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**Research article**

# On Hom-Spaces of Tame Algebras*

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**Abstract:** Let $\Lambda$ be a finite dimensional algebra over an algebraically closed field $k$ and $\Lambda$ has tame representation type. In this paper, the structure of Hom-spaces of all pairs of indecomposable $\Lambda$-modules having dimension smaller than or equal to a fixed natural number is described, and their dimensions are calculated in terms of a finite number of finitely generated $\Lambda$-modules and generic $\Lambda$-modules. In particular, such spaces are essentially controlled by those of the corresponding generic modules.


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## 1 Introduction

Let $\Lambda$ be a finite-dimensional $k$-algebra of tame representation type, $k$ an algebraically closed field. We recall that $\Lambda$ is of tame representation type if for all natural numbers $d$, there is a finite number of $\Lambda$-$k[x]$-bimodules $M_1, ..., M_n$ which are free of finite rank as right $k[x]$-modules and such that if $M$ is an indecomposable $\Lambda$-module of $k$-dimension equal to $d$, then $M \cong M_i \otimes_{k[x]} k[x]/(x-\lambda)$ for some $1 \leq i \leq n$ and $\lambda \in k$.

It is known from [6] that for each dimension $d$, almost all $\Lambda$-modules of dimension at most $d$ are controlled by finitely many isomorphism classes of generic modules in the sense of (i) of Theorem 1.2. A question arises naturally: are Hom-spaces of $\Lambda$-modules also controlled by those of generic modules? In this paper, we will give a positive answer.

If $G$ is a left $\Lambda$-module then $G$ can be regarded as a left $\mathrm{End}_\Lambda(G)$-module, and we call its length as $\mathrm{End}_\Lambda(G)$-module, the endolength of $G$. We say that $G$ is a generic module if it is indecomposable, of infinite dimension over $k$ but finite endolength. We recall that if $G$ is a generic $\Lambda$-module and $R$ a commutative principal ideal domain which is finitely generated over $k$, then a realization of $G$ over $R$ is a finitely generated $\Lambda$-$R$-bimodule $T$ such that if $K$ is the quotient field of $R$, then $G \cong T \otimes_R K$ and $\dim_K(T \otimes_R K)$ is equal to the endolength of $G$.

As an example consider, $\Lambda$=$kQ$, the Kronecker algebra defined by quiver $Q$, then $G$ is a generic module, and $T$ is a realization of $G$ over $R = k[x]$.

$$Q =: \cdot \underset{b}{\overset{a}{\rightrightarrows}} \cdot, \qquad G =: k(x) \underset{id}{\overset{x}{\rightrightarrows}} k(x), \qquad T =: k[x] \underset{id}{\overset{x}{\rightrightarrows}} k[x].$$

We denote by $\Lambda$-Mod the category of left $\Lambda$-modules, by $\Lambda$-mod the full subcategory of $\Lambda$-Mod consisting of the finite-dimensional $\Lambda$-modules, and by $\Lambda$-ind the full subcategory of $\Lambda$-mod consisting of the indecomposable $\Lambda$-modules.

We recall from Theorem 5.4 of [6] that if $\Lambda$ is of tame representation type then given any generic $\Lambda$-module there is a *good realization* of $G$ over some $R$ in the sense of the following definition:

**Definition 1.1.** Let $T$ be a realization of a generic module $G$ over some $R$, then $T$ is called a good realization if:

(i) $T$ is free as right $R$-module;

(ii) the functor $T \otimes_R - : R\text{-Mod} \to \Lambda\text{-Mod}$ preserves isomorphism classes and indecomposability;

(iii) if $p \in R$ is a prime, $n \geq 1$ and $S_{p,n}$ denotes the exact sequence

$$0 \to R/(p^n) \overset{(p,\pi)}{\to} R/(p^{n+1}) \oplus R/(p^{n-1}) \overset{\begin{pmatrix} \pi \\ -p \end{pmatrix}}{\to} R/(p^n) \to 0$$

where $\pi$ is the canonical projection, then $T \otimes_R S_{p,n}$ is an almost split sequence in $\Lambda$-mod.

We know from Theorem 4.6 of [6] that if $G$ is a generic $\Lambda$-module then there is a splitting $\mathrm{End}_\Lambda(G) = k(x) \oplus \mathrm{radEnd}_\Lambda(G)$. This splitting induces a structure of left $\Lambda^{k(x)} = \Lambda \otimes_k k(x)$-module for $G$ and such structure is called an *admissible structure*. The main aim of this paper is to prove of the following theorem:

**Theorem 1.2.** *Let $\Lambda$ be a finite-dimensional $k$-algebra of tame representation type, $k$ an algebraically closed field. Let $d$ be an integer greater than the dimension of $\Lambda$ over $k$. Then there are generic $\Lambda$-modules $G_1, ..., G_s$ with admissible structures of left $\Lambda^{k(x)}$-modules and good realizations $T_i$ over some $R_i$, finitely generated localization of $k[x]$, of each $G_i$ and indecomposable $\Lambda$-modules $L_1, ..., L_t$ with $\dim_k L_j \leq d$ for $j = 1, ..., t$ with the following properties:*

*(i) If $M$ is an indecomposable left $\Lambda$-module with $\dim_k M \leq d$, then either $M \cong L_j$ for some $j \in \{1, ..., t\}$ or $M \cong T_i \otimes_{R_i} R_i/(p^m)$ for some $i \in \{1, ..., s\}$ some prime element $p \in R_i$ and some natural number $m$. If $M$ is an indecomposable which is simple, projective or injective left $\Lambda$-module, then $M \cong L_j$ for some $j \in \{1, ..., t\}$.*

*(ii) If $M = T_i \otimes_{R_i} R_i/(p^m), N = T_j \otimes_{R_j} R_j/(q^n), L_u^{k(x)} = L_u \otimes_k k(x)$ with $i, j \in \{1, ..., s\}, u \in \{1, ..., t\}$, $p$ a prime in $R_i$, $q$ a prime in $R_j$, then*

$$\dim_k \mathrm{rad}^\infty_\Lambda(M, N) = mn\dim_{k(x)}\mathrm{rad}_{\Lambda^{k(x)}}(G_i, G_j),$$

$$\dim_k \mathrm{rad}^\infty_\Lambda(L_u, M) = m\dim_{k(x)}\mathrm{rad}_{\Lambda^{k(x)}}(L_u^{k(x)}, G_i),$$

$$\dim_k \mathrm{rad}^\infty_\Lambda(M, L_u) = m\dim_{k(x)}\mathrm{rad}_{\Lambda^{k(x)}}(G_i, L_u^{k(x)}).$$

*(iii) Suppose $M = T_i \otimes_{R_i} R_i/(p^m), N = T_j \otimes_{R_j} R_j/(q^n)$, then if $i = j$, $p = q$,*

$$\mathrm{Hom}_\Lambda(M, N) \cong \mathrm{Hom}_{R_i}(R_i/(p^m), R_i/(p^n)) \oplus \mathrm{rad}^\infty_\Lambda(M, N).$$

*And if $i \neq j$ or $(p) \neq (q)$:*

$$\mathrm{Hom}_\Lambda(M, N) = \mathrm{rad}^\infty_\Lambda(M, N).$$

*Moreover,* $\mathrm{Hom}_\Lambda(L_u, M) = \mathrm{rad}^\infty_\Lambda(L_u, M)$, $\mathrm{Hom}_\Lambda(M, L_u) = \mathrm{rad}^\infty_\Lambda(M, L_u)$.

For the proof of our main result we first study layered bocses of tame representation type (see Theorem 9.2). For this we use the method of reduction functors $F : \mathcal{B}_1$-Mod $\to \mathcal{B}_2$-Mod between the representation categories of two layered bocses $\mathcal{B}_1$ and $\mathcal{B}_2$ (see [5], [7] and section 7 of this paper). We prove that given a layered bocs $\mathcal{A}$ of tame representation type and a dimension vector $\mathbf{d}$ of $\mathcal{A}$ there is a composition of reduction functors $F : \mathcal{B}$-Mod $\to \mathcal{A}$-Mod with $\mathcal{B}$ a minimal bocs such that if $M \in \mathcal{A}$-Mod with $\mathbf{dim}M \leq \mathbf{d}$, then there is a $N \in \mathcal{B}$-Mod with $F(N) \cong M$. Observe that in Theorem A of [5] several minimal bocses are needed. In section 6 we study the Hom-spaces for minimal bocses. Consider now the category $P^1(\Lambda)$ of morphisms $f : P \to Q$ with $P$,$Q$ projective $\Lambda$-modules and $f(P) \subset \mathrm{rad}Q$. There is a layered bocs $\mathcal{D}(\Lambda)$, the Drozd's bocs, such that $\mathcal{D}(\Lambda)$-Mod is equivalent to $P^1(\Lambda)$. Using our results on Hom-spaces for minimal layered bocses we study the Hom-spaces in $P^1(\Lambda)$ obtaining a version of Theorem 1.2 for $P^1(\Lambda)$ (see Theorem 9.5). Finally, we use the relations between Hom-spaces in $P^1(\Lambda)$ and $\Lambda$-Mod collected in the results of sections 2 and 3.

## 2 Generalities

Here we state the general results needed in our work. We recall that an additive $k$-category $\mathcal{R}$ is a Krull-Schmidt category if each object is a finite direct sum of indecomposable objects with local endomorphism rings. In this case, the indecomposable objects coincide with those having local endomorphism rings.

Let $\mathcal{R}$ be a Krull-Schmidt category. A morphism $f : E \to M$ in $\mathcal{R}$ is called irreducible if it is neither a retraction nor a section and for any factorization $f = vu$, either $u$ is a section or $v$ is a retraction.

A morphism $f : E \to M$ in $\mathcal{R}$ is called right almost split if

(i)$f$ is not a retraction ;

(ii) if $g : X \to M$ is not a retraction, there is a $s : X \to E$ with $fs = g$.

Moreover, $f : E \to M$ a right almost split morphism is said to be minimal if $fu = f$ with $u \in \mathrm{End}_{\mathcal{R}}(E)$ implies $u$ is an isomorphism.

One has the dual concepts for left almost split morphisms and minimal left almost split morphisms.

**Remark**. Any minimal right almost split morphism $f : E \to M$ is an irreducible morphism. Moreover if $X \neq 0$, $g : X \to M$ is an irreducible morphism iff there is a section $\sigma : X \to E$ with $f\sigma = g$.

In particular if $h : F \to M$ is also a minimal right almost split morphism there is an isomorphism $u : F \to E$ with $fu = h$.

Similar properties hold for minimal left almost split morphisms.

**Definition 2.1.** A pair of composable morphisms in $\mathcal{R}$,

$$M \xrightarrow{f} E \xrightarrow{g} N$$

is said to be almost split if

(i) $g$ is a minimal right almost split morphism;

(ii) $f$ is a minimal left almost split morphism, and;

(iii) $gf = 0$

In the following, we use the following notation. If $f : E \to M$ and $f' : E' \to M'$ are morphisms in $\mathcal{R}$, a morphism from $f$ to $f'$ is a pair $(u, v)$ where $u : E \to E'$ and $v : M \to M'$ are morphisms such that $f'u = vf$. If $u, v$ are isomorphisms, we say that $f$ and $g$ are isomorphic. Similarly if $M \xrightarrow{f} E \xrightarrow{g} N$, $M' \xrightarrow{f'} E' \xrightarrow{g'} N'$ are pairs of composable morphisms, a morphism from $(f, g)$ into $(f', g')$ is a triple $(u_1, u_2, u_3)$ where $u_1 : M \to M'$,

$u_2 : E \to E'$, $u_3 : N \to N'$ are morphisms such that $u_2 f = f' u_1, u_3 g = g' u_2$. If $u_1, u_2, u_3$ are isomorphisms we say that the pair $(f, g)$ is isomorphic to the pair $(f', g')$. The pairs $(f, g)$ and $(f', g')$ are equivalent if $M = M'$, $N = N'$ and there is an isomorphism from the first pair into the second one of the form $(1_M, u, 1_N)$.

If $\mathcal{A}$ is an additive category with split idempotents a pair $(i, d)$ of composable morphisms $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $\mathcal{A}$ is said to be exact if $i$ is a kernel of $d$, and $d$ is a cokernel of $i$. Let $\mathcal{E}$ be a class of exact pairs closed under isomorphisms. The morphisms $i$ and $d$ appearing in a pair of $\mathcal{E}$ are called an inflation and a deflation of $\mathcal{E}$, respectively.

We recall from [9] that the class $\mathcal{E}$ is an exact structure for $\mathcal{E}$ if the following axioms are satisfied:

E.1 The composition of two deflations is a deflation.

E.2 If $f : Z' \to Z$ is a morphism in $\mathcal{A}$ for each deflation $d : Y \to Z$ there is a morphism $f' : Y' \to Y$ and a deflation $d' : Y' \to Z'$ such that $df' = fd'$.

E.3 Identities are deflations. If $de$ is deflation, then so is $d$.

E.$3^{op}$ Identities are inflations. If $ji$ is a inflation, then so is $i$.

If $\mathcal{E}$ is an exact structure for $\mathcal{A}$ then we denote by $\mathrm{Ext}_{\mathcal{A}}(X, Y)$ the equivalence class of the pairs $Y \xrightarrow{i} E \xrightarrow{d} X$ in $\mathcal{E}$. If $\mathcal{A}$ is a $k$-category, $\mathrm{Ext}_{\mathcal{A}}(?, -)$ is a bifunctor from $\mathcal{A}$ into the category of $k$-vector spaces, contravariant in the first variable and covariant in the second variable.

An object $X \in \mathcal{A}$ is called $\mathcal{E}$-projective if $\mathrm{Ext}_{\mathcal{A}}(X, -) = 0$, and it is called $\mathcal{E}$-injective if $\mathrm{Ext}_{\mathcal{A}}(-, X) = 0$.

**Definition 2.2.** An almost split pair $X \to Y \to Z$ in $\mathcal{A}$ which is in $\mathcal{E}$ is called an almost split $\mathcal{E}$-sequence.

As in the case of modules, one can prove that in the above definition, $X$ and $Z$ are indecomposables.

Now, consider $(\mathcal{A}, \mathcal{E})$ an exact category with $\mathcal{A}$ a Krull-Schmidt $k$-category such that for $X, Y \in \mathcal{A}, \dim_k \mathrm{Hom}_{\mathcal{A}}(X, Y)$ is finite. Let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$ having the following property:

(A) If $X$ is an indecomposable object in $\mathcal{C}$ there is a minimal left almost split morphism in $\mathcal{A}$, $f : X \to Y_1 \oplus ... \oplus Y_t$ with $Y_i \in \mathcal{C}$.

We recall that a morphism $f : M \to N$ with $M, N$ indecomposable objects in $\mathcal{A}$ is called a radical morphism if $f$ is not an isomorphism.

**Proposition 2.3.** *Let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$ with condition (A).*

*Suppose $h : M \to N$ is a morphism in $\mathcal{A}$ with $M, N$ indecomposable objects in $\mathcal{C}$ such that $h = \sum h_i$, where each $h_i$ is a composition of $m$ radical morphisms between indecomposables in $\mathcal{A}$, then $h = \sum g_j$ with each $g_j$ composition of $m$ radical morphisms between indecomposables in $\mathcal{C}$.*

**Proof.** By induction on $m$. If $m = 1$ our assertion is trivial. Assume our assertion is true for $m-1$. We may assume $h = s_m \cdots s_1$ with $s_i : M_i \to M_{i+1}$, $M_j$ indecomposable object of $\mathcal{A}$ for $j = 1, ..., m+1$, $M_1 = M, M_{m+1} = N$. By (A), there is a left almost split morphism $M = M_1 \xrightarrow{u} Y_1 \oplus ... \oplus Y_t$ with $Y_1, ..., Y_t \in \mathcal{C}$. We have $u = \begin{pmatrix} u_1 \\ \vdots \\ u_t \end{pmatrix}$. Then there is $v = (v_1, ..., v_t) : Y_1 \oplus ... \oplus Y_t \to M_2$ with $vu = s_1 = \sum_{i=1}^{t} v_i u_i$. Therefore,

$$h = s_m \cdots s_2 s_1 = \sum_{i=1}^{t} s_m \cdots s_2 v_i u_i.$$

Now, consider $g_i = s_m \cdots s_2 v_i : Y_i \to N$ which is a composition of $m-1$ radical morphisms. Then, by induction hypothesis, each $g_i$ is a sum of $m-1$ radical morphisms between indecomposables in $\mathcal{C}$. Consequently, $h$ is a sum of compositions of $m$ radical morphisms between objects in $\mathcal{C}$. This proves our claim. □

We recall that an ideal of a $k$-category $\mathcal{R}$ is a subfunctor of $\mathrm{Hom}_{\mathcal{R}}(-, ?)$. If $I, J$ are ideals of $\mathcal{R}$, $IJ$ is the ideal such that for $X, Y \in \mathcal{R}$, $IJ(X, Y)$ consists of sums of compositions $gf$ with $f \in J(X, Z), g \in I(Z, Y)$ for some $Z \in \mathcal{R}$. We denote by $I^2$ the ideal $II$ and, by induction, $I^n = I^{n-1}I$. For $\mathcal{R}$ a Krull-Schmidt $k$-category we define the ideal $\mathrm{rad}_{\mathcal{R}}$ such that for $X$ and $Y$ indecomposable objects of $\mathcal{R}$, $\mathrm{rad}_{\mathcal{R}}(X, Y) =$ the morphisms which are not isomorphisms. The infinity radical is defined by

$$\mathrm{rad}_{\mathcal{R}}^{\infty} = \bigcap_n \mathrm{rad}_{\mathcal{R}}^n.$$

**Corollary 2.4.** *With the hypothesis of proposition 2.3, for $X, Y \in \mathcal{C}$,*

$$\mathrm{rad}_{\mathcal{C}}^{\infty}(X, Y) = \mathrm{rad}_{\mathcal{A}}^{\infty}(X, Y).$$

**Proof.** We may assume $X$ and $Y$ are indecomposables. It follows from Proposition 2.3 that $\mathrm{rad}_{\mathcal{C}}^m(X, Y) = \mathrm{rad}_{\mathcal{C}}^m(X, Y)$ for all $m$. Hence,

$$\mathrm{rad}_{\mathcal{C}}^{\infty}(X, Y) = \bigcap_m \mathrm{rad}_{\mathcal{C}}^m(X, Y) = \bigcap_m \mathrm{rad}_{\mathcal{A}}^m(X, Y) = \mathrm{rad}_{\mathcal{A}}^{\infty}(X, Y).$$

□

Now, we recall the following definition of [5], section 2:

**Definition 2.5.** If $(\mathcal{A}, \mathcal{E})$ is an exact category with $\mathcal{A}$ a Krull-Schmidt category, we say that it has almost split sequences if

i) for any indecomposable $Z$ in $\mathcal{A}$ there is a right almost split morphism $Y \to Z$ and a left almost split morphism $Z \to X$;

ii) for each indecomposable $Z$ in $\mathcal{A}$ which is not $\mathcal{E}$-projective, there is an almost split $\mathcal{E}$-sequence ending in $Z$, and for each indecomposable $Z$ in $\mathcal{A}$ which is not $\mathcal{E}$-injective, there is an almost split $\mathcal{E}$-sequence starting in $Z$.

