nu - t)}. \tag{3.9}$$

With respect to (3.9) we consider the implicit equation

$$p\,(t)\sin\Gamma + \dot{p}\,(t + \Gamma) - \dot{p}\,(t)\cos\Gamma = 0.$$

Differentiating the above equation and using the formulas (1.2) and (1.4), we get

$$(R(t + \Gamma) - \lambda\sin\Gamma)\dot{\Gamma} = \lambda\sin\Gamma - R(t + \Gamma) + R\cos\Gamma,$$

and therefore

$$\dot{\Gamma} = \frac{R(t)\cos\Gamma}{R(t + \Gamma) - \lambda(t, \Gamma)\sin\Gamma} - 1. \tag{3.10}$$

If the maximal width is attained at $t = t_0$, then $\Gamma(t_0) = \pi$. We note that $\Gamma(t) > \frac{\pi}{2}$, since for the orthoptic curve we have $f = -\mu \neq 0$ and the considered function Γ is differentiable.

We associate with the curve C and $\nu \in \mathcal{N}$ a curve C_ν defined by

$$t \to w_\nu\,(t) = z\,(t) + \lambda\,(t, \nu\,(t) - t)\, ie^{it} \qquad \text{for} \qquad t \in [0, 2\pi]\,. \tag{3.11}$$

Theorem 3.1. *The integral formula*

$$\int_0^{2\pi} \left[w_\nu\,(t)\,,\ \dot{Q}\,(t)\right] dt = 0 \tag{3.12}$$

holds, where $[a + bi, c + di] = ad - bc$.

Proof. Let $\alpha\,(t) = \nu\,(t) - t$. Using (1.10), (1.7), and (1.8), we obtain

$$\begin{aligned}&\int_0^{2\pi} \left[w_\nu,\ \dot{Q}\right] dt\\ &\quad= \int_0^{2\pi} \left[pe^{it} + (\dot{p} + \lambda)\, ie^{it}, \dot{\nu}R(\nu)\cdot\sin\alpha e^{it} + (R - \dot{\nu}R(\nu)\cdot\cos\alpha)\, ie^{it}\right] dt\\ &\quad= \int_0^{2\pi} \left(p\,(R - \dot{\nu}R(\nu)\cdot\cos\alpha) - \frac{p(\nu) - p\cos\alpha}{\sin\alpha}\dot{\nu}R(\nu)\cdot\sin\alpha\right) dt\\ &\quad= \int_0^{2\pi} (pR - \dot{\nu}R(\nu)\cdot p(\nu))\, dt = 0.\end{aligned}$$

$\square$

4. The main theorem

Now we assume that a chord of constant length m slides around with both endpoints on C which is given by formula (1.1). The endpoints of the sliding chord determine an increasing function $\nu \in \mathcal{N}$. We assume that $|z(0)-z(t_0)| = m$ for some $t_0 \in (0, 2\pi)$. Thus the function ν satisfies the differential equation (3.9) with the initial condition $\nu(0) = t_0$. For a fixed $\xi \in [0,1]$ we consider a curve $C(m,\xi)$ given by the formula

$$t \to v_\xi(t) = z(t) - \xi Q(t) \quad \text{for} \quad t \in [0, 2\pi]. \tag{4.1}$$

Obviously, we have $|Q| \equiv m$. We note that

$$[v_\xi, \dot{v}_\xi] = [z, \dot{z}] - 2\xi[z, \dot{Q}] + \xi[z, Q]^{\cdot} + \xi^2[Q, \dot{Q}].$$

Hence we get immediately

$$\operatorname{Area} C - \operatorname{Area} C(m,\xi) = -\xi \int_0^{2\pi} [\dot{z}, Q]dt - \pi\xi^2 m^2. \tag{4.2}$$

We note that the graphs of the curves $C(m,0)$ and $C(m,1)$ coincide with the graph of C. Thus for $\xi = 1$ from (4.2) it follows immediately that $\int_0^{2\pi} [\dot{z}, Q]dt = -\pi m^2$. Now formula (4.2) can be rewritten in the form

$$\operatorname{Area} C - \operatorname{Area} C(m,\xi) = \pi m^2 \xi(1-\xi). \tag{4.3}$$

Letting $m = a+b$ and $\xi = \frac{a}{a+b}$, we get the well-known *Holditch formula*

$$\operatorname{Area} C - \operatorname{Area} C\left(a+b, \frac{a}{a+b}\right) = \pi ab.$$

Now we associate with C and C_ν a certain curve $D_{\nu,\gamma}$ defined as follows:

$$t \to v\left(t\right) = w_\nu\left(t\right) + \gamma Q\left(t\right) \text{ for } t \in \left[0, 2\pi\right], \tag{4.4}$$

where γ is a nonnegative constant.

Let us fix $\xi \in (0,1)$. We consider a curve $C_{\nu,\xi}$, $t \to v_{\nu,\xi}(t) = (1-\xi)z(t) + \xi z(\nu(t))$ for $t \in [0, 2\pi]$, see Fig. 4.

