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ON HOLOMORPHIC REPRESENTATIONS OF SYMPLECTIC GROUPS

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Let G denote the complex symplectic group which may be defined by the equation

$$G = \left\{ g \in \mathrm{GL}(2k, \mathbf{C}) \colon g s_k g^t = s_k,\ s_k = \begin{bmatrix} 0 & -I_k \\ I_k & 0 \end{bmatrix} \right\}.$$

In this paper we shall give a simple and concrete realization of a set of representatives of all irreducible holomorphic representations of G. This realization, which involves the G-module structure of a symmetric algebra of polynomial functions is inspired by the work of B. Kostant [1] and follows the general scheme formulated in [2]. Detailed proofs will appear elsewhere.

1. **The symmetric algebra $S(E^*)$.** Set $E = \mathbf{C}^{n\times 2k}$ with $k \geqslant n \geqslant 2$; then G acts linearly on E by right multiplication. Let $(\cdot\,,\cdot)$ denote the skew-symmetric bilinear form on E given by

$$(X, Y) = \mathrm{trace}(X s_k Y^t), \quad \forall X, Y \in E.$$

If $X \in E$, let X^* denote the linear form $Y \longrightarrow (X, Y)$ on E. The map $X \longrightarrow X^*$ establishes an isomorphism between E and its dual E^*. Let $S(E^*)$ denote the symmetric algebra of all complex-valued polynomial functions on E. The action of G on E induces a representation R of G on $S(E^*)$ defined by

$$(R(g)p)(X) = p(Xg), \quad \forall p \in S(E^*), \ \forall X \in E.$$

If $X \in E$, define a differential operator $X^*(D)$ on $S(E^*)$ by setting

$$(X^*(D)f)(Y) = \{(d/dt) f(Y + tX)\}_{t=0},$$

$$\text{for all } f \in S(E^*),\ t \in \mathbf{R}, \text{ and } X, Y \in E.$$

Define $(X_1^* \cdots X_n^*)(D)f = X_1^*(D)((X_2^* \cdots X_n^*)(D)f)$ inductively on n. If

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m and l are nonnegative integers and if S_m denotes the symmetric group on m elements, then it may be shown that

$$[X_1^* \cdots X_l^*(D)]Y_1^* \cdots Y_m^* = \begin{cases} 0, & \text{if } m < l, \\ \dfrac{(-1)^l}{(m-l)!} \displaystyle\sum_{\sigma \in S_m} X_1^*(Y_{\sigma(1)}) \cdots X_l^*(Y_{\sigma(l)}^*) Y_{\sigma(l+1)}^* \cdots Y_{\sigma(m)}^*, & \text{if } m \geq l. \end{cases}$$

It follows from the above equation and by linearity that the map $X^* \longrightarrow X^*(D)$ extends to an isomorphism $p \longrightarrow p(D)$ between $S(E^*)$ and the symmetric algebra $S(E)$ of differential operators on E.

A polynomial $f \in S(E^*)$ will be called *G-invariant* if $R(g)f = f$, $\forall g \in G$. A differential operator $p(D) \in S(E)$ will be called *G-invariant* if $R(g)(p(D)f) = p(D)(R(g)f)$, for all $g \in G$, $f \in S(E^*)$. It is then shown that $p \in S(E^*)$ is G-invariant if and only if $p(D)$ is G-invariant.

Let $J(E^*)$ (resp. $J(E)$) denote the subalgebra of $S(E^*)$ (resp. of $S(E)$) consisting of all G-invariant polynomials (resp. of all G-invariant differential operators). Let $J^+(E^*)$ denote the set of all G-invariant polynomials without constant terms; $J^+(E)$ is then defined in a similar fashion.

A polynomial $f \in S(E^*)$ is said to be *G-harmonic* if $p(D)f = 0$ for all $p \in J^+(E^*)$. Let $H(E^*)$ denote the subspace of $S(E^*)$ consisting of all G-harmonic polynomials. Let $J^+(E^*)S(E^*)$ be the ideal in $S(E^*)$ generated by $J^+(E^*)$, and denote by V the algebraic variety in E of common zeros of polynomials in the ideal $J^+(E^*)S(E^*)$. It follows from the theory of polynomial invariants (cf. [3, Chapter VI]) that $J(E^*)$ is generated by the constant function 1 and $n(n-1)/2$ polynomials p_{ij} defined by

$$p_{ij}(X) = \sum_{l=1}^{k} (X_{i,l+k} X_{jl} - X_{il} X_{j,l+k}), \qquad 1 \leq i < j \leq n;\ X = (X_{rs}) \in E.$$

Moreover, we have $V = \{X \in E; Xs_k X^t = 0\}$ and that $H(E^*) = \{f \in S(E^*): p_{ij}(D)f = 0, \forall i, j, 1 \leq i < j \leq n\}$. It is then shown that the ideal $J^+(E^*)S(E^*)$ is prime.