**Remark**. If the exact category $(\mathcal{A}, \mathcal{E})$ has almost split sequences one can consider the valued Auslander-Reiten quiver of $\mathcal{A}$ as in the case of the category of finitely generated modules over an artin algebra.

**Proposition 2.6.** *Suppose $(\mathcal{A}, \mathcal{E}_\mathcal{A})$ and $(\mathcal{B}, \mathcal{E}_\mathcal{B})$ are two exact categories such that the first category has almost split sequences and $F : \mathcal{B} \to \mathcal{A}$ is a full and faithful functor sending $\mathcal{E}_\mathcal{B}$-sequences into $\mathcal{E}_\mathcal{A}$-sequences. Let $\{E_i\}_{i\in\mathbb{N}}$ be a set of pairwise non-isomorphic objects in $\mathcal{B}$ which are not $\mathcal{E}_\mathcal{B}$-projectives, and almost split $\mathcal{E}_\mathcal{B}$-sequences:*

$$(e_1) : E_1 \xrightarrow{f_1} E_2 \xrightarrow{g_1} E_1$$

$$(e_i) : E_i \xrightarrow{\begin{pmatrix} f_i \\ g_{i-1} \end{pmatrix}} E_{i+1} \oplus E_i \xrightarrow{(g_i, f_{i-1})} E_i,$$

*for $i > 1$. Then, if there is an almost split $\mathcal{E}_\mathcal{A}$- sequence ending in $F(E_1)$ which is the image under $F$ of a sequence in $\mathcal{E}_\mathcal{B}$, then the image $F(e_i)$ of the sequence $e_i$ is an $\mathcal{E}_\mathcal{A}$-almost split sequence for all $i \in \mathbb{N}$.*

**Proof.** There is a sequence in $\mathcal{E}_\mathcal{B}$, $(a) : M \xrightarrow{u} E \xrightarrow{v} E_1$ whose image under $F$ is an almost split $\mathcal{E}_\mathcal{A}$-sequence. Since $F$ is a full and faithful functor, then $(a)$ is an almost split sequence. This implies that $(a)$ is isomorphic to $(e_1)$. Therefore, the image under $F$ of $(e_1)$ is isomorphic to the image under $F$ of $(a)$ which is an almost split sequence, and so, the image of $(e_1)$ under $F$ is an almost split sequence.

Suppose that $F(e_l)$ is an almost split sequence for all $l \leq i$. By hypothesis, $(e_{i+1})$ is a non-trivial $\mathcal{E}_\mathcal{B}$-sequence, since $F$ is a full and faithful functor. Then $F(e_{i+1})$ is a non-trivial $\mathcal{E}_\mathcal{A}$-sequence. Thus, $F(E_{i+1})$ is not $\mathcal{E}_\mathcal{A}$-projective. Then there is an almost split sequence

$$L_{i+1} \to M_{i+1} \to F(E_{i+1}).$$

Here $F(e_i)$ is an almost split sequence. Then we have an almost split sequence:

$$F(E_i) \to F(E_{i+1}) \oplus F(E_{i-1}) \to F(E_i),$$

and so, we have an irreducible morphism $F(E_i) \to F(E_{i+1})$. Therefore, $M_{i+1} \cong F(E_i) \oplus Y$. Thus, we have an irreducible morphism $L_{i+1} \to F(E_i)$. This implies that $L_{i+1} \cong F(E_{i+1})$ or $L_{i+1} \cong F(E_{i-1})$. But we have an almost split sequence starting and ending in $F(E_{i-1})$. Therefore, if $L_{i+1} \cong F(E_{i-1})$, then $F(E_{i+1}) \cong F(E_{i-1})$ implies $E_{i+1} \cong E_{i-1}$, which is not the case, therefore $L_{i+1} \cong F(E_{i+1})$. Then the socle of $\mathrm{Ext}_\mathcal{A}(F(E_{i+1}), F(E_{i+1}))$ as $\mathrm{End}_\mathcal{A}(F(E_{i+1}))$-module is simple. As previously stated, $F(e_{i+1})$ is a non-zero element of the above socle, and; therefore, $F(e_{i+1})$ is an almost split sequence. □

## 3 The categories $P(\Lambda)$ and $P^1(\Lambda)$

Let $\Lambda$ be a finite-dimensional algebra over an arbitrary field $k$. We denote by $\Lambda$-Proj the full subcategory of $\Lambda$-Mod whose objects are projective $\Lambda$-modules, and by $\Lambda$-proj, the full subcategory of $\Lambda$-mod whose objects are projective $\Lambda$-modules.

Here $\Lambda$-proj has only a finite number of isoclasses of indecomposable objects, then for any indecomposable projective $\Lambda$-module $P$ there are morphisms

$$\rho(P) : r(P) \rightarrow P, \quad \lambda(P) : P \rightarrow l(P)$$

such that they are a minimal right almost split in $\Lambda$-proj and a minimal left almost split in $\Lambda$-proj, respectively. Observe that $\rho(P)$ and $\lambda(P)$ are also a minimal right almost split and a minimal left almost split morphism, respectively, in the category $\Lambda$-Proj.

Denote by $P(\Lambda)$ the category whose objects are morphisms $X = f_X : P_X \rightarrow Q_X$, with $P_X, Q_X \in \Lambda$-Proj. The morphisms from $X$ to $Y$, objects of $P(\Lambda)$, are pairs $u = (u_1, u_2)$ with $u_1 : P_X \rightarrow P_Y$, $u_2 : Q_X \rightarrow Q_Y$ such that $u_2 f_X = f_Y u_1$. If $u = (u_1, u_2) : X \rightarrow Y$ and $v = (v_1, v_2) : Y \rightarrow Z$ are morphisms, its composition is defined by $vu = (v_1 u_1, v_2 u_2)$.

We denote by $\mathcal{E}$ the class of pairs of composable morphisms $X \overset{u}{\rightarrow} Y \overset{v}{\rightarrow} Z$ such that the sequences of $\Lambda$-modules:

$$0 \rightarrow P_X \overset{u_1}{\rightarrow} P_Y \overset{v_1}{\rightarrow} P_Z \rightarrow 0$$

$$0 \rightarrow Q_X \overset{u_2}{\rightarrow} Q_Y \overset{v_2}{\rightarrow} Q_Z \rightarrow 0$$

are exact and then split exact.

**Proposition 3.1.** *The pair $(P(\Lambda), \mathcal{E})$ is an exact category.*

**Proof.** See [1]. □

For $P$ any projective $\Lambda$-module consider $J(P) = (P \overset{id_P}{\rightarrow} P)$, $Z(P) = (P \overset{0}{\rightarrow} 0)$, $T(P) = (0 \overset{0}{\rightarrow} P)$. It is easy to see that the objects $J(P)$ and $T(P)$ are $\mathcal{E}$-projectives and the objects $J(P), Z(P)$ are $\mathcal{E}$-injectives. One can see without difficulty that the exact category $(P(\Lambda), \mathcal{E})$ has enough projectives and enough injectives.

**Proposition 3.2.** *The indecomposable $\mathcal{E}$-projectives in $P(\Lambda)$ are the objects $J(P)$ and $T(P)$ for $P$ indecomposable projective $\Lambda$-module.*

*The indecomposable $\mathcal{E}$-injectives in $P(\Lambda)$, are the objects $J(P)$ and $Z(P)$ for $P$ indecomposable projective $\Lambda$-module.*

We denote by $\overline{P(\Lambda)}$ the category having the same objects as $P(\Lambda)$ and morphisms those of $P(\Lambda)$ modulo the morphisms which factorizes through $\mathcal{E}$-injective objects.

We have a full and dense functor $Cok : P(\Lambda) \rightarrow \Lambda$-Mod which in objects is given by $Cok(f_X : P_X \rightarrow Q_X) = \mathrm{Coker} f_X$.

**Proposition 3.3.** *The functor $Cok : P(\Lambda) \rightarrow \Lambda$-Mod induces an equivalence $\overline{Cok} : \overline{P(\Lambda)} \rightarrow \Lambda$-Mod.*

**Proof.** One can prove (see [1] ) that if $f : X \rightarrow Y$ is a morphism in $P(\Lambda)$ then $Cok(f) = 0$ iff $f$ factorizes through some $\mathcal{E}$-injective object in $P(\Lambda)$. □

We consider now $p(\Lambda)$, the full subcategory of $P(\Lambda)$ whose objects are morphisms between finitely generated $\Lambda$-modules.

**Proposition 3.4.** *The exact category $(p(\Lambda), \mathcal{E})$ has almost split $\mathcal{E}$- sequences.*

**Proof.** See [1]. □

Now consider $P^1(\Lambda)$ the full subcategory of $P(\Lambda)$ whose objects are those $X = f_X : P_X \to Q_X$ with $\mathrm{Im}(f_X) \subset \mathrm{rad}(Q_X)$. We denote by $\mathcal{E}_1$ the class of composable morphisms in $P^1(\Lambda)$ which are in $\mathcal{E}$ . By $p^1(\Lambda)$ we denote the full subcategory of $P^1(\Lambda)$, whose objects are morphisms between finitely generated projective $\Lambda$-modules.

**Proposition 3.5.** *The pair $(P^1(\Lambda), \mathcal{E}_1)$ is an exact category.*

**Proof.** See [1]. □

For an indecomposable projective $\Lambda$-module $P$ denote by $R(P)$ the object $\rho(P) : r(P) \to P$ and by $L(P)$ the object $\lambda(P) : P \to l(P)$. Observe that $P$ a left $\Lambda$-module is in $\Lambda$-proj if $P$ is indecomposable and projective.

**Lemma 3.6.** *The morphism*

$$\sigma(P) = (\rho(P), id_P) : R(P) \to J(P)$$

*is a minimal right almost split morphism in $P(\Lambda)$, the morphism*

$$\tau(P) = (id_P, \lambda(P)) : J(P) \to L(P)$$

*is a minimal left almost split morphism in $P(\Lambda)$.*

**Proposition 3.7.** *Suppose $u : X \to Y$ is a morphism in $P^1(\Lambda)$ such that $Cok(u) = 0$, then $u = gh$ with $h : X \to W$, $g : W \to Y$ and $W$ a sum of objects of the form $Z(P)$ and $R(Q)$.*

**Proof.** It follows from Proposition 3.3 and Lemma 3.6. □

**Proposition 3.8.** *The indecomposable $\mathcal{E}_1$-projectives in $P^1(\Lambda)$ are the objects $T(P)$ and $L(P)$ with $P$ indecomposable projective $\Lambda$-module. The indecomposable $\mathcal{E}_1$-injectives are the objects $Z(P)$ and $R(P)$ with $P$ an indecomposable projective $\Lambda$-module.*

**Proof.** It follows from Proposition 3.2 and Lemma 3.6. □

**Proposition 3.9.** *For $X, Y \in P^1(\Lambda)$, there is an exact sequence*

$$0 \to \mathrm{Hom}_{P^1(\Lambda)}(X, Y) \xrightarrow{i} \mathrm{Hom}_\Lambda(P_X, P_Y) \oplus \mathrm{Hom}_\Lambda(Q_X, Q_Y)$$

$$\xrightarrow{\delta} \mathrm{rad}_\Lambda(P_X, Q_Y) \xrightarrow{\eta} \mathrm{Ext}_{P^1(\Lambda)}(X,Y) \to 0$$

**Proof.** See Proposition 5.1 of [1]. □

Now, if $X = (P_X \xrightarrow{f_X} Q_X) \in P(\Lambda)$ choose some minimal projective cover $P_2 \xrightarrow{g} P_1 \xrightarrow{\eta}$ $\mathrm{Ker} h \to 0$ with $h = D(\Lambda) \otimes f_X : D(\Lambda) \otimes_\Lambda P_X \to D(\Lambda) \otimes_\Lambda Q_X$. We put $\tau X = (P_2 \xrightarrow{g} P_1)$.

**Proposition 3.10.** *If $X$ is an indecomposable which is not $\mathcal{E}_1$-projective in $p^1(\Lambda)$, then there is an almost split $\mathcal{E}_1$-sequence:*

$$(1) \quad Y \to E \to X$$

*with $Y \cong \tau X$. Dually if $Y$ is indecomposable non $\mathcal{E}_1$-injective, then there is an almost split $\mathcal{E}_1$-sequence (1).*

**Proof.** See [10] for $k$ a perfect field and [1] for the general case. □

**Proposition 3.11.** *For $X, Y \in p^1(\Lambda)$, there is an isomorphism of $k$-modules*

$$\mathrm{Ext}_{P^1(\Lambda)}(X,Y) \cong D\overline{\mathrm{Hom}}_{P^1(\Lambda)}(Y, \tau(X)).$$

*Here $\overline{\mathrm{Hom}}_{p^1(\Lambda)}(Z, W)$ stands for the morphisms from $Z$ to $W$ modulo those morphisms which are factorized through $\mathcal{E}_1$-injectives objects.*

**Proof.** It follows from Corollary 9.4 of [9]. □

As a consequence we obtain:

**Proposition 3.12.** *(See [3] and [1]) For $X, Y \in p^1(\Lambda)$, there is an isomorphism of $k$-modules:*

$$\mathrm{Ext}_{P^1(\Lambda)}(X,Y) \cong D(\mathrm{Hom}_\Lambda(Cok(Y), DtrCok(X))/\mathcal{S}(Cok(Y), Dtr(Cok(X)))$$

*where $\mathcal{S}(M, N)$ are the morphisms which factorizes through semisimple $\Lambda$-modules.*

**Proposition 3.13.** *If $Y \xrightarrow{v} E \xrightarrow{u} X$ is an almost split sequence in $p(\Lambda)$ with $Cok(Y) \neq 0$ and $Cok(X) \neq 0$, then*

$$0 \to Cok(Y) \xrightarrow{Cok(v)} Cok(E) \xrightarrow{Cok(u)} Cok(X) \to 0$$

*is an almost split sequence in $\Lambda$-mod. Moreover, if $Cok(Y)$ is not a simple $\Lambda$-module, then the sequence $Y \xrightarrow{v} E \xrightarrow{u} X$ lies in $p^1(\Lambda)$.*

**Proof.** For the first part of our statement see Proposition 5.6 of [1], for the second part see Theorem 2.6 of [10] and Proposition 5.7 of [1]. □

Suppose now that $\Lambda$ is a basic finite-dimensional $k$-algebra, and $1_\Lambda = \sum_{i=1}^n e_i$ is a decomposition into pairwise orthogonal primitive idempotents. Moreover, assume that $\dim_k(\Lambda/\mathrm{rad}\Lambda)e_i = 1$ for all $i = 1, ..., n$. For $M \in \Lambda$-mod we put

$$\mathbf{dim}M = (\dim_k e_1 M, ..., \dim_k e_n M).$$

For $X = f_X : P_X \to Q_X$ an object in $p^1(\Lambda)$ we put

$$\mathbf{dim}X = (\mathbf{dim}(P_X/\mathrm{rad}P_X), \mathbf{dim}(Q_X/\mathrm{rad}Q_X)) \in \mathbb{Z}^{2n}.$$

In the following, we consider three bilinear forms defined on $\mathbb{Z}^{2n}$:
For $\mathbf{x} = (x_1, ..., x_n; x'_1, ..., x'_n), \mathbf{y} = (y_1, ..., y_n; y'_1, ..., y'_n)$, we put

$$h_\Lambda(\mathbf{x}, \mathbf{y}) = \sum_{i,j}(x_i y_j + x'_i y'_j)\dim_k(e_i \Lambda e_j) - \sum_{i,j} x_i y'_j \dim_k(e_i \mathrm{rad}\Lambda e_j),$$

$$s_\Lambda(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i y'_i, \quad g_\Lambda(x, y) = \sum_{i,j}(x_i y_j + x'_i y'_j - x_i y'_j)(\dim_k e_i \Lambda e_j).$$

Clearly $g_\Lambda(\mathbf{x}, \mathbf{y}) = h_\Lambda(\mathbf{x}, \mathbf{y}) - s_\Lambda(\mathbf{x}, \mathbf{y})$.

**Proposition 3.14.** *For $X, Y \in p^1(\Lambda)$ we have:*

$$(1)\dim_k \mathrm{Hom}_{p^1(\Lambda)}(X, Y) - \dim_k \mathrm{Ext}_{p^1(\Lambda)}(X, Y) = h_\Lambda(\boldsymbol{dim}X, \boldsymbol{dim}Y);$$

$$(2)\dim_k \mathrm{Ext}_{p^1(\Lambda)}(X, Y) = \dim_k \mathrm{Hom}_\Lambda(Cok(Y), DtrCok(X)) - s_\Lambda(\boldsymbol{dim}X, \boldsymbol{dim}Y);$$

$$(3)\dim_k \mathrm{Hom}_\Lambda(Cok(Y), DtrCok(X)) = \dim_k \mathrm{Hom}_{p^1(\Lambda)}(X, Y) - g_\Lambda(\boldsymbol{dim}X, \boldsymbol{dim}Y).$$

**Proof.** The part (1) follows from Proposition 3.9, part (2) follows from Proposition 3.12 and from the equalities:

$$\dim_k \mathcal{S}(Cok(Y), DtrCok(X)) = \dim_k \mathrm{Hom}_\Lambda(\mathrm{top}Cok(Y), \mathrm{soc}DtrCok(X))$$

$$= s_\Lambda(\mathbf{dim}X, \mathbf{dim}Y).$$

Finally, (3) follows from (1) and (2). □

# 4 Bocses

We recall that a coalgebra over a $k$-category $A$ is an $A$-bimodule $V$ endowed with two bimodule homomorphisms, a comultiplication $\mu : V \to V \otimes_A V$ and a counit $\epsilon : V \to A$, subject to the conditions

$$(\mu \otimes 1)\mu = (1 \otimes \mu)\mu$$

$$(\epsilon \otimes 1)\mu = i_l, \quad (1 \otimes \epsilon)\mu = i_r$$

with $i_l : V \cong A \otimes_A V$ and $i_r : V \cong V \otimes_A A$ the natural isomorphisms. Observe that $A$ is a coalgebra over $A$ with comultiplication $A \cong A \otimes_A A$ the natural isomorphism and the counit the identity morphism $id_A : A \to A$.

A bocs is a pair $\mathcal{A} = (A, V)$ with $A$ a skeletally small $k$-category and $V$ a coalgebra over $A$.

The bocs $(A, A)$ is called the principal bocs.