Theorem 4.1. *If $C \in \mathcal{C}$, $\nu \in \mathcal{N}$, $\xi \in (0,1)$ and $\gamma \neq 0$, then the following formula holds:*

$$\frac{\operatorname{Area} D_{\nu,\gamma} - \operatorname{Area} C_\nu}{\operatorname{Area} C - \operatorname{Area} C_{\nu,\xi}} = \frac{\gamma^2}{\xi(1-\xi)}. \tag{4.5}$$

Proof. We have

$$[v, \dot{v}] = [w_\nu, \dot{w_\nu}] - 2\gamma\left[w_\nu, \dot{Q}\right] + \gamma\left[w_\nu, Q\right]^{\cdot} + \gamma^2\left[Q, \dot{Q}\right].$$

Applying Theorem 3.1, we obtain

$$2\operatorname{Area} D_{\nu,\gamma} - 2\operatorname{Area} C_\nu = \gamma^2 \int_0^{2\pi} \left[Q\left(t\right),\ \dot{Q}\left(t\right)\right] dt. \tag{4.6}$$

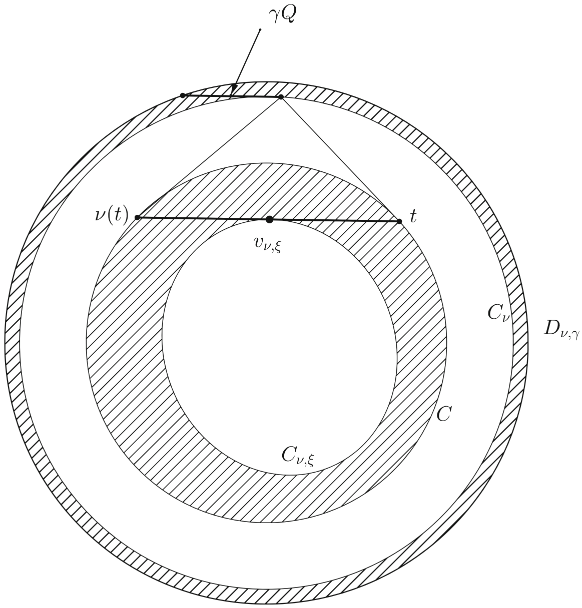

FIGURE 4 The curves C, $C_{\nu,\xi}$, C_ν, $D_{\nu,\gamma}$

It was proved in [5] that

$$2\,\mathrm{Area}\,C - 2\,\mathrm{Area}\,C_{\nu,\xi} = \xi(1-\xi)\int_0^{2\pi}\left[Q\left(t\right),\ \dot{Q}\left(t\right)\right]dt. \tag{4.7}$$

Comparing the formulas (4.6) and (4.7), we get (4.5). □

As corollaries of Theorem 4.1 we have the following Holditch-type formulas.

Corollary 4.2. *If $a+b < ww(C)$, then*

$$\mathrm{Area}\,D_{\frac{b}{a+b}} - \mathrm{Area}\,C_{a+b} = \pi b^2. \tag{4.8}$$

Corollary 4.3. *If $m < ww\left(C\right)$, then*

$$\mathrm{Area}\,D_{\frac{1}{m}} - \mathrm{Area}\,C_m = \pi. \tag{4.9}$$

Figure 5 illustrates Corollary 4.2

5. Crofton-type integral formulas

In this section we provide an interesting application of the developed theory and derive a new, geometrically justified Crofton-type formula.

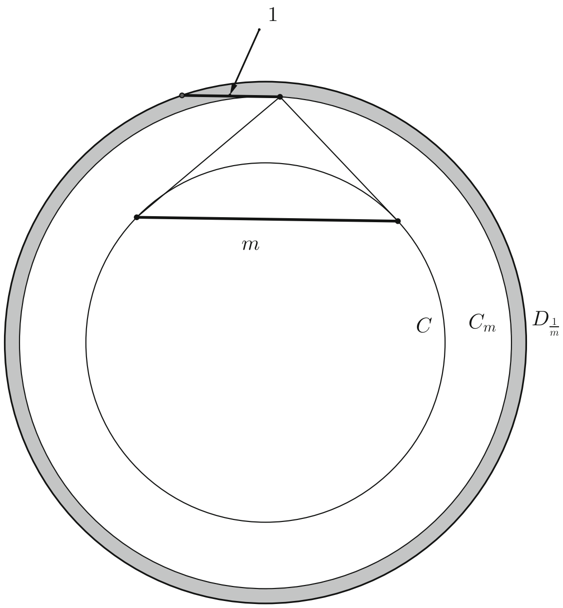

FIGURE 5 Curves C, C_m, $D_{\frac{1}{m}}$

Let us fix $C \in \mathcal{C}$ and $r < ww\,(C)$. The function $\nu\,(t,m)$ determined by $m \in (0,r)$ satisfies the condition (3.4). Differentiating (3.4) with respect to m, we obtain

$$m = \left\langle Q, \frac{\partial Q}{\partial m} \right\rangle = -R(\nu) \cdot \frac{\partial \nu}{\partial m} \left\langle Q, ie^{i\nu} \right\rangle = R(\nu) \cdot \frac{\partial \nu}{\partial m} \left[Q, e^{i\nu}\right].$$

Let $\alpha\,(t,m) = \nu\,(t,m) - t$. With respect to (3.5) and (1.15) we have

$$\left[Q, e^{i\nu}\right] = \left[Be^{it} - bie^{it}, \cos\alpha \cdot e^{it} + \sin\alpha \cdot ie^{it}\right] = B\sin\alpha + b\cos\alpha = F\sin\alpha.$$

The above calculations imply that

$$\frac{\partial \nu}{\partial m} = \frac{m}{R(\nu) \cdot F\sin\alpha}. \tag{5.1}$$

Now, we consider the ring domain CC_r , and we introduce the notations as in Fig. 6, maintaining at the same time the notations of Santaló from [17].