THEOREM 1.1. *The space $S(E^*)$ is decomposed into a direct sum as $S(E^*) = J^+(E^*)S(E^*) \oplus H(E^*)$. Moreover, $S(E^*) = J(E^*) \otimes H(E^*)$ and $H(E^*)$ is spanned by all polynomials $(X^*)^m$, $m = 1, 2, \ldots$, for all $X \in V$.*

COROLLARY 1.2. *If $S(V)$ denotes the ring of functions on V obtained by restricting elements of $S(E^*)$ to V, then the restriction mapping $f \longrightarrow f/V$ ($f \in H(E^*)$) is a G-module isomorphism of $H(E^*)$ onto $S(V)$.*

2. **The irreducible holomorphic representations of** G. Let B denote the lower triangular subgroup of $GL(n, \mathbf{C})$ and define a holomorphic character $\xi =$

$\xi(m_1, \ldots, m_n)$ of B by setting

$$\xi(b) = b_{11}^{m_1} b_{22}^{m_2} \cdots b_{nn}^{m_n} \qquad (b \in B),$$

where the m_i's $(1 \leqslant i \leqslant n)$ are integers satisfying $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n \geqslant 0$. A polynomial $f \in S(E^*)$ will be called ξ-covariant if $f(bX) = \xi(b) f(X)$, $\forall (b, X) \in B \times E$. Let $H(E, \xi)$ denote the subspace of $H(E^*)$ consisting of all ξ-covariant G-harmonic polynomials.

THEOREM 2.1. *If $R(\cdot, \xi)$ denotes the representation of G which is obtained by right translation on $H(E, \xi)$ then $R(\cdot, \xi)$ is irreducible and its highest weight is indexed by $(m_1, m_2, \ldots, m_n, 0, \ldots, 0)$ (k factors).*

PROOF. Let

$$C = \left\{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} \in \mathrm{GL}(2k, \mathbf{C}) \colon c \text{ diagonal } k \times k \text{ matrix} \right\}$$

and

$$U = \left\{ \begin{bmatrix} u_1 & 0 \\ u_2 & u_1^v \end{bmatrix} \colon u_1^v = (u_1^t)^{-1},\ u_1^v u_2^t - u_2 u_1^{-1} = 0,\ u_1 \text{ lower triangular unipotent} \right\};$$

then CU is a Borel subgroup of G. Define a holomorphic character ζ on CU by setting

$$\zeta(cu) = c_{11}^{m_1} \cdots c_{nn}^{m_n}, \quad \forall\, cu \in CU.$$

Let $\mathrm{Hol}(G, \zeta)$ denote the space of all ζ-covariant holomorphic functions on G. Then by the Borel-Weil-Bott theorem the representation $\pi(\cdot, \zeta)$ of G which is obtained by right translation on $\mathrm{Hol}(G, \zeta)$ is irreducible (see also [4, Chapter XVI]). Let $\mathbf{I} = [I_n \ 0] \in E$, then $\mathrm{Orb}(\mathbf{I}) = \{\mathbf{I}g \colon g \in G\}$ is a dense subset of V. Define a map Φ from $H(E, \xi)$ into $\mathrm{Hol}(G, \zeta)$ by the equation $(\Phi f)(g) = f(\mathbf{I}g)$, $\forall f \in H(E, \xi)$, $\forall g \in G$. Then it follows from Corollary 1.2 that Φ is a G-module isomorphism. □

When $k = n$, the following theorem is an immediate consequence of Theorem 2.1.

THEOREM 2.2. *Suppose that*

$$E = \mathbf{C}^{k \times 2k} \ (k \geqslant 2) \quad \text{and} \quad \xi = \xi(m_1, m_2, \ldots, m_k);$$

then the representations $R(\cdot, \xi)$ of G on the various spaces $H(E, \xi)$ realize up to equivalence all irreducible holomorphic representations of G when the m_i's $(1 \leqslant i \leqslant k)$ are allowed to take all integral values subject to the condition $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_k \geqslant 0$. Moreover, to each representation $R(\cdot, \xi)$ corresponds a highest weight vector $f_\xi \in S(E^*)$ defined by the equation

$$f_\xi(X) = \Delta_1^{m_1 - m_2}(X)\Delta_2^{m_2 - m_3}(X) \cdots \Delta_{k-1}^{m_{k-1} - m_k}(X)\, \Delta_k^{m_k}(X), \quad \forall X \in E$$

where the $\Delta_i(X)$ $(1 \leqslant i \leqslant k)$ *are the principal minors of* X.

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