The category $\mathcal{A}$-Mod has the same objects as $A$-Mod, the covariant functors $A \to k$-Mod. Then, if $M, N$ are in $\mathcal{A}$-Mod, a morphism in $\mathcal{A}$-Mod is given by an $A$-module morphism from $V \otimes_A M$ to $N$. The composition of $f : V \otimes_A M \to N$ and $g : V \otimes_A N \to L$ is given by the composition

$$V \otimes_A M \stackrel{\mu\otimes 1}{\to} V \otimes_A V \otimes_A M \stackrel{1\otimes f}{\to} V \otimes_A N \stackrel{g}{\to} L,$$

the identity morphism for $M$ in $\mathcal{A}$-Mod is given by the composition:

$$V \otimes_A M \stackrel{\epsilon\otimes 1}{\to} A \otimes_A M \stackrel{\sigma}{\to} M,$$

where $\sigma$ is given by $\sigma(a \otimes m) = am$ for $a \in A$, $m \in M$. We identify $A$-Mod with $(A, A)$-Mod.

Suppose now $\mathcal{A} = (A, V)$ and $\mathcal{B} = (B, W)$ are two bocses, denote by $\epsilon_V, \mu_V, \epsilon_W$, $\mu_W$ the corresponding counits and comultiplications. A morphism of bocses $\theta : \mathcal{A} \to \mathcal{B}$ is a pair $(\theta_0, \theta_1)$ where $\theta_0 : A \to B$ is a functor and $\theta_1 : V \to {}_{\theta_0}W_{\theta_0}$ is a morphism of $A$-$A$ bimodules such that

$$\epsilon_W \theta_1 = \theta_0 \epsilon_V, \text{and} \quad \pi(\theta_1 \otimes \theta_1)\mu_V = \mu_W \theta_1,$$

where $\pi$ is the natural map $W \otimes_A W \to W \otimes_B W$. A morphism of bocses $\theta : \mathcal{A} \to \mathcal{B}$ induces a functor $\theta^* : \mathcal{B}$-Mod $\to \mathcal{A}$-Mod. For $M \in \mathcal{B}$-Mod we put $\theta^* M = {}_{\theta_0}M$ and if $f : W \otimes_B M \to N$ is a morphism in $\mathcal{B}$-Mod then $\theta^*(f)$ is the composition:

$$V \otimes_A ({}_{\theta_0}M) \stackrel{\theta_1\otimes 1}{\to} W \otimes_A ({}_{\theta_0}M) \stackrel{\pi}{\to} W \otimes_B M \stackrel{f}{\to} N$$

where $\pi$ is the natural morphism.

Observe that if

$$\mathcal{A} \stackrel{(\theta_0,\theta_1)}{\to} \mathcal{B} \stackrel{(\phi_0,\phi_1)}{\to} \mathcal{C}$$

are morphisms of bocses then $(\phi_0\theta_0, \phi_1\theta_1) = \phi\theta : \mathcal{A} \to \mathcal{C}$ is a morphism of bocses. Clearly $(\phi\theta)^* = (\theta)^*(\phi)^*$.

**Lemma 4.1.** *If $\theta = (\theta_0, \theta_1) : \mathcal{A} = (A, V) \to \mathcal{B} = (B, W)$ is a morphism of bocses then*

$$(\theta)^*(1, \epsilon_W)^* = (1, \epsilon_V)^*(\theta_0, \theta_0)^*.$$

**Proof.** It follows from the definition of morphism of bocses and the above. □

Let $\mathcal{A} = (A, V)$ be a bocs and $A'$ a subcategory of $A$ with the same objects as $A$. A morphism $\omega : A' \to {}_{A'}V_{A'}$ of $A'$-$A'$ bimodules is said to be a grouplike of $\mathcal{A}$ relative to

$A'$ if $(i,\omega):(A',A')\to\mathcal{A}$ is a morphism of bocses, where $i:A'\to A$ is the inclusion. If the induced functor $(i,\omega)^*:\mathcal{A}\text{-Mod}\to A'\text{-Mod}$ reflects isomorphisms we say that $\omega$ is a reflector. If $\omega:_{A'}A'_{A'}\to -_{A'}V_{A'}$ is a grouplike we have that $\omega$ is completely determined by the elements $\omega_X=\omega(id_X)$ for all $X\in \text{ind}A'$ such that $\mu(\omega_X)=\omega_X\otimes\omega_X$.

If $\mathcal{A}=(A,V)$ is a bocs $\overline{V}=\text{Ker}\epsilon$ is called the kernel of $\mathcal{A}$. Then there is the following exact sequence of $A$-$A$ bimodules:

$$0\to\overline{V}\xrightarrow{\sigma}V\xrightarrow{\epsilon}A\to 0$$

where $\sigma$ is the inclusion.

We recall that if $\omega:A'\to {}_{A'}V_{A'}$ is a grouplike, it determines two morphisms $\delta_1:{}_{A'}A_{A'}\to {}_{A'}\overline{V}_{A'}$ and $\delta_2:{}_{A'}\overline{V}_{A'}\to {}_{A'}\overline{V}\otimes_A\overline{V}_{A'}$, given for $a\in\text{Hom}_A(X,Y)$ and $v\in V(X,Y)$ by :

$$\delta_1(a)=a\omega_X-\omega_Y a,\quad \delta_2(v)=\mu(v)-\omega_Y\otimes v-v\otimes\omega_X.$$

Observe that $(id_A,\epsilon):\mathcal{A}\to(A,A)$ is a morphism of bocses. Therefore, it induces a functor $(id_A,\epsilon)^*:A\text{-Mod}\to\mathcal{A}\text{-Mod}$. For $M\in A\text{-Mod}$, $(id_A,\epsilon)^*(M)=M$, and for $h:M\to N$ a morphism of $A$-modules $(id_A,\epsilon)^*h:V\otimes{}_AM\to N$ is given by $(id_A,\epsilon)^*(h)(v\otimes m)=h(\epsilon(v)m)$ for $m\in M, v\in V$.

For $M\in\mathcal{A}\text{-Mod}$, $(i,\omega)^*(M)={}_{A'}M$ and if $f:V\otimes_A M\to N$ is a morphism in $\mathcal{A}$-Mod, $f^0=(i,\omega)^*f:{}_{A'}M\to{}_{A'}N$ is given by $f^0(m)=f(\omega_X\otimes m)$ for $m\in M(X)$.

Given $\mathcal{A}=(A,V)$ a bocs with a grouplike $\omega$ relative to some $A'$ subcategory of $A$, for any morphism, $f:V\otimes_A M\to N$ we have the morphisms $f^0=(i,\omega)^*f\in\text{Hom}_{A'}(M,N)$, $f^1=f(\sigma\otimes 1):\overline{V}\otimes_A M\to N$. The pair of morphisms $(f^0,f^1)$ satisfies the following property:

$$(A)\quad f^0(am)=af^0(m)+f^1(\delta_1(a)\otimes m).$$

Now, for any object $Y\in A$ we have :

$$(V\otimes_A M)(Y)=V(-,Y)\otimes_A M=\omega_Y\otimes M(Y)\oplus(\overline{V}\otimes_A M)(Y),$$

therefore, a pair of morphisms $(f^0,f^1)$ with

$$f^0\in\text{Hom}_{A'}(M,N)\quad\text{and}\quad f^1\in\text{Hom}_A(\overline{V}\otimes_A M,N)$$

which satisfies the condition $(A)$ determines a morphism of $A$-modules $f:V\otimes_A M\to N$. Thus, any morphism $f:V\otimes_A M\to N$ is completely determined by the pair $(f^0,f^1)$ satisfying property $(A)$. In the rest of the paper, we put $f=(f^0,f^1)$.

**Proposition 4.2.** *If $f=(f^0,f^1):M\to N$, $g=(g^0,g^1):N\to L$ are morphisms in $\mathcal{A}$-Mod then $gf=(g^0f^0,(gf)^1)$ with*

$$(gf)^1(v\otimes m)=g^1(v\otimes f^0(m))+g^0(f^1(v\otimes m))+\sum_i g^1(v_i^1\otimes f^1(v_i^2\otimes m)),$$

*where $v\in V, m\in M$ and $\delta_2(v)=\sum_i v_i^1\otimes v_i^2$.*

**Proof.** It follows from the fact that $(i,\omega)^*$ is a functor and from the definitions. □

Following [5], if $A$ is a $k$-category a morphism $a \in A(X,Y)$ is called indecomposable if both $X$ and $Y$ are indecomposable objects of $A$. Similarly, if $W$ is an $A$-$A$ bimodule an element of $W$ is an element $w \in W(X,Y)$ for some $X,Y$. In case both $X$ and $Y$ are indecomposable, $w$ will be called indecomposable. If $X$ and $Y$ are objects of $A$ , then we denote by $F_{X,Y}$ the $A$-$A$ bimodule given by

$$F_{X,Y} = \mathrm{Hom}_A(-,X) \otimes_k \mathrm{Hom}_A(Y,-).$$

We say that the $A$-$A$ bimodule $W$ is freely generated by the elements $w_i \in W(X_i,Y_i), i = 1,...,n$ if there is an isomorphism of $A$-$A$ bimodules

$$\psi : F_{X_1,Y_1} \oplus ... \oplus F_{X_n,Y_n} \longrightarrow W$$

such that $\psi(id_{X_i} \otimes id_{Y_i}) = w_i$, for $i = 1,...,n$.

Now, suppose that $A'$ has the same objects as $A$, and $T$ is an $A'$-$A'$ -subimodule of ${}_{A'}A_{A'}$, denote by $T^{\otimes n}$ the tensor product $T \otimes_{A'} T \otimes_{A'} ... \otimes_{A'} T$ of $n$ copies of $T$ and set $T^0 = A'$. Then the direct sum of $A'$-$A'$-bimodules:

$$T^{\otimes} = \bigoplus_{n=0}^{\infty} T^{\otimes n}$$

can be regarded as a category with the same objects as $A$ and product given by the natural isomorphisms $T^{\otimes n} \otimes_A T^{\otimes m} \to T^{\otimes m+n}$.

We recall from Definition 2.5 of [5] that if $A'$ has the same objects as $A$, we say that $A$ is freely generated over $A$ by morphisms $a_1,...,a_n$ in $A$ if the $a_i$ freely generate an $A'$-$A'$ subimodule $T$ of ${}_{A'}A_{A'}$ such that the functor $T^{\otimes} \to A$ induced by the inclusion of $A'$ and $T$ in $A$ is an isomorphism.

**Definition 4.3.** A $k$-category $A$ is called minimal if it is skeletal and is equivalent to

$$\mathrm{mod}(k) \times ... \times \mathrm{mod}(k) \times P(R_1) \times ... \times P(R_n)$$

where $R_i = k[x, f_i(x)^{-1}]$ with $f_i(x)$ is a nonzero element of $k[x]$ and $P(R)$ denotes the category of finitely generated projective left $R$-modules. We denote by ind$A$ the set of indecomposable objects of a minimal category $A$.

**Definition 4.4.** Let $\mathcal{A} = (A,V)$ be a bocs with kernel $\overline{V}$. A collection $L = (A';\omega;a_1,...,a_n;v_1,...v_m)$, is a layer for $\mathcal{A}$, if

(L1) $A'$ is a minimal category;

(L2) $A$ is freely generated over $A'$ by indecomposable elements $a_1,...,a_n$;

(L3) $\omega$ is a reflector for $\mathcal{A}$ relative to $A'$;

(L4) $\overline{V}$ is freely generated as an $A$-$A$ bimodule by indecomposable elements $v_1,...v_m$;

(L5) let $\delta_1 : A \to \overline{V}$ be the morphism induced by $\omega$, $A_0 = A'$ and for $i \in \{1,...,n-1\}$, $A_i$ the subcategory of $A$ generated by $A'$ and $a_1,...a_i$, then for any $0 \leq i < n$, $\delta_1(a_{i+1})$ is contained in the $A_i$-$A_i$ subimodule of $\overline{V}$ generated by $v_1,...v_m$.

A bocs having a layer will be called layered.

Suppose $\mathcal{A} = (A, V)$ is a bocs with layer $L = (A'; \omega; a_1, ..., a_n; v_1, ...v_m)$. Throughout this paper, we denote by $\mathcal{A}$-mod the full subcategory of $\mathcal{A}$-Mod whose objects are representations $M$ such that $\sum_{X \in \text{ind}A'} \dim_k M(X) < \infty$.

For $\mathcal{A}$ as before we have

$$\overline{V} \otimes_A M \cong \bigoplus_{v_i} A(-, Y_i) \otimes_k M(X_i)$$

for $M \in A$-Mod. Thus, for $M, N \in A$-Mod we have an isomorphism:

$$\phi_{M,N} : \bigoplus_{v_i} \text{Hom}_k(M(X_i), N(Y_i)) \to \text{Hom}_A(\overline{V} \otimes_A M, N).$$

Therefore, in this case a morphism $f : M \to N$ in $\mathcal{A}$-Mod is given by a pair of morphisms

$$(f^0, \phi_{M,N}(f_1^1, ..., f_m^1)), f^0 \in \text{Hom}_{A'}(M, N), f_i^1 \in \text{Hom}_k(M(X_i), N(Y_i)),$$

$i = 1, ..., m$ such that for all $a_j : X_j \to Y_j, j = 1, ..., n$ and $u \in M(X_j)$

$$f_{Y_j}^0(a_j u) = a_j f_{X_j}^0(u) + \phi_{M,N}(f_1^1, ..., f_m^1)(\delta_1(a_j) \otimes u).$$

Observe that $\phi_{M,N}(f_1^1, ..., f_m^1)(v_i \otimes u) = f_i^1(u)$ for $u \in M(X_i), i = 1, ..., m$.

**Lemma 4.5.** *With the above notations, if* $(f, 0) : M \to N$ *and* $(h^0, \phi_{N,L}(h_1, ..., h_m)) : N \to L$ *are morphisms in* $\mathcal{A}$-Mod *then:*

$$(h^0, \phi_{N,L}(h_1, ..., h_m))(f, 0) = (h^0 f, \phi_{M,L}(g_1, ..., g_m)) \quad \text{with} \quad g_i = h_i f_{X_i}.$$

*Similarly, if* $(h^0, \phi_{M,N}(h_1, ..., h_m)) : M \to N$, $(f, 0) : N \to L$ *are morphisms in* $\mathcal{A}$-Mod*, then:*

$$(f, 0)(h^0, \phi_{M,N}(h_1, ..., h_m)) = (f h^0, \phi_{M,N}(g_1, ..., g_m)), \quad \text{with} \quad g_i = f_{Y_i} h_i.$$

In later sections we need the following.

**Definition 4.6.** Let $\mathcal{A} = (A, V)$ be a bocs with layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$. Then a sequence of morphisms in $\mathcal{A}$-Mod,

$$M \xrightarrow{f} E \xrightarrow{g} N$$

is called proper exact if $gf = 0$ and the sequence of morphisms

$$0 \to M \overset{(i,\omega)^* f}{\to} E \overset{(i,\omega)^* g}{\to} N \to 0$$

in $A'$-Mod is exact. An almost split sequence in $\mathcal{A}$-mod which is also a proper exact sequence is called a proper almost split sequence.

**Definition 4.7.** With the notation of Definition 4.6 an indecomposable object $X \in A'$ is called marked if $A'(X, X) \neq k id_X$.

## 5 Hom-spaces of Minimal Bocses

We recall from [5] that a minimal bocs is a bocs $\mathcal{A} = (A, V)$ with layer

$$L = (A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$$

such that $A' = A$. Therefore in this case the $a_1, ..., a_n$ do not appear.

Throughout this section, $\mathcal{B} = (B, W)$ is a minimal bocs with layer

$$L = (B; \omega; w_1, ..., w_m), \quad \text{where} \quad w_i \in \overline{W}(X_i, Y_i).$$

For $M, N \in \mathcal{B}$-Mod we put $\mathrm{Hom}_{\mathcal{B}}(M, N)^1 = \{f : M \to N | (1, \omega)^*(f) = 0\}$.

**Proposition 5.1.** *Let $\mathcal{B} = (B, W)$ be a minimal bocs and $\epsilon : W \to B$ the counit of $W$. Then for $M, N \in \mathcal{B}$-Mod we have*

$$\mathrm{Hom}_{\mathcal{B}}(M, N) = (1, \epsilon)^*(\mathrm{Hom}_B(M, N)) \oplus \mathrm{Hom}_{\mathcal{B}}(M, N)^1.$$

**Proof.** We have $(1, \epsilon)^*(1, \omega)^* \cong id_{B\text{-Mod}}$. □

Observe that if we have any pair of morphisms $(f, \phi_{M,N}(h_1, ..., h_m))$ with $f \in \mathrm{Hom}_B(M, N)$, $h_i \in \mathrm{Hom}_k(M(X_i), N(Y_i))$ where $w_i : X_i \to Y_i$, this pair is a morphism from $M$ to $N$ in $\mathcal{B}$-Mod, because in a minimal bocs $\delta_1 = 0$ and condition (A) before Proposition 4.2 is trivially satisfied. Then we have:

**Corollary 5.2.** *For $M, N \in \mathcal{B}$-mod :*

$$\dim_k \mathrm{Hom}^1_{\mathcal{B}}(M, N) = \sum_{w_i} \dim_k \mathrm{Hom}_k(M(X_i), N(Y_i)).$$

The morphisms in the image of $(1, \epsilon)^*$ have the form $(f, 0)$ where the morphism $f$ is in $\mathrm{Hom}_B(M, N)$.

**Lemma 5.3.** *(Compare Definition 3.8 in [5]) Let $M, N$ be two objects in $\mathcal{B}$-Mod, then $M \cong N$ in $\mathcal{B}$-Mod iff $M \cong N$ in $B$-Mod.*

**Proof.** If $h : M \to N$ is an isomorphism in $\mathcal{B}$-Mod then $(1, \omega)^*(h)$ is an isomorphism in $B$-Mod. Conversely, if $g : M \to N$ is an isomorphism in $B$-Mod then $(1, \epsilon)^*(g)$ is an isomorphism in $\mathcal{B}$-Mod. □

Clearly, Lemma 5.3 implies that indecomposable objects in $B$-Mod and $\mathcal{B}$-Mod coincide.

We have $B(Z, Z') = 0$ for $Z \neq Z' \in \mathrm{ind}B$ and for $Z \in \mathrm{ind}B$, $B(Z, Z) = R_Z = k[x, h(x)^{-1}]id_Z$ with $h(x) \in k[x]$ or $B(Z, Z) = kid_Z$. Take $M$ an indecomposable object in $B$-mod, then there is only one $Z \in \mathrm{ind}B$ such that $M(Z) \neq 0$. Here $M$ is a covariant functor of $B$ into $k$-Mod, $M(Z)$ is a left $R_Z$- module. Therefore if $B(Z,Z) = R_Z \neq kid_Z$, $M(Z) \cong R_Z/(p^n)$ with $p = x - \lambda$ a prime element in $R_Z$, if $B(Z,Z) = kid_Z$, $M(Z) = k$.