Let $R_1(x,y)$, $R_2(x,y)$ denote the radii of curvature of C at the tangent points A_1, A_2, respectively.

Crofton proved the integral formula

$$\iint\limits_{\operatorname{ext} C} \frac{\sin\omega}{t_1 t_2} dxdy = 2\pi^2,$$

where $\operatorname{ext} C$ denotes the exterior of C, see [13]. We will prove some Crofton-type theorem, namely

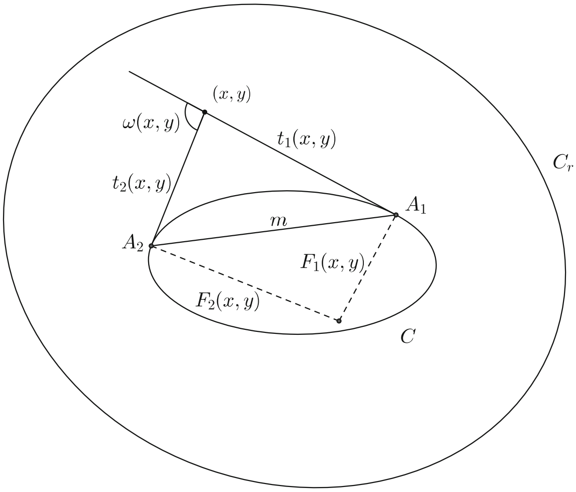

Figure 6 Notations for Theorem 5.1

Theorem 5.1. *If $C \in \mathcal{C}$ and $r < ww\,(C)$, then the following integral formula holds:*

$$\iint\limits_{CC_r} R_2 F_1 \frac{\sin^3 \omega}{t_1 t_2} dxdy = \pi r^2. \tag{5.2}$$

Proof. We consider a mapping $T\colon (0, 2\pi) \times (0, r) \to$ interior of $CC_r \setminus \{$some segment$\}$ defined by the formula

$$T\,(t, m) = z\,(t) + \lambda\,(t, \nu\,(t, m) - t)\, ie^{it}.$$

We note that T is a bijection and

$$\frac{\partial T}{\partial m} = \frac{\partial \lambda}{\partial m} ie^{it},$$
$$\frac{\partial T}{\partial t} = -\lambda e^{it} + hie^{it},$$

where h is some function. Thus the Jacobian JT of T at (t, m) is given by the formula

$$JT\,(t, m) = \left[\frac{\partial T}{\partial t}\,(t, m)\,, \frac{\partial T}{\partial m}\,(t, m)\right] = -\lambda\,(t, m)\,\frac{\partial \lambda}{\partial m}\,(t, m)\,.$$

On the other hand, since $\alpha\,(t, m) = \nu\,(t, m) - t$, so $\frac{\partial \nu}{\partial m} = \frac{\partial \alpha}{\partial m}$ and

$$\begin{aligned}\sin^2 \alpha \frac{\partial \lambda}{\partial m} &= \sin^2 \alpha \frac{\partial}{\partial m}\left(\frac{p(\nu) - p\cos\alpha}{\sin\alpha} - \dot{p}\right) \\ &= \left(\dot{p}(\nu)\cdot\frac{\partial \nu}{\partial m} + p\sin\alpha\frac{\partial \alpha}{\partial m}\right)\sin\alpha - (p(\nu) - p\cos\alpha)\cos\alpha\frac{\partial \alpha}{\partial m}\end{aligned}$$

$$
\begin{aligned}
&= (\dot{p}(\nu) \cdot \sin\alpha - p(\nu) \cdot \cos\alpha + p) \frac{\partial \nu}{\partial m} \\
&= -\mu \frac{\partial \nu}{\partial m} = \frac{-\mu m}{R(\nu) \cdot F \sin\alpha}.
\end{aligned}
$$

Thus the Jacobian of T at (t, m) has the form

$$
JT\,(t, m) = \frac{-\mu\lambda m}{R(\nu) \cdot F \sin^3\alpha}.
$$

Now we have

$$
\iint\limits_{CC_r} R_2 F_1 \frac{\sin^3\omega}{t_1 t_2} dx dy = \int\limits_0^{2\pi} \int\limits_0^{r} R(\nu) \cdot F \frac{\sin^3\alpha}{-\mu\lambda} \frac{-\mu\lambda m}{R(\nu) \cdot F \sin^3\alpha} dm dt = \pi r^2.
$$

$\square$

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