For $Z \in \text{ind}B$ with $B(Z,Z) = R_Z \neq kid_Z$ and $p = x - \lambda$, a prime element in $R_Z$ we define $M(Z,p,n) \in B$-Mod by

$$M(Z,p,n)(W) = 0 \quad \text{for} \quad W \neq Z, W \in \text{ind}B, \quad M(Z,p,n)(Z) = R_Z/(p^n).$$

If $B(Z,Z) = kid_Z$ we define $S_Z \in B$-mod by

$$S_Z(W) = 0 \quad \text{for} \quad W \neq Z, W \in \text{ind}B, \quad S_Z(Z) = k.$$

**Lemma 5.4.** *If $M$ is an indecomposable object in $B$-mod then $M \cong M(Z,p,n)$ or $M \cong S_Z$ for some $Z \in \text{ind}B$.*

**Lemma 5.5.** *Let $(f,0) : M \to N$ be a morphism in $\mathcal{B}$-Mod such that for all $Z \in \text{ind}B$, $f_Z : M(Z) \to N(Z)$ is surjective. Then if $h : L \to N$ is a morphism in $\mathcal{B}$-Mod with $(1,\omega)^*(h) = 0$, there is a morphism $g : L \to M$ in $\mathcal{B}$-Mod with $(f,0)g = h$.*

**Proof.** Take $h : L \to N$ with $(1,\omega)^*(h) = 0$, then $h = (0, \phi_{L,N}(h_1, ..., h_m))$. We may assume that there is a $j$ with $0 \neq h_j \in \text{Hom}_k(M(X_j), N(Y_j))$ and $h_i = 0$ for $i \neq j$.

We have that $f_{Y_j} : M(Y_j) \to N(Y_j)$ is an epimorphism. Consequently, there is a $k$-linear map $\sigma : N(Y_j) \to M(Y_j)$ with $f_{Y_j}\sigma = id_{N(Y_j)}$. Take now $g_j = \sigma h_j \in \text{Hom}_k(L(X_j), M(Y_j))$, and $0 = g_i \in \text{Hom}_k(L(X_i), M(Y_i))$, for $i \neq j$. Take now the morphism

$$g = (0, \phi_{L,M}(g_1, ..., g_m)) : L \to M$$

then by Lemma 4.5 $(f,0)g = (0, \phi_{L,N}(\lambda_1, ..., \lambda_m))$ with $\lambda_i = f_{Y_i}g_i$. Therefore, $\lambda_i = 0$ for $i \neq j$ and $\lambda_j = f_{Y_j}g_j = f_{Y_j}\sigma h_j = h_j$. Consequently, $(f,0)g = (0, \phi_{L,N}(\lambda_1, ..., \lambda_m)) = (0, \phi_{L,N}(h_1, ..., h_m)) = h$. □

Similarly, we have the dual version of the above result.

**Lemma 5.6.** *Let $(f,0) : M \to N$ be a morphism in $\mathcal{B}$-Mod such that for all $Z \in \text{ind}B$, $f_Z : M(Z) \to N(Z)$ is an injection. Then if $u : M \to L$ is a morphism with $(1,\omega)^*(u) = 0$ there is a morphism $v : N \to L$ with $v(f,0) = u$.*

For $Z, Z' \in \text{ind}B$ we denote by $t(Z,Z')$ the number of $w_i \in \overline{W}(Z,Z')$.

**Lemma 5.7.** *Suppose $M, N$ are indecomposable objects in $\mathcal{B}$-mod with $M(Z) \neq 0, N(Z') \neq 0$, $Z, Z' \in \text{ind}B$. Then*

$$\dim_k \text{Hom}_{\mathcal{B}}(M,N)^1 = t(Z,Z')\dim_k M(Z)\dim_k N(Z').$$

**Proof.** It follows from Corollary 5.2. □

**Lemma 5.8.** *If $M, N$ are indecomposable objects in $\mathcal{B}$-mod, then*

$$\mathrm{rad}^{\infty}_{\mathcal{B}}(M,N) \subset \mathrm{Hom}_{\mathcal{B}}(M,N)^1.$$

**Proof.** Suppose there is a $h \in \mathrm{rad}^{\infty}_{\mathcal{B}}(M,N)$ with $(1,\omega)^*(h) \neq 0$. Then there is a $Z \in \mathrm{ind}B$ with $M(Z) \neq 0, N(Z) \neq 0$. Since $(1,\omega)^*$ reflects isomorphisms, then $(1,\omega)^*(h)$ is not an isomorphism. Consequently, $B(Z,Z) = R_Z \neq kid_Z$ and $M \cong M(Z,p,m), N \cong M(Z,p,n)$.

Here $\mathrm{rad}^{\infty}_{B}(M,N) \cong \mathrm{rad}^{\infty}_{R_Z}(R_Z/(p^m), R_Z/(p^n)) = 0$. Then there is a $s$ with $\mathrm{rad}^{s}_{B}(M,N) = 0$.

On the other hand, there is a chain of non-isomorphisms between indecomposables:

$$M \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \rightarrow \ldots \rightarrow X_{s-1} \xrightarrow{f_s} N$$

with $g = (1,\omega)^*(f_s \cdots f_2 f_1) \neq 0$.

But $g = (1,\omega)^*(f_s) \cdots (1,\omega)^*(f_1) \in \mathrm{rad}^{s}_{B}(M,N) = 0$, a contradiction. This proves our claim. □

Consider $M = M(Z,p,m), N = M(Z,p,n)$ indecomposables in $B$-mod. If $f : R_Z/(p^m) \rightarrow R_Z/(p^n)$ is a morphism of $R_Z$-modules, we put $u(f) : M \rightarrow N$ given by $u(f)_Z = f$ and $u(f)_W = 0$ for $W \neq Z$.

**Proposition 5.9.** *Let $M, N$ be indecomposables in $\mathcal{B}$-mod with $M(Z) \neq 0$ or $N(Z) \neq 0$ for some $Z \in \mathrm{ind}B$ with $B(Z,Z) \neq kid_Z$, then*

$$\mathrm{rad}^{\infty}_{\mathcal{B}}(M,N) = \mathrm{Hom}_{\mathcal{B}}(M,N)^1.$$

**Proof.** By Lemma 5.8, it is enough to prove that if $f : M \rightarrow N$ is a morphism in $\mathcal{B}$-mod with $(1,\omega)^*(f) = 0$ then $f \in \mathrm{rad}^{\infty}_{\mathcal{B}}(M,N)$. Suppose $M(Z) \neq 0$ with $B(Z,Z) = R_Z \neq id_Z k$. Then we may assume $M = M(Z,p,m)$. Take any natural number $n$. Consider the monomorphism $i_l : R_Z/(p^l) \rightarrow R_Z/(p^{l+1})$ given by $i_l(\eta_l(a)) = \eta_{l+1}(pa)$ for $a \in R_Z$ and $\eta_j : R_Z \rightarrow R_Z/(p^j)$ the quotient map. Take $(u,0) = (u(i_{n+m-1}),0)...(u(i_{m+1}),0)(u(i_m),0) : M(Z,p,m) \rightarrow M(Z,p,m+n)$. Here $u_Z : M(Z,p,m)(Z) \rightarrow M(Z,p,m+n)(Z)$ is a monomorphism . By Lemma 5.6, there is a morphism $t : M(Z,p,m+n) \rightarrow N$ in $\mathcal{B}$-Mod such that $t(u,0) = f$.

Now, $(u,0) \in \mathrm{rad}^{n}_{\mathcal{B}}(M, M(Z,p,m+n))$, and, therefore, $f = t(u,0) \in \mathrm{rad}^{n}_{\mathcal{B}}(M,N)$ for all $n$, then $f \in \mathrm{rad}^{\infty}_{\mathcal{B}}(M,N)$.

For the case in which $N(Z) \neq 0$ with $B(Z,Z) \neq kid_Z$ one proceeds in a similar way. □

**Corollary 5.10.** *If $M, N$ are indecomposable objects in $\mathcal{B}$-mod, and $Z, Z' \in \mathrm{ind}B$ with $M(Z) \neq 0$, $N(Z') \neq 0$, and $B(Z,Z) \neq kid_Z$ or $B(Z',Z') \neq kid_{Z'}$, then*

$$\dim_k \mathrm{rad}^{\infty}_{\mathcal{B}}(M,N) = \dim_k M(Z) \dim_k N(Z') t(Z,Z').$$

**Corollary 5.11.** *Let $M = M(Z, p, m), N = M(Z', q, n), S = S_W$ be indecomposables in $\mathcal{B}$-mod, with $B(Z, Z) \neq kid_Z$, $B(Z', Z') \neq kid_{Z'}$, $B(W, W) = kid_W$. Then if $Z = Z'$, $p = q$,*

$$\mathrm{Hom}_{\mathcal{B}}(M, N) \cong \mathrm{Hom}_B(M, N) \oplus \mathrm{rad}^{\infty}_{\mathcal{B}}(M, N),$$

*with $\dim_k(\mathrm{Hom}_B(M, N)) = min\{m, n\}$.*
*And if $Z \neq Z'$ or $Z = Z'$, and $(p) \neq (q)$*

$$\mathrm{Hom}_{\mathcal{B}}(M, N) = \mathrm{rad}^{\infty}_{\mathcal{B}}(M, N).$$

*Moreover,*

$$\mathrm{Hom}_{\mathcal{B}}(M, S) = \mathrm{rad}^{\infty}_{\mathcal{B}}(M, S) \quad \textit{and} \quad \mathrm{Hom}_{\mathcal{B}}(S, M) = \mathrm{rad}^{\infty}_{\mathcal{B}}(S, M).$$

**Lemma 5.12.** *If $0 \to M \xrightarrow{f^0} E \xrightarrow{g^0} N \to 0$ is a short exact sequence in $B$-Mod, then the pair of morphisms in $\mathcal{B}$-Mod, $M \xrightarrow{(f^0,0)} E \xrightarrow{(g^0,0)} N$ is an exact pair of morphisms.*

**Proof.** We claim that $f = (f^0, 0)$ is a kernel of $(g^0, 0)$. Assume there is a morphism $u = (u^0, u^1) = (u^0, 0) + (0, u^1) : L \to E$ such that $gu = (g^0u^0, (gu)^1) = 0$. Here $g^0u^0 = 0$, then there is a unique morphism in $B$-Mod, $v^0 : L \to M$ with $f^0v^0 = u^0$. Now, $u^1 = \phi_{L,E}(u_1, ..., u_m)$, with $u_i : L(X_i) \to E(Y_i)$ where $w_i \in \overline{W}(X_i, Y_i)$. Then $(gu)^1 = \phi_{L,N}(g^0_{Y_1}u_1, ..., g^0_{Y_m}u_m)$. Therefore, for $i = 1, ..., m$, $g^0_{Y_i}u_i = 0$. Thus, there are linear maps $v_i : L(X_i) \to M(Y_i)$ with $f^0_{Y_i}v_i = u_i$ for $i = 1, ..., m$. Then taking $v = (v^0, \phi_{L,M}(v_1, ..., v_m))$ we have $fv = u$. Clearly $v$ is unique with this property. This proves our claim. In a similar way one can prove that $g$ is a cokernel of $f$. □

**Lemma 5.13.** *Suppose $(a) : M \xrightarrow{f} E \xrightarrow{g} N$ is a proper exact sequence in $\mathcal{B}$-Mod. Then $(a)$ is isomorphic to the sequence: $M \xrightarrow{(f^0,0)} E \xrightarrow{(g^0,0)} N$.*

**Proof.** By Lemma 5.5 and its proof, there is a morphism $u = (0, u^1) : E \to E$ such that $(g^0, 0)u = (0, g^1)$. Then $(g^0, 0)(1_E, u^1) = g$, with $\sigma = (1_E, u^1)$ an isomorphism. Thus, $(g^0, 0)\sigma f = gf = 0$. But by the above Lemma, $(f^0, 0)$ is a kernel of $(g^0, 0)$, then there is a morphism $\lambda = (\lambda^0, \lambda^1) : M \to M$ with $(f^0, 0)\lambda = \sigma f$. Here $f^0\lambda^0 = f^0$, since $f^0$ is a monomorphism then $\lambda^0 = 1_M$. Therefore, $\lambda : M \to M$ is an isomorphism. This proves our claim. □

From Lemma 5.12 and Lemma 5.13, we deduce that proper exact sequences are exact pairs of morphisms. Denote by $\mathcal{E}_p$ the class of proper exact sequences in $\mathcal{B}$-Mod, then we have the following.

**Proposition 5.14.** *The pair $(\mathcal{B}\text{-Mod}, \mathcal{E}_p)$ is an exact category.*

**Proof.** Observe first that $g = (g^0, g^1) : E \to M$ is a deflation if and only if $g^0$ is an epimorphism. In fact, if $g$ is a deflation, by definition of proper exact sequence $g^0$ is an epimorphism. Conversely, suppose $g^0$ is an epimorphism, then as in the proof of Lemma 5.5 there is an isomorphism $\tau : E \to E$ such that $(g^0, 0) = g\tau$. Taking $f^0 : N \to E$ the kernel of $g^0$ in $B$-Mod, we see that $(g^0, 0)$ is a deflation, thus $g$ is a deflation too. Similarly, one can prove that $f : N \to E$ is an inflation if and only if $f^0$ is a monomorphism. From this, it is clear that conditions E.1, E.3 and E.$3^{op}$ hold. For proving E.2, assume $g : E \to N$ is a deflation and $h : L \to N$ is an arbitrary morphism. Then we have the morphism $(g, h) : E \oplus L \to N$. Now, $(g, h) = ((g^0, h^0), (g^1, h^1))$, here $g^0$ is an epimorphism, then $(g^0, h^0)$ is also an epimorphism, thus $(g, h)$ is a deflation, therefore it has a kernel, $M \xrightarrow{u} E \oplus L$. Take $u_1 : M \to E$ equal to $u$ composed with the projection on $E$ and $-u_2 : M \to L$, the composition of $u$ with the projection on $L$. Now, one can see that $u_2$ is a deflation and $gu_1 = hu_2$. Therefore, E.2 holds. □

Let $Z_1, ..., Z_s$ be all marked objects in ind$B$. For $i = 1, ..., s$ take $R_i = B(Z_i, Z_i)$ and the $B$-$R_i$-bimodule $B_i = B(Z_i, -)$. Then if $p$ is a prime element of $R_i$ and $n$ a positive integer, $M(Z_i, p, n) \cong B_i \otimes_{R_i} R_i/(p^n)$. We denote by $S^i_{p,n}$ the exact sequence in $R_i$-mod:

$$0 \to R_i/(p^n) \overset{(p,\pi)}{\to} ((R_i/(p^{n+1}) \oplus R_i/(p^{n-1})) \overset{\begin{pmatrix}\pi \\ -p\end{pmatrix}}{\to} R/(p^n) \to 0.$$

**Proposition 5.15.** *The sequence $B_i \otimes_{R_i} S^i_{p,n}$:*

$$B_i \otimes_{R_i} R_i/(p^n) \overset{id\otimes(p,\pi)}{\to} B_i \otimes_{R_i} ((R_i/(p^{n+1}) \oplus R_i/(p^{n-1}))$$

$$\overset{id\otimes\begin{pmatrix}\pi \\ -p\end{pmatrix}}{\to} B_i \otimes_{R_i} R_i/(p^n)$$

*is a proper almost split sequence in $\mathcal{B}$-*mod*.*

**Proof.** The sequence $S^i_{p,n}$ is an almost split sequence in $R_i$-mod. Now, using Lemma 5.5 and Lemma 5.6 one can prove that $B_i \otimes_{R_i} S^i_{p,n}$ is a proper almost split sequence. □

## 6 Hom-spaces between $\mathcal{A}$-$k(x)$-bimodules

Let $\mathcal{A} = (A, V)$ be a bocs with layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$. We recall from [6] that an $\mathcal{A}$-$k(x)$-bimodule is an object $M \in \mathcal{A}$-Mod with a morphism $\alpha_M : k(x) \to \mathrm{End}_{\mathcal{A}}(M)$. If $M$ and $N$ are $\mathcal{A}$-$k(x)$-bimodules, a morphism $f : M \to N$ in $\mathcal{A}$-Mod is a morphism of $\mathcal{A}$-$k(x)$-bimodules if for all $q \in k(x)$, $f\alpha_M(q) = \alpha_N(q)f$.

We denote by $\mathcal{A}$-$k(x)$-Mod the category whose objects are the $\mathcal{A}$-$k(x)$-bimodules and the morphisms are morphisms of $\mathcal{A}$-$k(x)$-bimodules. If $F : \mathcal{B}$-Mod $\to \mathcal{A}$-Mod is a functor with $\mathcal{A}, \mathcal{B}$ layered bocses, then $F$ induces a functor $F^{k(x)} : \mathcal{B}$-$k(x)$-Mod $\to \mathcal{A}$-$k(x)$-Mod. If

$M$ is a $\mathcal{B}$-$k(x)$-bimodule, with $\alpha_M : k(x) \to \mathrm{End}_{\mathcal{B}}(M)$ then $F(M)$ is an $\mathcal{A}$-$k(x)$-bimodule with $\alpha_{F(M)} = F\alpha_M : k(x) \to \mathrm{End}_{\mathcal{A}}(F(M))$. Observe that if $f : M \to N$ is a morphism of $\mathcal{B}$-$k(x)$-bimodules, then $F(f)$ is a morphism of $\mathcal{A}$-$k(x)$-bimodules. Now, if $F$ is full and faithful then $F(f) : F(M) \to F(N)$ is a morphism of $\mathcal{A}$-$k(x)$-bimodules if and only if for all $q \in k(x)$ , $F(f)F(\alpha_M(q)) = F(\alpha_N(q))F(f)$ and this is true if and only if $f\alpha_M(q) = \alpha_N(q)f$ for all $q \in k(x)$. Thus, $F$ induces a full and faithful functor

$$F^{k(x)} : \mathcal{B}\text{-}k(x)\text{-Mod} \to \mathcal{A}\text{-}k(x)\text{-Mod}.$$

The $\mathcal{A}$-$k(x)$-bimodule $M$ is called proper if there is a $\beta_M : k(x) \to \mathrm{End}_A(M)$ such that $\alpha_M = (1, \epsilon)^*\beta_M$, thus $\alpha_M(q) = (\beta_M(q), 0)$ for all $q \in k(x)$. Observe that if $M$ is a proper $\mathcal{A}$-$k(x)$-bimodule then $M$ is an $A$-$k(x)$-bimodule. We denote by $\mathcal{A}$-$k(x)$-Mod$^p$, the full subcategory of $\mathcal{A}$-$k(x)$-Mod whose objects are the proper bimodules. Suppose $\theta : \mathcal{A} \to \mathcal{B}$ is a morphism of bocses with $\epsilon_{\mathcal{B}}$ the counit of $\mathcal{B}$ and $\epsilon_{\mathcal{A}}$ the counit of $\mathcal{A}$ , then $\theta^* : \mathcal{B}$-Mod $\to \mathcal{A}$-Mod is a full and faithful functor. Observe that if $M$ is a proper $\mathcal{B}$-$k(x)$-bimodule then $\alpha_M = (1, \epsilon_{\mathcal{B}})^*\beta_M$ with $\beta_M : k(x) \to \mathrm{End}_B(M)$. Then $\theta^*(M)$ is a $\mathcal{A}$-$k(x)$-bimodule, using Lemma 4.1 we have

$$\alpha_{\theta^*(M)} = (\theta_0, \theta_1)^*(1, \epsilon_{\mathcal{B}})^*\beta_M = (1, \epsilon_{\mathcal{A}})^*(\theta_0, \theta_0)^*\beta_M,$$

thus $\theta^*(M)$ is a proper $\mathcal{B}$-$k(x)$-bimodule, consequently $\theta^*$ induces a full and faithful functor $(\theta^*)^{k(x)} : \mathcal{B}$-$k(x)$-Mod$^p \to \mathcal{A}$-$k(x)$-Mod$^p$.

**Proposition 6.1.** *Let $M, N$ be proper $\mathcal{A}$-$k(x)$-bimodules. Then $f = (f^0, \phi_{M,N}(f_1, ..., f_m)) : M \to N$ is a morphism of $\mathcal{A}$-$k(x)$-bimodules if and only if $f^0$ is a morphism of $A'$-$k(x)$-bimodules and $f_i \in \mathrm{Hom}_{k(x)}(M(X_i), N(Y_i))$ for all $v_i \in \overline{V}(X_i, Y_i)$.*

**Proof.** We have that $M$ and $N$ are proper bimodules so, $\alpha_M(q) = (\beta_M(q), 0)$ and $\alpha_N(q) = (\beta_N(q), 0)$ with morphisms of $k$-algebras $\beta_M : k(x) \to \mathrm{End}_A(M)$ and $\beta_N : k(x) \to \mathrm{End}_A(N)$. Then a morphism $f : M \to N$ in $\mathcal{A}$-Mod is a morphism of $\mathcal{A}$-$k(x)$-bimodules if and only if $f\alpha_M(q) = \alpha_N(q)f$ for all $q \in k(x)$. Then, by Proposition 4.2, the above holds if and only if $f^0\beta_M(q) = \beta_N(q)f^0$ for all $q \in k(x)$, and for all $v_i$ and all $q \in k(x)$, $u \in M(X_i)$: $\beta_N(q)\phi_{M,N}(f_1, ..., f_m)(v_i \otimes u) = \phi_{M,N}(f_1, ..., f_m)(v_i \otimes \beta_M(q)(u))$. Using the relations given in Lemma 4.5, we obtain that the latter equality is equivalent to $\beta_N(q)f_i(u) = f_i(\beta_M(q)(u))$. From here we obtain our result. □

**Corollary 6.2.** *Let $\mathcal{B} = (B, W)$ be a minimal bocs with layer $(B; \omega_B; w_1, ..., w_m)$, with $w_i \in \overline{W}(X_i, Y_i)$. Then if $M$ and $N$ are proper $\mathcal{B}$-$k(x)$-bimodules we have:*

$$\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(M, N) \cong \mathrm{Hom}_{B\text{-}k(x)}(M, N) \oplus \bigoplus_i \mathrm{Hom}_{k(x)}(M(X_i), N(Y_i)).$$

Let $\mathcal{B} = (B, W)$ be a minimal bocs with layer $(B; \omega; w_1, ..., w_m)$, for $Z$ a marked object in ind$B$ we define $Q_Z \in \mathcal{B}$-Mod as follows: $Q_Z(Z) = k(x)$ where $B(Z, Z) = k[x, f(x)^{-1}]id_Z$ and the action of $x$ on $Q_Z(Z)$ is the multiplication by $x$, $Q_Z(W) = 0$ for

$Z \neq W$. The action of $k(x)$ is the multiplication on the right by the elements of $k(x)$. Here $Q_Z$ is a proper $\mathcal{B}$-$k(x)$-bimodule. Using the notation of section 5, we have as a consequence of the above corollary:

**Corollary 6.3.** *If $Z, Z'$ are marked objects and $W$ is a non-marked object in* ind$B$*, write $S_W^{k(x)} = S_W \otimes_k k(x)$. We have:*

$$\dim_{k(x)}\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(Q_Z, Q_{Z'}) = \delta(Z, Z') + t(Z, Z')$$

*where $\delta(Z, Z') = 1$ if $Z = Z'$ and zero otherwise. Moreover*

$$\dim_{k(x)}(\mathrm{rad}_{\mathcal{B}\text{-}k(x)}(Q_Z, S_W^{k(x)})) = t(Z, W),$$

$$\dim_{k(x)}(\mathrm{rad}_{\mathcal{B}\text{-}k(x)}(S_W^{k(x)}, Q_Z)) = t(W, Z).$$

**Corollary 6.4.** *With the notations in Corollary 6.3 we have :*

$$\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(Q_Z, Q_{Z'}) = k(x) \oplus \mathrm{rad}_{\mathcal{B}\text{-}k(x)}(Q_Z, Q_{Z'}) \quad \textit{when} \quad Z = Z',$$

$$\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(Q_Z, Q_{Z'}) = \mathrm{rad}_{\mathcal{B}\text{-}k(x)}(Q_Z, Q_{Z'}) \quad \textit{when} \quad Z \neq Z'.$$

*Moreover:*

$$\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(Q_Z, S_W^{k(x)}) = \mathrm{rad}_{\mathcal{B}\text{-}k(x)}(Q_Z, S_W^{k(x)}),$$
$$\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(S_W^{k(x)}, Q_Z) = \mathrm{rad}_{\mathcal{B}\text{-}k(x)}(S_W^{k(x)}, Q_Z).$$

From the above corollaries, we obtain the next proposition.

**Proposition 6.5.** *Let $\mathcal{B} = (B, W)$ be a minimal bocs with layer $(B; \omega; w_1, ..., w_m)$. Suppose $Z$, $Z'$, and $W$ are objects in* ind $B$ *with $B(W, W) = id_W k$, $B(Z, Z) \neq id_Z k$, $B(Z', Z') \neq id_{Z'} k$. Take $M = M(Z, p, m)$, $N = M(Z', q, n)$, $L = S_W$ with $p, q$ prime elements in $B(Z, Z)$ and $B(Z', Z')$, respectively. Then*

$$\dim_k \mathrm{rad}_{\mathcal{B}}^{\infty}(M, N) = mn(\dim_{k(x)}\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(Q_Z, Q_{Z'}) - \delta(Z, Z'));$$

$$\dim_k \mathrm{rad}_{\mathcal{B}}^{\infty}(M, L) = m\dim_{k(x)}\mathrm{rad}_{\mathcal{B}\text{-}k(x)}(Q_Z, L^{k(x)});$$

$$\dim_k \mathrm{rad}_{\mathcal{B}}^{\infty}(L, M) = m\dim_{k(x)}\mathrm{rad}_{\mathcal{B}\text{-}k(x)}(L^{k(x)}, Q_Z).$$

## 7 $\mathcal{D}$-isolated Objects

Let $\mathcal{A} = (A, V)$ be a bocs with layer $L = (A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$. We recall that an object $X \in \mathrm{ind}A'$ is called marked if $A'(X, X) \neq kid_X$, we denote by $m(A')$, the set of marked objects of $A'$. For $M \in \mathcal{A}$-mod we define its dimension vector

$$\mathbf{dim}M : \mathrm{ind}A' \to \mathbb{N} \quad \text{by} \quad \mathbf{dim}M(X) = \dim_k M(X).$$

By Dim$\mathcal{A}$ we denote the set of functions $\mathbf{d} : \mathrm{ind}A' \to \mathbb{N}$. If $\mathbf{d}, \mathbf{d}' \in \mathrm{Dim}\mathcal{A}$ we have $\mathbf{d}+\mathbf{d}'$, defined by $(\mathbf{d}+\mathbf{d}')(X) = \mathbf{d}(X) + \mathbf{d}'(X)$ for all $X \in \mathrm{ind}A'$. The norm of $\mathbf{d} \in \mathrm{Dim}\mathcal{A}$ is defined by $||\mathbf{d}|| = \sum_{i=1}^{n} \mathbf{d}(X_i)\mathbf{d}(Y_i) + \sum_{X \in m(A')} \mathbf{d}(X)^2$, where $a_i : X_i \to Y_i$. For $M \in \mathcal{A}$-mod we define the norm of $M$, $||M|| = ||\mathbf{dim}M||$.

If $\mathbf{d} \in \mathrm{Dim}(\mathcal{A})$ we define $|\mathbf{d}| = \sum_{X \in \mathrm{ind}A'} \mathbf{d}(X)$. For $M \in \mathcal{A}$-mod, we put $|M| = |\mathbf{dim}M|$ which is called the dimension of $M$.

Take $\theta : A \to B$ a functor with $B$ a skeletally small category, the induced bocs $\mathcal{A}^B = (B, W)$ is given as follows: $W = B \otimes_A V \otimes_A B$ with counit

$$\epsilon_B : W \to B$$

given by $\epsilon_B(b_1 \otimes v \otimes b_2) = b_1\theta(\epsilon(v))b_2$ for $b_1, b_2$ morphisms in $B$, $v \in V$. The coproduct

$$\mu_B : W \to W \otimes_B W$$

is given by $\mu_B(b_1 \otimes v \otimes b_2) = \sum_i b_1 \otimes v_i^1 \otimes 1 \otimes 1 \otimes v_i^2 \otimes b_2$, where $b_1, b_2$ are morphisms in $B$ and $v \in V$ with $\delta(v) = \sum_i v_i^1 \otimes v_i^2$.

There is a morphism of $A$-$A$-bimodules

$$\theta_1 : V \to W$$

given by $\theta_1(v) = 1 \otimes v \otimes 1$, for $v \in V$. Then we obtain a morphism of bocses $(\theta, \theta_1) : \mathcal{A} \to \mathcal{A}^B$ which induces a full and faithful functor $\theta^* : \mathcal{A}^B\text{-Mod} \to \mathcal{A}\text{-Mod}$.

Assume $\mathcal{A}^B$ has layer

$$L^\theta = (B'; \omega'; b_1, ..., b_{n'}; w_1, ..., w_{m'}).$$

There is an additive function $t^\theta : \mathrm{Dim}(\mathcal{A}^B) \to \mathrm{Dim}(\mathcal{A})$, given by $t^\theta(\mathbf{d})(X) = \sum_j \mathbf{d}(Y_j)$ with $\theta(X) = \bigoplus_j Y_j$, $Y_j \in \mathrm{ind}B'$. We have $\mathbf{dim}\theta^*(M) = t^\theta(\mathbf{dim}M)$, for $M \in \mathcal{A}^B$-mod.

Following [6], we say that that the bocs $\mathcal{A} = (A, V)$ with counit $\epsilon : V \to A$ and layer $L = (A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$ is of wild representation type or simply wild if there is a functor $F : A \to \Sigma$, where $\Sigma$ are the finitely generated free $k\langle x, y\rangle$-modules such that the induced functor:

$$(F, F\epsilon)^* : \Sigma\text{-Mod} \to \mathcal{A}\text{-Mod}$$

preserves isomorphism classes and indecomposables.

From [7], we know that a layered bocs $\mathcal{A} = (A, V)$ which is not of wild representation type is of tame representation type. This is, for each natural number $d$, there are a finite number of $A$-$k[x]$-bimodules $M_1, ..., M_s$ free of finite rank as right $k[x]$-modules, and such that every indecomposable $M$ in $\mathcal{A}$-Mod with $|\mathbf{dim}M| \le d$ is isomorphic to $M_i \otimes_{k[x]} k[x]/(x - \lambda)$ for some $1 \le i \le s$ and $\lambda \in k$.

This section is devoted to find some subset $\mathcal{D}$ of Dim$\mathcal{A}$ with $\mathcal{A}$ a bocs of tame representation type such that the marked indecomposable objects of $A$ become $\mathcal{D}$-isolated objects in the sense of Definition 7.4. For this we need the following specific functors (see section 4 of [5]):

1. **Regularization**. Suppose $a_1 : X_1 \to Y_1$ with $\delta(a_1) = v_1$. Then $B$ is freely generated by $A'$ and $a_2, ..., a_n$. The functor $\theta : A \to B$ is the identity on $A'$, $\theta(a_1) = 0$, $\theta(a_i) = a_i$ for $i = 2, ..., n$. The bocs $\mathcal{A}^B = (B, W)$ has layer $(A'; \omega_B; a_2, ..., a_n; \theta_1(v_2), ..., \theta_1(v_m))$.

The functor $\theta^* : \mathcal{A}^B$-Mod $\to \mathcal{A}$-Mod is an equivalence of categories, $\mathrm{Dim}(\mathcal{A}^B) = \mathrm{Dim}(\mathcal{A})$ and $t^\theta = id$. In this case $||t^\theta(\mathbf{d})|| \geq ||\mathbf{d}||$, and one has the equality if and only if $\mathbf{d}(X_1)\mathbf{d}(Y_1) = 0$.

2. **Deletion of objects** . Let $C$ be a subcategory of $A$. Let $B'$ be the full subcategory of $A'$ whose objects have no non-zero direct summand isomorphic to a direct summand of an object of $C$. Take $I_0$ the set of $i \in \{1, ..., n\}$ such that $a_i \in A(X_i, Y_i)$ with $X_i, Y_i$ in $B'$, and $I_1$ the set of $j \in \{1, ..., m\}$ such that $v_j \in V(X_j, Y_j)$ with $X_j, Y_j$ in $B'$. Then $B$ is freely generated by $B'$ and the $a_i$ with $i \in I_0$. The functor $\theta : A \to B$ is the identity on $B'$ and $\theta(X) = 0$ for all $X \in C$. The bocs $\mathcal{A}^B$ has layer $(B'; \omega_B; (a_i)_{i\in I_0}; (\theta_1(v_j))_{j\in I_1})$. Here $M \in \mathcal{A}-$Mod is isomorphic to some $\theta^*(N)$ if and only if $M(X) = 0$ for all $X$ indecomposable objects of $C$. The function $t^\theta : \mathrm{Dim}(\mathcal{A}^B) \to \mathrm{Dim}(\mathcal{A})$ is an inclusion, $\mathbf{d} \in \mathrm{Dim}(\mathcal{A})$ is in the image of $t^\theta$ if and only if $\mathbf{d}(X) = 0$ for all $X$ indecomposable objects of $C$. In this case $||t^\theta(\mathbf{d})|| = ||\mathbf{d}||$.

3. **Edge reduction** . Suppose $a_1 : X_1 \to Y_1$ with $X_1 \neq Y_1$ is such that $\delta(a_1) = 0$, and $A'(X_1, X_1) = kid_{X_1}$, $A'(Y_1, Y_1) = kid_{Y_1}$. Let $C$ be the full subcategory of $A'$ whose objects have no direct summands isomorphic to $X_1$ or $Y_1$. Now denote by $D$ a minimal category with three indecomposable objects $Z_1, Z_2, Z_3$, $D(Z_i, Z_i) = kid_{Z_i}$ for $i = 1, 2, 3$. Take $B' = C \times D$. The category $B$ is freely generated by $B'$ and elements $b_1, ..., b_s$. The number of arrows $b_j : W_j \to W'_j$ with $W_j$ and $W'_j$ different from $Z_2$ is $n - 1$, where $n$ is the number of $a_i$.

The functor $\theta : A \to B$ is the identity on $C$ and $\theta(X_1) = Z_1 \oplus Z_2$, $\theta(Y_1) = Z_2 \oplus Z_3$.

The bocs $\mathcal{A}^B = (B, W)$ has a layer of the form $(B', \omega_B; b_1, ..., b_s; w_1, ...w_u)$. Moreover, if $M \in \mathcal{A}^B$-Mod, $\theta^*(M)(a_i) = 0$ for all $i \in \{1, ..., n\}$ if and only if $M(b_j) = 0$ for all $j \in \{1, ..., s\}$ and $M(Z_2) = 0$. The functor $\theta^*$ is an equivalence of categories. Moreover $||t^\theta(\mathbf{d})|| > ||\mathbf{d}||$ if and only if $(t^\theta(\mathbf{d}))(X_1)(t^\theta(\mathbf{d})(Y_1)) \neq 0$. If $||t^\theta(\mathbf{d})|| = ||\mathbf{d}||$ and $||t^\theta(\mathbf{d}')|| = ||\mathbf{d}'||$, then $t^\theta(\mathbf{d}) = t^\theta(\mathbf{d}')$ implies $\mathbf{d} = \mathbf{d}'$.

4. **Unraveling** . Let $X$ be an indecomposable object in $A'$ with $A'(X, X) = k[x, f(x)^{-1}]id_X$. Suppose $S = \{\lambda_1, ..., \lambda_t\}$ is a set of elements of $k$ which are not roots of $f(x)$. For $r$ a positive integer there is a functor $\theta : A \to B$, where $B$ is freely generated by $B'$ and elements $b_1, ..., b_s$, $B' = C \times D$, where $C$ is the full subcategory of $A'$ whose objects have no direct summands isomorphic to $X$. The category $D$ is the minimal category with indecomposable objects $Y, Z_{i,j}$ with $i \in \{1, ..., r\}, j \in \{1, ..., t\}$, $D(Z_{i,j}, Z_{i,j}) = kid_{Z_{i,j}}$, $D(Y, Y) = k[x, f(x)^{-1}, g(x)^{-1}]id_Y$, where $g(x) = (x-\lambda_1)...(x-\lambda_t)$. The functor $\theta : A \to B$ acts as the identity on $C$ and $\theta(X) = Y \oplus \bigoplus_{j=1}^t \bigoplus_{i=1}^r Z_{i,j}^i$, where $Z_{i,j}^i$ is the direct sum of $i$ copies of $Z_{i,j}$.

The bocs $\mathcal{A}^B = (B, W)$ has a layer of the form $(B'; \omega_B; b_1, ..., b_s; w_1, ..., w_u)$.

Moreover for $N \in \mathcal{A}^B$-mod we have the following:

(a) $||N|| \leq ||\theta^*(N)||$, with strict inequality if $\theta^*(N)(g(x))$ is not invertible.

(b) If $M \in \mathcal{A}$-mod and for all $Z \in \mathrm{ind}A'$, $\dim_k M(Z) \leq r$ then there is a $N \in \mathcal{A}^B$-mod such that $\theta^*(N) \cong M$.

(c) $\theta^*(N)(x) = N(x) \oplus \bigoplus_{j=1}^{s} \bigoplus_{i=1}^{r} N(Z^i_{i,j})(x)$ with eigenvalues of $N(x)$ not in $S$, and $N(Z^i_{i,j})(x) = J_i(\lambda_j)$, the Jordan block of size $i$ and eigenvalue $\lambda_j$.

(d) Suppose $M \in \mathcal{A}$-mod is an indecomposable with $M(X) \neq 0$ and $M(W) = 0$ for all $W \neq X$, $W \in \text{ind}A'$, $M(a_i) = 0$ for $i \in \{1, ..., n\}$. Then if the unique eigenvalue of $M(x)$ is not in the set $S$, there is a $N \in \mathcal{A}^B$-mod with $N(W) = 0$ for all $W \in \text{ind}B'$, with $W \neq Y$, $N(b_j) = 0$ for all $j \in \{1, ..., s\}$ and $\theta^*(N) \cong M$.

(e) The number of $b_j : Y_1 \to Y_2$ with $Y_1, Y_2$ non isomorphic to $Z_{i,j}$ is equal to $n$, the number of $a_i$.

**Definition 7.1.** Let $\mathcal{A} = (A, V)$ be a bocs with layer $(A'; \omega; a_1, ..., a_n; v_1, ...v_m)$. We say that $M \in \mathcal{A}$-Mod is concentrated in the indecomposable $X \in A'$ if $M(X) \neq 0$, $M(Y) = 0$ for $Y$ indecomposable in $A'$, $Y \neq X$ and $M(a_i) = 0$ for all $i \in \{1, ..., n\}$.

**Proposition 7.2.** *Let $\mathcal{A} = (A, V)$ be a bocs which is not wild, with layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$. Let $X$ be an indecomposable object in $A'$ with $A'(X, X) = k[x, f(x)^{-1}]$. Then given a fixed dimension vector $\mathbf{d}$ with $\mathbf{d}(X) \neq 0$, there is a finite subset $S(X, \mathbf{d})$ of $k$ such that if $M$ is indecomposable in $\mathcal{A}$-mod with $\mathbf{dim}M = \mathbf{d}$ and $\lambda$ in $k$ but not in $S(X, \mathbf{d})$ is an eigenvalue of $M(x)$, then $M \cong M'$, with $M'$ concentrated in $X$.*

**Proof.** We may assume $\mathbf{d}$ is sincere. We prove our assertion by induction on $||\mathbf{d}||$. If $||\mathbf{d}|| = 1$, take $S(X, \mathbf{d})$ the set of roots of $f(x)$. Then if $M$ is an indecomposable in $\mathcal{A}$-mod, $M(X) \neq 0$, $\mathbf{dim}M = \mathbf{d}$, clearly $M$ is concentrated in $X$.

Suppose our result proved for all non-wild layered bocses and dimension vectors with norm smaller than $r$. We may assume that for all $a_i : X_i \to Y_i$ with $\delta(a_i) = 0$, $Y_i$ is not equal to $X_i$, since if $X_i = Y_i$, then because $\mathcal{A}$ is not wild and by Proposition 9 of [7] we have $A'(X_i, X_i) = kid_{X_i}$, so we may move $a_i$ into $A'$, such that $A'(X_i, X_i) = k[z]$, with $z = a_i$.

Take $a_1 : X_1 \to Y_1$ the first arrow. By condition $L.5$ of a layered bocs we have

$$\delta(a_1) = \sum_{j \in T} c_j v_j d_j,$$

where $c_j \in A'(Y_1, Y_1), d_j \in A'(X_1, X_1)$ and $T$ is the set of all $j \in \{1, ..., m\}$ such that $v_j : \overline{V}(X_1, Y_1)$. We have then the following possibilities: $\delta(a_1) = 0$ or $\delta(a_1) = \sum_j c_j v_j d_j$ with some $c_j v_j d_j \neq 0$. If all $c_i, d_i \in k$, we may assume $d_i = 1$ for all $i \in T$. In this case we put $v'_i = v_i$ for $i \neq j$ and $v'_j = \sum_j c_j v_j$. Taking $\{v'_j, v'_1, ..., v'_m\}$ instead of $\{v_1, ..., v_m\}$ we have again a layer for $\mathcal{A}$, thus in this case we may assume $\delta(a_1) = v_1$. In case that for some $j \in T$, $c_j$ is not in $k$ or $d_j$ is not in $k$, we have $A'(Y_1, Y_1) \neq kid_{Y_1}$ or $A'(X_1, X_1) \neq kid_{X_1}$.

**Case 1.** $\delta(a_1) = v_1$. Take $\theta^* : \mathcal{A}^B$-Mod $\to \mathcal{A}$-Mod the regularization of $a_1$. Here $\theta^*$ is an equivalence and the norm of $\mathbf{d}$ in $\mathcal{A}^B$ is smaller than $r$. Our claim is true for $X$ and the norm $r'$ of $\mathbf{d}$ in $\mathcal{A}^B$. Take $S(X, \mathbf{d}) = S'(X, \mathbf{d})$, with $S'(X, \mathbf{d})$ the subset of $k$ for which our claim is true in $\mathcal{A}^B$.

Then if $M \in \mathcal{A}$-mod is indecomposable with $\mathbf{dim}M = \mathbf{d}$ and $\lambda$ is an eigenvalue of $M(x)$ which is not in $S(X, \mathbf{d})$, we may assume $M = \theta^*(N)$. Here $M(x) = N(x)$, thus $N \cong N'$, with $N'$ concentrated in $X$, but this implies that $\theta^*(N')$ is concentrated in $X$, thus $\theta^*(N') \cong \theta^*(N) = M$, proving our claim.

**Case 2.** $\delta(a_1) = 0$. Since $\mathcal{A}$ is not wild, by Proposition 9 of [7], $A'(X_1, X_1) = kid_{X_1}$ and $A'(Y_1, Y_1) = kid_{Y_1}$. Here $X_1$ is not equal to $Y_1$. We have the edge reduction of $a_1$, $\theta^* : \mathcal{A}^B$-Mod $\to \mathcal{A}$-Mod, with $\mathcal{A}^B = (B, W)$. Consider the dimension vectors $\mathbf{d}_1, ..., \mathbf{d}_l$ of those $N \in \mathcal{A}^B$-mod such that $\mathbf{dim}\theta^*(N) = \mathbf{d}$.

The norms of the $\mathbf{d}_i$ are smaller than $r$. Here $X$ is not equal to $X_1$ and to $Y_1$. Therefore $X$ is an indecomposable object of $B'$. We may consider the subsets $S(X, \mathbf{d}_1), ..., S(X, \mathbf{d}_l)$. Take $S(X, \mathbf{d}) = S(X, \mathbf{d}_1) \cup ... \cup S(X, \mathbf{d}_l)$.

Let $M$ be an indecomposable in $\mathcal{A}$-mod with $\mathbf{dim}M = \mathbf{d}$. Suppose $\lambda$ is an eigenvalue of $M(x)$ which is not in $S(X, \mathbf{d})$. Since $\theta^*$ is an equivalence there is a $N \in \mathcal{A}^B$-mod such that $\theta^*(N) \cong M$. We may assume $\theta^*(N) = M$, then $M(X) = N(X)$ and $M(x) = N(x)$. Here $\mathbf{dim}N = \mathbf{d_i}$ for some $i \in [1, l]$. Therefore, since $\lambda$ is an eigenvalue of $N(x)$ which is not in $S(X, \mathbf{d}_i)$, $N \cong N'$, with $N'$ concentrated in $X$, consequently $\theta^*(N')$ is concentrated in $X$ and $\theta^*(N') \cong M$.

**Case 3.** $a_1 : X_1 \to Y_1$ with $A'(X_1, X_1) \neq kid_{X_1}$ or $A'(Y_1, Y_1) \neq kid_{Y_1}$.

Using the notation of [5], we have an unraveling in $X_1$ or in $Y_1$, for $r$ and some elements of $k$, $\lambda_1, ..., \lambda_s$ followed by regularization of $b : Y \to Y_1$ or of $b : X_1 \to Y$, with $b$ the generator corresponding to $a_1$. Let $\theta^* : \mathcal{A}^B$-Mod $\to \mathcal{A}$-Mod be the unraveling functor followed by the corresponding regularization, with $\mathcal{A}^B = (B, W)$ and layer $(B', \omega_B; b_1, ..., b_v; w_1, ..., w_u)$.

In case $X$ is not equal to $X_1$ and to $Y_1$ we proceed as in Case 2.

Suppose now that the unraveling is in $X$ with $X = X_1$ or $X = Y_1$, such that $\theta(X) = Y \oplus (\bigoplus_{i,j} Z^i_{i,j})$. Take all dimension vectors $\mathbf{d}_1, ..., \mathbf{d}_l$ of those $N \in \mathcal{A}^B$-mod with $\mathbf{dim}\theta^*(N) = \mathbf{d}$.

The norms of all $\mathbf{d_i}$ are smaller than $r$. Then we may take $S(Y, \mathbf{d}_i)$. We put $S(X, \mathbf{d}) = S(Y, \mathbf{d}_1) \cup ... \cup S(Y, \mathbf{d}_l) \cup \{\lambda_1, ..., \lambda_s\}$.

Let $M$ be an indecomposable in $\mathcal{A}$-mod with $\mathbf{dim}M = \mathbf{d}$, $M(X) \neq 0$ and $\lambda$ an eigenvalue of $M(x)$ which is not in $S(X, \mathbf{d})$.

There is a $N \in \mathcal{A}^B$ with $\theta^*(N) \cong M$. We may assume $\theta^*(N) = M$. There is a $\mathbf{d}_i$ with $i \in [1, l]$ such that $\mathbf{dim}N = \mathbf{d}_i$.

Here $M(x) = N(x) \oplus M'(x)$ with eigenvalues of $M'(x)$ contained in $\{\lambda_1, ..., \lambda_s\}$. The eigenvalue $\lambda$ of $M(x)$ is not in $S(X, \mathbf{d})$, therefore, $\lambda$ is an eigenvalue of $N(x)$. But $\lambda$ is not in $S(Y, \mathbf{d}_i)$, then $N \cong N'$, with $N'$ concentrated in $Y$. This implies that $\theta^*(N')$ is concentrated in $X$ and $M \cong \theta^*(N')$. □

**Notation 7.3.** We recall that if $\mathbf{d}$ and $\mathbf{d}'$ are dimension vectors of the bocs $\mathcal{A} = (A, V)$ we say that $\mathbf{d} \leq \mathbf{d}'$ if for all indecomposable objects $X$ of $A'$, $\mathbf{d}(X) \leq \mathbf{d}'(X)$. Then if $\mathcal{D}$ is a finite set of dimension vectors of $\mathcal{A}$, we denote by $s(\mathcal{D})$ the set consisting of all vectors in $\mathcal{D}$, all sums $\mathbf{d} + \mathbf{d}'$ with $\mathbf{d}, \mathbf{d}' \in \mathcal{D}$, and all vectors $\mathbf{e}$ with $\mathbf{e} \leq \mathbf{f}$ with $\mathbf{f}$ one of the above dimension vectors. Clearly $s(\mathcal{D})$ is also a finite set.

**Definition 7.4.** Let $\mathcal{A} = (A, V)$ be a bocs with layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$ and $\mathcal{D}$ be a finite set of dimension vectors of $\mathcal{A}$. We say that $X$, an indecomposable object in $A'$, with $A'(X, X) = k[x, f(x)^{-1}]id_X$ is $\mathcal{D}$-isolated if for any indecomposable $M \in \mathcal{A}$-mod with $\mathbf{dim}M \in s(\mathcal{D})$ and $M(X) \neq 0$, there is a $M' \in \mathcal{A}$-mod, concentrated in $X$ with $M \cong M'$.

**Lemma 7.5.** *Let $\mathcal{A} = (A, V)$ be a layered bocs as above, which is not of wild representation type, and $\mathcal{D}$ be a finite set of dimension vectors of $\mathcal{A}$ such that for all indecomposable $X \in A'$ there is a $\mathbf{d} \in \mathcal{D}$ with $\mathbf{d}(X) \neq 0$, and $a_1 : X_1 \to Y_1$. Then*

*(1) if $X_1$ and $Y_1$ are both $\mathcal{D}$-isolated and $\delta(a_1) \in \mathcal{I}_2\overline{V} + \overline{V}\mathcal{I}_1$ with $\mathcal{I}_1$ an ideal of $A'(X_1, X_1)$, $\mathcal{I}_2$ an ideal of $A'(Y_1, Y_1)$, then $\mathcal{I}_1 = A'(X_1, X_1)$ or $\mathcal{I}_2 = A'(Y_1, Y_1)$;*

*(2) if $X_1$ is $\mathcal{D}$-isolated, $A'(Y_1, Y_1) = kid_{Y_1}$, $\delta(a_1) \in \overline{V}\mathcal{I}_1$ with $\mathcal{I}_1$ an ideal of $A'(X_1, X_1)$, then $\mathcal{I}_1 = A'(X_1, X_1)$;*

*(3) if $Y_1$ is $\mathcal{D}$-isolated, $A'(X_1, X_1) = kid_{X_1}$, $\delta(a_1) \in \mathcal{I}_2\overline{V}$ with $\mathcal{I}_2$ an ideal of $A'(Y_1, Y_1)$, then $\mathcal{I}_2 = A'(Y_1, Y_1)$.*

**Proof.** We have

$$(*) \quad \delta(a_1) = \sum_{s \in T_1} h_s v_s + \sum_{s \in T_2} v_s g_s$$

with $h_s \in \mathcal{I}_2, g_s \in \mathcal{I}_1$.

(1) Suppose our claim is not true, then we may assume $\mathcal{I}_1$ and $\mathcal{I}_2$ are maximal ideals. Then $A'(X_1, X_1)/\mathcal{I}_1 \cong k$ and $A'(Y_1, Y_1)/\mathcal{I}_2 \cong k$. First assume $X_1 = Y_1$. Take the representation $M$ of $A$ such that $M(X_1) = M_1 \oplus M_2$ with $M_i = A'(X_1, X_1)/\mathcal{I}_i$ for $i = 1, 2$, $M(W) = 0$ for $W \neq X_1$. Take $M(a_1)$ such that $0 \neq M(a_1)(M_1) \subset M_2$, $M(a_1)(M_2) = 0$ and $M(a_j) = 0$ for $j > 1$. Here $\mathbf{dim}M \in s(\mathcal{D})$, then if $M$ is indecomposable, $M \cong M'$ with $M'$ concentrated in $X_1$, but this implies that $M'$ is indecomposable as $A'$-module, which is not the case because as $A'$-modules, we have $M' \cong M \cong M_1 \oplus M_2$. Therefore, $M \cong L_1 \oplus L_2$, with $L_1, L_2$ indecomposables, and $\mathbf{dim}L_1, \mathbf{dim}L_2$ are in $s(\mathcal{D})$. Then $L_1 \cong L_1', L_2 \cong L_2'$, with $L_1', L_2'$ concentrated in $X_1$, thus $M \cong L = L_1' \oplus L_2'$, and $L(a_1) = 0$. There is an isomorphism $f = (f^0, f^1) : M \to L$. Then from $(*)$ we obtain

$$L(a_1)f^0_{X_1} - f^0_{Y_1}M(a_1) = \sum_{s \in T_1} L(h_s)f^1(v_s) + \sum_{s \in T_2} f^1(v_s)M(g_s),$$

then, since $L(a_1) = 0$ and $\mathcal{I}_1 M_1 = 0$, from the above formula we obtain

$$f^0_{Y_1}M(a_1)(M) = f^0_{Y_1}M(a_1)(M_1) \subset \mathcal{I}_2 L,$$

then if $\mathcal{I}_1 = \mathcal{I}_2$, $\mathcal{I}_2 L = 0$, so $f^0_{Y_1}M(a_1)(M) = 0$. If $\mathcal{I}_1 \neq \mathcal{I}_2$, $A'(X_1, X_1) = \mathcal{I}_1 + \mathcal{I}_2$. We have

$$\mathcal{I}_1 f^0_{Y_1} M(a_1)(M) \subset \mathcal{I}_1\mathcal{I}_2 L = 0,$$

$$\mathcal{I}_2 f^0_{Y_1} M(a_1)(M) \subset f^0_{Y_1}(\mathcal{I}_2 M_2) = 0.$$

Consequently, $f^0_{Y_1}M(a_1) = 0$, a contradiction to $M(a_1) \neq 0$. Thus we obtain our statement in this case.

Now, assume $X_1 \neq Y_1$, take $M$ the representation of $A$ such that $M(X_1) = A'(X_1, X_1)/\mathcal{I}_1$, $M(Y_1) = A'(Y_1, Y_1)/\mathcal{I}_2$, $M(Z) = 0$ for $Z$ indecomposable non-isomorphic to $X_1$ or $Y_1$; $M(a_1) \neq 0$ and $M(a_j) = 0$ for all $j > 1$. Clearly $\mathbf{dim}M \in s(\mathcal{D})$. We claim that $M \cong L$ with $L(a_1) = 0$. In fact if $M$ is indecomposable then $M \cong M'$ with $M'$ concentrated in $X_1$ since $M(X_1) \neq 0$, and $M \cong M''$ with $M''$ concentrated in $Y_1$, since $M(Y_1) \neq 0$. Thus $X_1 = Y_1$ a contradiction, therefore $M$ is decomposable $M \cong L = L_1 \oplus L_2$ with $L_1(X_1) \cong M(X_1)$, $L_1(Y_1) = 0$ and $L_2(X_1) = 0, L_2(Y_1) \cong M(Y_1)$, consequently, $L_1(a_1) = 0$ and $L_2(a_1) = 0$, and, therefore $L(a_1) = 0$, proving our claim.

Then there is an isomorphism $(f^0, f^1) : M \to L$. Here $f^0_{X_1} : M(X_1) \to L(X_1)$ and $f^0_{Y_1} : M(Y_1) \to L(Y_1)$ are isomorphisms. From $(*)$ we obtain

$$L(a_1)f^0_{X_1} - f^0_{Y_1}M(a_1) = \sum_{s \in T_1} L(h_s)f^1(v_s) + \sum_{s \in T_2} f^1(v_s)M(g_s) = 0,$$

consequently, $f^0_{Y_1}M(a_1) = 0$, so $M(a_1) = 0$, a contradiction.

(2) We are assuming that $X_1$ is $\mathcal{D}$-isolated, by Definition 7.4, $A'(X_1, X_1) \neq kid_{X_1}$. Here we suppose $A'(Y_1, Y_1) = kid_{Y_1}$, then $X_1 \neq Y_1$. If our claim is not true, we may assume that $\mathcal{I}_1$ is a maximal ideal and $A'(X_1, X_1)/\mathcal{I}_1 = k$. Consider now $M$, the representation of $A$, such that $M(X_1) = A'(X_1, X_1)/\mathcal{I}_1, M(Y_1) = k$, $M(Z) = 0$ for $Z$ indecomposable non-isomorphic to $X_1$ and to $Y_1$, $M(a_1) \neq 0, M(a_j) = 0$ for all $j \geq 2$. If $M$ is indecomposable, then $M \cong M'$ with $M'$ concentrated in $X_1$, since $M(X_1) \neq 0$, a contradiction to $M(Y_1) \neq 0$. If $M$ is decomposable, we may construct a module $L = L_1 \oplus L_2$ and lead to a contradiction similar to (1).

(3) The proof is similar to (2). □

**Remark 7.6.** Let $\mathcal{A}$ be a non wild bocs and $\theta : A \to B$ any of our reduction functors such that it does not delete marked indecomposable objects. If $\mathcal{A}$ has layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$ and $\mathcal{A}^B$ has layer $(B'; \omega_B; b_1, ..., b_{n'}; w_1, ...w_{m'})$, then to each marked $X \in \text{ind}A'$ corresponds a marked $X^m \in B'$ such that $\theta(X) = X^m \oplus Y$ with $Y$ either 0 or a sum of non-marked indecomposables. Conversely each marked object in $B'$ is equal to some $X^m$. Moreover,

i) if $N \in \mathcal{A}^B$-Mod is concentrated in $X^m$ then $\theta^*(N)$ is concentrated in $X$.

ii) Suppose $N \in \mathcal{A}^B$-Mod is indecomposable with $N(X^m) \neq 0$ and $\theta^*(N) \cong M$ with $M$ concentrated in $X$, then there exists $N' \in \mathcal{A}^B$-Mod concentrated in $X^m$ such that $N' \cong N$.

**Lemma 7.7.** *If $\theta : A \to B$ is a reduction functor and $(e) : M \xrightarrow{f} E \xrightarrow{g} N$ is a proper exact sequence in $\mathcal{A}^B$-mod, then $\theta^*(e) : \theta^*(M) \xrightarrow{\theta^*(f)} \theta^*(E) \xrightarrow{\theta^*(g)} \theta^*(N)$ is a proper exact sequence in $\mathcal{A}$-mod (see Definition 4.6).*

**Proof.** Let $f : L \to H$ be a morphism in $\mathcal{A}^B$-Mod. From the explicit description of $\theta^*$

for each of the reduction functors given in section 4 of [5] one can see that if $(i, \omega_B)^*(f)$ is a monomorphism (respectively an epimorphism), then $(i, \omega)^*\theta^*(f)$ is a monomorphism (respectively an epimorphism). We have $\mathbf{dim}E = \mathbf{dim}M + \mathbf{dim}N$, then $\mathbf{dim}\theta^*(E) = t^\theta(\mathbf{dim}E) = \mathbf{dim}\theta^*(M) + \mathbf{dim}\theta^*(N)$. Therefore, $\dim_k\theta^*(E)(X) = \dim_k\theta^*(M)(X) + \dim_k\theta^*(N)(X)$, for each $X \in \mathrm{ind}A'$. From this and our first observation we may conclude that $\theta^*(e)$ is a proper exact sequence, proving our claim. □

## 8 An improvement of the Tame Theorem

In this section, we prove in Theorem 8.5 that given a tame layered bocs $\mathcal{A}$ and a positive integer $r$, then there is a minimal layered bocs $\mathcal{B}$ and a functor $F : \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$, which is a composition of the reduction functors of section 7, such that for any $M$ representation of $\mathcal{A}$, with dimension smaller than or equal to $r$ there is a representation $N$ of $\mathcal{B}$ with $F(N) \cong M$. This is an improvement of Theorem A in [5] which needs several minimal bocses.

We recall that if $\mathcal{A} = (A, V)$ is a bocs, then a family $\mathcal{F}$ of non-isomorphic indecomposable objects in $\mathcal{A}$-mod is called a one-parameter family if there is $T$ an $A$-$k[x, f(x)^{-1}]$-bimodule free of finite rank as right $k[x, f(x)^{-1}]$-module, such that for all $\lambda \in k$ which is not a root of $f(x)$, there is a $N \in \mathcal{F}$ with $T \otimes_{k[x,f(x)^{-1}]} k[x]/(x-\lambda) \cong N$ and for each $N \in \mathcal{F}$ there is an unique $\lambda \in k$ which is not a root of $f(x)$ with $N \cong T \otimes_{k[x,f(x)^{-1}]} k[x]/(x-\lambda)$.

Two one-parameter families $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be equivalent if there is only a finite number of elements in $\mathcal{F}_1$ which are not isomorphic to objects in $\mathcal{F}_2$. It follows from Theorem 5.6 of [6] that if $\mathcal{A}$ is not of wild representation type and $\mathcal{D}$ is a finite set of dimension vectors there is only a finite number $m(\mathcal{A}, \mathcal{D})$ of non-equivalent one-parameter families of objects in $\mathcal{A}$-mod having dimension vectors in $s(\mathcal{D})$. Observe that the number of $\mathcal{D}$-isolated objects $X$ in $A'$ is smaller than or equal to $m(\mathcal{A}, \mathcal{D})$.

In the following, $\mathcal{A}_0 = (A_0, V_0)$ is a fixed layered bocs which is not of wild representation type and $\mathcal{D}_0$ a fixed finite set of dimension vectors of $\mathcal{A}_0$. Consider the family $\mathcal{P}$ of pairs $(\mathcal{A}, \mathcal{D})$ with $\mathcal{A}$ a bocs with layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$, $\mathcal{D}$ a finite set of dimension vectors of $\mathcal{A}$ such that there exists $\theta : A_0 \to A$ a composition of reduction functors with $\mathcal{A}_0^A = \mathcal{A}$ and $t^\theta(\mathcal{D}) \subset \mathcal{D}_0$. We denote by $m_0$ the number $m(\mathcal{A}_0, s(\mathcal{D}_0))$. Observe that since $\theta^*$ is a full and faithful functor and $\mathcal{A}_0$ is not of wild representation type, then $\mathcal{A}$ is not of wild representation type.

If $(\mathcal{A}, \mathcal{D}) \in \mathcal{P}$, for each $X \in \mathrm{ind}A'$ which is $\mathcal{D}$-isolated we have a one-parameter family of representations of $\mathcal{A}$. To different $\mathcal{D}$-isolated indecomposables in $\mathrm{ind}A'$ correspond non-equivalent one-parameter families of representations of $\mathcal{A}$. By the definition of $\mathcal{P}$, there exists a composition of reduction functors $\theta : A_0 \to A$ with $t^\theta(\mathcal{D}) \subset \mathcal{D}_0$. Therefore, the image under $\theta^*$ of the one-parametric family corresponding to a $\mathcal{D}$-isolated indecomposable in $A'$ is a one-parametric family of $\mathcal{A}_0$ with dimension vector in $s(\mathcal{D}_0)$. Therefore, the number of $\mathcal{D}$-isolated indecomposables in $A'$ is smaller or equal to $m_0$.

**Notation.** Suppose $\mathcal{A}$ is a layered bocs which is not of wild representation type and $\mathcal{D}$ is a finite set of dimension vectors of $\mathcal{A}$. For $j$ a non-negative integer, we denote by

$\mathcal{S}(\mathcal{A},\mathcal{D})(j)$ the subset of $\mathcal{D}$ consisting of the $\mathbf{d}$ in $\mathcal{D}$ with $||\mathbf{d}|| = j$.
Take $(\mathcal{A},\mathcal{D})$ a pair in $\mathcal{P}$, we define a function $c(\mathcal{A},\mathcal{D}) : \{-1, 0, 1, 2, ..., \infty\} \to \{0, 1, 2, ...\}$ in the following way:

$$c(\mathcal{A},\mathcal{D})(\infty) = m_0 - i(\mathcal{A},\mathcal{D})$$

with $i(\mathcal{A},\mathcal{D})$ the number of indecomposables in $A'$ which are $\mathcal{D}$-isolated.

$$c(\mathcal{A},\mathcal{D})(-1) = n$$

where $n$ is the number of $a_i$ in the layer of $\mathcal{A}$. For $j$ a non-negative integer we put

$$c(\mathcal{A},\mathcal{D})(j) = \mathrm{Card}\mathcal{S}(\mathcal{A},\mathcal{D})(j).$$

The functions $c(\mathcal{A},\mathcal{D})$ belong to $\mathcal{H}$, the set of functions

$$f : \{-1, 0, 1, ..., \infty\} \to \{0, 1, ..., \}$$

with $f(x) = 0$ for almost all $x \in \{-1, 0, 1, ..., \infty\}$.

If $f, g$ are elements in $\mathcal{H}$ we put $f < g$ if there is a $s$ in $\{-1, 0, 1, ..., \infty\}$ such that $f(s) < g(s)$ and $f(u) = g(u)$ for $u \in \{-1, 0, 1, ..., \infty\}, u > s$. Clearly if we have an infinite sequence of elements in $\mathcal{H}$ with:

$$f_1 \geq f_2 \geq .... \geq f_m \geq f_{m+1} \geq ....$$

then there exists $l$ such that for all $m > l$, $f_m = f_l$.

**Notation**. If $\theta : A \to B$ is any of our reduction functors and $\mathcal{D}$ is a finite set of dimension vectors of $\mathcal{A}$, we say that $\theta^*$ is $\mathcal{D}$-covering if for each $M \in \mathcal{A}$-mod with $\mathbf{dim}M \in \mathcal{D}$ there exists a $N \in \mathcal{A}^B$-mod with $\theta^*(N) \cong M$. If $\theta : A \to B$ is a composition of our reduction functors, we denote by $\mathcal{D}^B$ the set of $\mathbf{d}' \in \mathrm{Dim}(\mathcal{A}^B)$ such that $t^\theta(\mathbf{d}') \in \mathcal{D}$.

In the statement of the following Lemma, we use the notation of Remark 7.6.

**Lemma 8.1.** *Let $\theta : A \to B$ be any of our reduction functors such that it does not delete marked objects. Then if $X$ is $\mathcal{D}$-isolated, one has that $X^m$ is $\mathcal{D}^B$-isolated. Conversely if $\theta$ is a regularization or the deletion of an object $W$ such that $\mathbf{d}(W) = 0$ for all $\mathbf{d} \in \mathcal{D}$ and $X^m$ is $\mathcal{D}^B$-isolated then $X$ is $\mathcal{D}$-isolated.*

**Proof.** Suppose $X$ is $\mathcal{D}$-isolated in $\mathcal{A}$. We shall prove that $X^m$ is $\mathcal{D}^B$-isolated in $\mathcal{A}^B$. For this take an indecomposable $N \in \mathcal{A}^B$-mod, with $\mathbf{dim}N \in s(\mathcal{D}^B)$ and $N(X^m) \neq 0$. Consider $M = \theta^*(N)$, then following the notation of Remark 7.6, $M(X) = N(X^m) \oplus N(Y)$, thus $M(X) \neq 0$, moreover $\mathbf{dim}M \in s(\mathcal{D})$. Since $X$ is $\mathcal{D}$-isolated, then there exists $M' \in \mathcal{A}$-mod, with $M \cong M'$ and $M'$ concentrated in $X$. Therefore, by Remark 7.6 there is a $N'$ concentrated in $X^m$ such that $N \cong N'$. From here we conclude that $X^m$ is $\mathcal{D}^B$-isolated. This proves the first part of our claim.

Suppose now that $\theta$ is a regularization. In this case $t^\theta = id$ and $\mathcal{D}^B = \mathcal{D}$. Suppose $X^m$ is $\mathcal{D}^B$-isolated, let us prove that $X$ is $\mathcal{D}$-isolated. Let $M$ be an indecomposable in

$\mathcal{A}$-mod, with $\mathbf{dim}M \in s(\mathcal{D})$ and $M(X) \neq 0$. Since $\theta^*$ is an equivalence of categories, there is a $N \in \mathcal{A}^B$-mod with $\theta^*(N) \cong M$. We have $N(X^m) = M(X)$, and, therefore, $N(X^m) \neq 0$. Moreover, $\mathbf{dim}N \in s(\mathcal{D}^B)$. Since $X^m$ is $\mathcal{D}^B$-isolated, there is a $N' \in \mathcal{A}^B$-mod, concentrated in $X^m$ such that $N' \cong N$. We have $M' = \theta^*(N')$ is concentrated in $X$, clearly $M \cong M'$, proving our claim.

A similar proof is done for the case $\theta$ is the deletion of an indecomposable $W$ with $\mathbf{d}(W) = 0$ for all $\mathbf{d} \in \mathcal{D}$. □

**Lemma 8.2.** *Let $\theta : A \to B$ be a reduction functor which is not an unraveling or the deletion of some $X$ for which there is a $\mathbf{d} \in \mathcal{D}$ with $\mathbf{d}(X) \neq 0$. Suppose there is a $\mathbf{d}'$ with $t^\theta(\mathbf{d}') \in \mathcal{D}$ and $||t^\theta(\mathbf{d}')|| > ||\mathbf{d}'||$. Let*

$$r = \max\{||t^\theta(\mathbf{d}')|| \mid t^\theta(\mathbf{d}') \in \mathcal{D}, \textit{ and } ||t^\theta(\mathbf{d}')|| > ||\mathbf{d}'||\}.$$

*Then for $j > r$,*

$$c(\mathcal{A}^B, \mathcal{D}^B)(j) = c(\mathcal{A}, \mathcal{D})(j) \quad \textit{and} \quad c(\mathcal{A}^B, \mathcal{D}^B)(r) < c(\mathcal{A}, \mathcal{D})(r).$$

**Proof.** Let us prove first that for $j \geq r$, $t^\theta$ induces an injective function

$$t^\theta_j : \mathcal{S}(\mathcal{A}^B, \mathcal{D}^B)(j) \to \mathcal{S}(\mathcal{A}, \mathcal{D})(j).$$

Take $\mathbf{d}' \in \mathcal{S}(\mathcal{A}^B, \mathcal{D}^B)(j)$, then $||t^\theta(\mathbf{d}')|| \geq ||\mathbf{d}'|| = j \geq r$. By definition of $r$, $||t^\theta(\mathbf{d}')|| = ||\mathbf{d}'|| = j$. Thus, $t^\theta$ induces a function $t^\theta_j$. If $t^\theta_j(\mathbf{d}') = t^\theta_j(\mathbf{d}'')$, we have $||t^\theta(\mathbf{d}')|| = ||\mathbf{d}'||$ and $||t^\theta(\mathbf{d}'')|| = ||\mathbf{d}''||$, therefore $\mathbf{d}' = \mathbf{d}''$. Consequently, $t^\theta_j$ is an injective function.

Suppose $j > r$. Take $\mathbf{d} \in \mathcal{S}(\mathcal{A}, \mathcal{D})(j)$, since $\theta^*$ does not delete indecomposable objects $X \in \text{ind}A'$ for which there is a $\mathbf{f} \in \mathcal{D}$ with $\mathbf{f}(X) \neq 0$ then there is a $\mathbf{d}' \in \mathcal{S}(\mathcal{A}^B, \mathcal{D}^B)$ with $t^\theta(\mathbf{d}') = \mathbf{d}$. We have $r < ||\mathbf{d}|| = ||t^\theta(\mathbf{d}')|| \geq ||\mathbf{d}'||$. By definition of $r$, $||t^\theta(\mathbf{d}')|| = ||\mathbf{d}'|| = j$. Thus $\mathbf{d}' \in \mathcal{S}(\mathcal{A}^B, \mathcal{D}^B)(j)$. Consequently, $t^\theta_j$ is a bijective function and we have proved the first part of our claim.

For the second part of our claim, take $\mathbf{d}' \in \mathcal{D}^B$ such that $r = ||t^\theta(\mathbf{d}')|| > ||\mathbf{d}'||$. We have $\mathbf{d} = t^\theta(\mathbf{d}')$ in $\mathcal{S}(\mathcal{A}, \mathcal{D})(r)$. Let us prove that $\mathbf{d}$ is not in the image of $t^\theta_r : \mathcal{S}(\mathcal{A}^B, \mathcal{D}^B)(r) \to \mathcal{S}(\mathcal{A}, \mathcal{D})(r)$. If $\theta$ is a regularization or deletion of objects, $t^\theta$ is an injective function and if $\mathbf{d} = t^\theta_r(\mathbf{d}'')$, with $||\mathbf{d}''|| = r$, since $t^\theta$ is injective we have $\mathbf{d}' = \mathbf{d}''$, a contradiction. We only need consider the case in which $\theta$ is an edge reduction of $a_1 : X_1 \to Y_1$. Since $||\mathbf{d}|| = ||t^\theta(\mathbf{d}')|| > ||\mathbf{d}'||$, $\mathbf{d}(X_1)\mathbf{d}(Y_1) \neq 0$ and if $\mathbf{d} = t^\theta(\mathbf{d}'')$ then $r = ||t^\theta(\mathbf{d}'')|| > ||\mathbf{d}''||$, proving our claim. □

**Lemma 8.3.** *Suppose $(A, \mathcal{D})$ is a pair in $\mathcal{P}$. Let $\theta : A \to B$ be the deletion of a non-marked indecomposable $X \in A'$, such that for all $\mathbf{d} \in \mathcal{D}$, $\mathbf{d}(X) = 0$, then $c(\mathcal{A}^B, \mathcal{D}^B)(u) = c(\mathcal{A}, \mathcal{D})(u)$ for all $u \in \{0, 1, .., \infty\}$.*

**Proof.** By Lemma 8.1 $c(\mathcal{A}^B, \mathcal{D}^B)(\infty) = c(\mathcal{A}, \mathcal{D})(\infty)$. On the other hand, by our hypothesis, $t^\theta$ induces a bijective function $t^\theta : \mathcal{D}^B \to \mathcal{D}$ and $||t^\theta(\mathbf{d})|| = ||\mathbf{d}||$, for all $\mathbf{d} \in \mathcal{D}^\theta$.

Therefore, $c(\mathcal{A}^B, \mathcal{D}^B)(j) = c(\mathcal{A}, \mathcal{D})(j)$ for all non-negative integers $j$. This proves our claim. □

**Lemma 8.4.** *Let $(\mathcal{A}, \mathcal{D})$ be a pair in $\mathcal{P}$. Suppose that for each $X \in \mathrm{ind}A'$ there exists $\mathbf{d} \in \mathcal{D}$ with $\mathbf{d}(X) \neq 0$. Then, if $\mathcal{A}$ is not a minimal bocs, there is a composition of reduction functors $\theta : A \to B$, with $\theta^*$ a $s(\mathcal{D})$-covering functor, such that $c(\mathcal{A}^B, \mathcal{D}^B) < c(\mathcal{A}, \mathcal{D})$, or there is a change of layer of $\mathcal{A}$ such that if $c'(\mathcal{A}, \mathcal{D})$ is the corresponding function we have $c'(\mathcal{A}, \mathcal{D}) < c(\mathcal{A}, \mathcal{D})$.*

**Proof.** (1) Suppose $a_1 : X_1 \to X_1$ and $\delta(a_1) = 0$. Since $\mathcal{A}$ is not of wild representation type, then by Proposition 9 of [7] we have $A'(X_1, X_1) = kid_{X_1}$. Take $B' = A'(a_1)$ and change the layer $(A'; \omega; a_1, ..., a_n; v_1, ...v_m)$ by the layer $(B'; \omega; a_2, ..., a_n; v_1, ..., v_m)$. We have $B'(X_1, X_1) = k[a_1]id_{X_1}$. Clearly if $W$ is an object non isomorphic to $X_1$ in $\mathrm{ind}A'$, this object is $\mathcal{D}$-isolated with respect to the original layer of $\mathcal{A}$ if and only if it is $\mathcal{D}$-isolated with respect to the new layer. Here it is possible that $X_1$, which is not marked with respect to the original layer of $\mathcal{A}$, becomes a $\mathcal{D}$ -isolated object with respect to the new layer. Therefore, if we denote by $c'(\mathcal{A}, \mathcal{D})$ the corresponding function with respect to the new layer we have $c'(\mathcal{A}, \mathcal{D})(\infty) \leq c(\mathcal{A}, \mathcal{D})(\infty)$.

The norm of a dimension vector does not depend of the choice of the layer, therefore, $c'(\mathcal{A}, \mathcal{D})(j) = c(\mathcal{A}, \mathcal{D})(j)$ for all non-negative integers $j$. Moreover,

$$c'(\mathcal{A}, \mathcal{D})(-1) = c(\mathcal{A}, \mathcal{D})(-1) - 1.$$

Therefore, $c'(\mathcal{A}, \mathcal{D}) < c(\mathcal{A}, \mathcal{D})$.

(2) Suppose there is a marked $X \in \mathrm{ind}A'$ which is not $\mathcal{D}$-isolated. Take $S = \bigcup_{\mathbf{d} \in s(\mathcal{D})} S(X, \mathbf{d})$, with $S(X, \mathbf{d})$ the sets of Proposition 7.2. Take $r$ the maximal of the numbers $\mathbf{d}(X)$ with $\mathbf{d} \in s(\mathcal{D})$. Consider now the unraveling $\theta : A \to B$ in $X$ with respect to $r$ and $S$. Clearly, the functor $\theta^* : \mathcal{A}^B\text{-Mod} \to \mathcal{A}\text{-Mod}$ is a $s(\mathcal{D})$-covering functor. We have $\theta(X) = X^m \oplus \bigoplus_{i,j} Z^i_{i,j}$. We shall see that $X^m$ is $\mathcal{D}^B$-isolated. Take $N$ an indecomposable in $\mathcal{A}^B$-mod with $N(X^m) \neq 0$ and $\mathbf{dim}N \in s(\mathcal{D}^B)$, then $\mathbf{dim}\theta^*(N) \in s(\mathcal{D})$. We have $\theta^*(N)(X) = N(X^m) \oplus \bigoplus_{i,j} N(Z_{i,j})^i \neq 0$. Take any eigenvalue of $N(x)$, this is an eigenvalue of $\theta^*(N)(x)$ which is not in $S$, therefore, it is not in $S(X, \mathbf{d})$ with $\mathbf{d} = \mathbf{dim}\theta^*(N)$. Therefore, by Proposition 7.2, $\theta^*(N) \cong M$, with $M$ concentrated in $X$. But this implies that $M(x)$ has only one eigenvalue which is not in $S$. Therefore, $M \cong \theta^*(N')$ with $N'$ concentrated in $X^m$. But $N \cong N'$, this proves that $X^m$ is $\mathcal{D}^B$-isolated. We have

$$c(\mathcal{A}^B, \mathcal{D}^B)(\infty) \leq c(\mathcal{A}, \mathcal{D})(\infty) - 1.$$

Therefore, $c(\mathcal{A}^B, \mathcal{D}^B) < c(\mathcal{A}, \mathcal{D})$.

(3) Suppose $a_1 : X_1 \to Y_1$ with $\delta(a_1) = 0$ and $X_1 \neq Y_1$. Take $\theta : A \to B$ the reduction of $a_1$. By Lemma 8.1, $c(\mathcal{A}^B, \mathcal{D}^B)(\infty) \leq c(\mathcal{A}, \mathcal{D})(\infty)$. If there is a $\mathbf{d}' \in \mathcal{D}^B$ such that $||t^\theta(\mathbf{d}')|| > ||\mathbf{d}'||$, by Lemma 8.2, $c(\mathcal{A}^B, \mathcal{D}^B) < c(\mathcal{A}, \mathcal{D})$. On the other hand if for all $\mathbf{d}' \in \mathcal{D}^B$, $||t^\theta(\mathbf{d}')|| = ||\mathbf{d}'||$, then again by Lemma 8.2, $c(\mathcal{A}^B, \mathcal{D}^B)(j) = c(\mathcal{A}, \mathcal{D})(j)$ for all non-negative integers $j$. We have that for all $\mathbf{d} \in \mathcal{D}$, $\mathbf{d}(X_1)\mathbf{d}(Y_1) = 0$. This implies

that for all $\mathbf{d}' \in \mathcal{D}^B$, $\mathbf{d}'(Z_2) = 0$. Take $\theta : B \to C$ the deletion of $Z_2$. By Lemma 8.3 we have $c(((\mathcal{A})^B)^C, (\mathcal{D}^B)^C)(u) = c(\mathcal{A}^B, \mathcal{D}^B)(u) = c(\mathcal{A}, \mathcal{D})(u)$ for all $u \neq -1$. Moreover, $c(((\mathcal{A})^B)^C, (\mathcal{D}^B)^C)(-1) = c(\mathcal{A}, \mathcal{D})(-1) - 1$, therefore, $c((\mathcal{A}^B)^C), (\mathcal{D}^B)^C) < c(\mathcal{A}, \mathcal{D})$.

(4) $\delta(a_1) = v_1$. In this case take $\theta : A \to B$ the regularization of $a_1$. As in the above case if there is a $\mathbf{d}' \in \mathcal{D}^B$ with $||t^\theta(\mathbf{d}')|| > ||\mathbf{d}'||$, then $c(\mathcal{A}^B, \mathcal{D}^B) < c(\mathcal{A}, \mathcal{D})$. On the other hand if for all $\mathbf{d}' \in \mathcal{D}^B$, $||t^\theta(\mathbf{d}')|| = ||\mathbf{d}'||$, by Lemma 8.1 $c(\mathcal{A}^B, \mathcal{D}^B)(\infty) = c(\mathcal{A}, \mathcal{D})(\infty)$. By Lemma 8.2, $c(\mathcal{A}^B, \mathcal{D}^B)(j) = c(\mathcal{A}, \mathcal{D})(j)$ for all non-negative integers $j$. Moreover, $c(\mathcal{A}^B, \mathcal{D}^B)(-1) = c(\mathcal{A}, \mathcal{D})(-1) - 1$. Therefore, $c(\mathcal{A}^B, \mathcal{D}^B) < c(\mathcal{A}, \mathcal{D})$.

(5) $\delta(a_1) = \sum_{s \in T} r_s v_s$ with $a_1 : X_1 \to Y_1$, $T$ the set of $s$ such that $v_s \in \overline{V}(X_1, Y_1)$ and $r_s \in A'(Y_1, Y_1) \otimes_k (A'(X_1, X_1))^{op} = H$. If there is a marked object in ind$A'$ which is not $\mathcal{D}$-isolated we may proceed as in (2). Therefore, we may assume that all marked objects in ind$A'$ are $\mathcal{D}$-isolated. The ring $H$ is isomorphic either to $k$, or to $k[x, f(x)^{-1}]$, or to $k[x, y, f(x)^{-1}, g(y)^{-1}]$. Let $\mathcal{I}$ be the ideal of $H$ generated by the elements $\{r_s\}_{s \in T}$. If $\mathcal{I} \neq H$, then $A'(X_1, X_1) \neq kid_{X_1}$ or $A'(Y_1, Y_1) \neq id_{Y_1}$. Moreover there are ideals $\mathcal{I}_2 \subset A'(Y_1, Y_1)$ and $\mathcal{I}_1 \subset A'(X_1, X_1)$ with $\mathcal{I} \subset \mathcal{I}_2 \otimes_k (A'(X_1, X_1))^{op} + A'(Y_1, Y_1) \otimes_k \mathcal{I}_1$, $\mathcal{I}_2 \neq A'(Y_1, Y_1)$ and $\mathcal{I}_1 \neq A'(X_1, X_1)$. Thus, $\delta(a_1) \in \mathcal{I}_2 \overline{V}(X_1, Y_1) + \overline{V}(X_1, Y_1)\mathcal{I}_1$ with $\mathcal{I}_2 \neq A'(Y_1, Y_1)$ and $\mathcal{I}_1 \neq A'(X_1, X_1)$.

Then if $A'(X_1, X_1) \neq kid_{X_1}$ and $A'(Y_1, Y_1) \neq kid_{Y_1}$, both $X_1$ and $Y_1$ are $\mathcal{D}$-isolated. But this contradicts (1) of Lemma 7.5 (recall that $\mathcal{A}$ is not of wild representation type).

If $A'(X_1, X_1) \neq kid_{X_1}$ and $A'(Y_1, Y_1) = kid_{Y_1}$, then $X_1$ is marked, so it is $\mathcal{D}$-isolated, we have $\mathcal{I}_1 \neq A'(X_1, X_1)$, and $\mathcal{I}_2 = 0$, but this contradicts (2) of Lemma 7.5. In case $A'(X_1, X_1) = kid_{X_1}$, then $Y_1$ is a marked object in ind$A'$, so it is $\mathcal{D}$-isolated and this contradicts (3) of Lemma 7.5.

Therefore, $\mathcal{I} = H$ and $1 = \sum_{s \in T} u_i r_i$. This implies that there is a free basis of $\overline{V}(X_1, Y_1)$, with one of their elements equal to $\delta(a_1)$, then we may apply case (4). □

**Theorem 8.5.** *Let $\mathcal{A}_0 = (A_0, V_0)$ be a layered bocs which is not of wild representation type. Then given a positive integer $r$ there is a composition of reduction functors $\theta : A_0 \to B$ with $\mathcal{A}^B$ a minimal layered bocs such that for all $M \in \mathcal{A}_0$-*mod *with $|M| \leq r$ there exists $N \in \mathcal{B}$-*Mod *with $\theta^*(N) \cong M$.*

**Proof.** Take $\mathcal{D}_0$ the set of $\mathbf{d} \in \mathrm{Dim}(\mathcal{A}_0)$ such that $\sum_{X \in \mathrm{ind} A_0'} \mathbf{d}(X) \leq r$, $\mathcal{D}_0$ is a finite set. Denote by $\mathcal{P}$ the family of pairs $(\mathcal{A}, \mathcal{D})$, with $\mathcal{A}$ a layered bocs, $\mathcal{D}$ a finite subset of $\mathrm{Dim}(\mathcal{A})$ such that there is a functor, composition of reduction functors $\theta : A_0 \to B$ with $t^\theta(\mathcal{D}) \subset \mathcal{D}_0$ and $\theta^*$ a $s(\mathcal{D}_0)$-covering functor.

Let $\mathcal{A} = (A, V)$ be a bocs with layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$ and $\mathcal{D}$ be a set of dimension vectors of $\mathcal{A}$, such that $(\mathcal{A}, \mathcal{D})$ is in $\mathcal{P}$.

For $X \in$ ind$A'$ we denote by $\mathbf{d}_X$ the dimension vector of $\mathcal{A}$ such that $\mathbf{d}_X(X) = 1$ and $\mathbf{d}_X(Z) = 0$ for $Z \in$ ind$A'$ with $Z \neq X$.

We will consider non-empty sets $\mathcal{D}$ of dimension vectors of $\mathcal{A}$ with the following two conditions:

(a) If $\mathbf{d} \in \mathcal{D}$ and $\mathbf{d}' < \mathbf{d}$, then $\mathbf{d}' \in \mathcal{D}$.

(b) If $X$ is a marked object in ind$A'$ then $\mathbf{d}_X \in \mathcal{D}$.

Let $\theta : A \to B$ be a reduction functor which does not delete marked objects of ind$A'$ and such that $\theta^* :$ Mod-$\mathcal{A}^B \to$ Mod-$\mathcal{A}$ is a $s(\mathcal{D})$-covering functor, we claim that if $\mathcal{D}$ satisfies properties $(a)$ and $(b)$, then $\mathcal{D}^B$ also satisfies these properties. Let $(B'; \omega; b_1, ..., b_t; w_1, ..., w_s)$ be a layer for $\mathcal{A}^B$.

Here $\theta^*$ is a $s(\mathcal{D})$-covering functor, then $\mathcal{D}^B$ is a non-empty set. Suppose now that $\mathcal{D}$ satisfies properties $(a)$ and $(b)$. Property $(a)$ for $\mathcal{D}^B$, follows from the fact that $\mathbf{d}' < \mathbf{d}$ in $\mathcal{D}$ implies $t^\theta(\mathbf{d}') \leq t^\theta(\mathbf{d})$.

For proving property $(b)$ of $\mathcal{D}^B$, suppose $W$ is a marked object in $B'$. Then following the notation of Lemma 7.6, $W = X^m$ for some marked object $X \in$ ind$A'$. Consider $\mathbf{d}_{X^m}$, dimension vector of $\mathcal{A}^B$. Then for $Z \in$ ind$A'$, $Z \neq X$ we have $\theta(Z) = \bigoplus_i Z_i$ with $Z_i \in$ ind$B'$, $Z_i \neq X^m$. Then $t^\theta(\mathbf{d}_{X^m})(Z) = \sum_i \mathbf{d}_{X^m}(Z_i) = 0$. We have $\theta(X) = X^m \oplus \bigoplus_j Y_j$ with $Y_j \in$ ind$B'$, $Y_j \neq X^m$, then $t^\theta(\mathbf{d}_{X^m})(X) = \mathbf{d}_{X^m}(X^m) = 1$. Consequently, $t^\theta(\mathbf{d}_{X^m}) = \mathbf{d}_X \in \mathcal{D}$, thus $\mathbf{d}_{X^m} \in \mathcal{D}^B$, proving our claim.

Now, suppose $\mathcal{D}$ satisfies properties $(a)$ and $(b)$, and $\theta : A \to B$ is the deletion of all objects $Z \in$ ind$A'$ such that $\mathbf{d}(Z) = 0$ for all $\mathbf{d} \in \mathcal{D}$. Since $\mathcal{D}$ satisfies property $(b)$, then $\theta$ does not delete marked objects. Therefore, $\mathcal{D}^B$ satisfies properties $(a)$ and $(b)$.

Now, if $\mathcal{A}^B$ is not a minimal bocs, by Lemma 8.4 there is a reduction functor $\rho : B \to A_1$ such that $\rho^*$ is a $s(\mathcal{D}^B)$-covering functor with

$$c((\mathcal{A}^B)^{A_1}, (\mathcal{D}^B)^{A_1}) < c(\mathcal{A}^B, \mathcal{D}^B),$$

or there exists a new layer for $\mathcal{A}^B$ such that

$$c'(\mathcal{A}^B, \mathcal{D}^B) < c(\mathcal{A}^B, \mathcal{D}^B).$$

By the proof of Lemma 8.4, we know that $\rho$ does not delete marked objects, then $(\mathcal{D}^B)^{A_1}$ satisfies properties $(a)$ and $(b)$. Now for any $Z \in$ ind$B'$ there exists some $\mathbf{d} \in \mathcal{D}^B$ with $\mathbf{d}(Z) \neq 0$, thus $\mathbf{d}_Z \leq \mathbf{d}$, so by property $(a)$, $\mathbf{d}_Z \in \mathcal{D}^B$, then $\mathcal{D}^B$ also satisfies property $(b)$ with respect to the new layer.

Then starting from $(\mathcal{A}_0, \mathcal{D}_0)$, we can construct a sequence of composition of reduction functors:

$$A_0 \overset{\theta_0}{\to} A_1 \overset{\theta_1}{\to} A_2 \to ... \overset{\theta_{l-1}}{\to} A_l,$$

with sets of dimension vectors $\mathcal{D}_i = (\mathcal{D}_{i-1})^{A_i}$ of $\mathcal{A}_i = (\mathcal{A}_{i-1})^{A_i}$ having conditions $(a)$ and $(b)$, such that all functors $\theta_i^*$ are $s(\mathcal{D}_i)$-covering functors. Moreover, we have a strictly decreasing sequence in $\mathcal{H}$,

$$c(\mathcal{A}_0, \mathcal{D}_0) > c(\mathcal{A}_1, \mathcal{D}_1) > ... > c(\mathcal{A}_l, \mathcal{D}_l).$$

In $\mathcal{H}$ we can not have infinite strictly decreasing sequences, so there is a sequence of reduction functors as before with $\mathcal{A}_l$ a minimal bocs, proving our result. $\square$

## 9 Hom-spaces in $\mathcal{D}(\Lambda)$-mod and in $P(\Lambda)$

We may observe that if $\Lambda_1$ and $\Lambda_2$ are two Morita-equivalent finite-dimensional $k$-algebras, then Theorem 1.2 is valid for $\Lambda_1$ if and only if it is valid for $\Lambda_2$. Therefore, without loss of generality, we assume in the rest of the paper that $\Lambda$ is a basic algebra.

Assume $k$ is an algebraically closed field and $1 = \sum_{i=1}^{n} e_i$ is a decomposition of the unit element of $\Lambda$ as a sum of pairwise orthogonal primitive idempotents. Then we have ${}_\Lambda\Lambda = \bigoplus_{i=1}^{n} \Lambda e_i$ a decomposition as sum of indecomposable projective $\Lambda$ -modules and $\Lambda = S \oplus J$ a decomposition as a direct sum of $S$-$S$-bimodules, with $J = \mathrm{rad}(\Lambda)$, $S = ke_1 \oplus ... \oplus ke_n$ a basic semisimple algebra. We can construct a basis $T = \{\alpha_1, ..., \alpha_m\}$ of $J$ with $\alpha_j \in e_{s(j)}\mathrm{rad}\Lambda e_{t(j)}$, inductively extending a basis of $J^i$ to $J^{i-1}$ by adding elements each of which lies in $e_s J e_t$ for some $s$ and $t$. In the following, if $L$ is a right $S$-modulo we denote its dual with respect to $S$ by $L^* = \mathrm{Hom}_S(L, S)$. For each element $\alpha_j \in e_{s(j)} T e_{t(j)}$ we define the element $\alpha_j^* \in J^*$, by $\alpha_j^*(\alpha_i) = 0$ for $\alpha_i \neq \alpha_j$ and $\alpha_j^*(\alpha_j) = e_{t(j)}$, clearly $\alpha_j^* \in e_{t(j)} J^* e_{s(j)}$ the elements $\alpha_j^*$ form a basis for $J^*$.

In the following, if $U_1, U_2, U_3$ are $k$-vector spaces we denote by $\begin{pmatrix} U_1 & 0 \\ U_2 & U_3 \end{pmatrix}$, the set of matrices of the form $\begin{pmatrix} u_1 & 0 \\ u_2 & u_3 \end{pmatrix}$, with $u_i \in U_i$, $i = 1, 2, 3$. With the usual sum of matrices and multiplication of scalars in $k$ by matrices, the above set is a $k$-vector space.

In order to define the Drozd's bocs of $\Lambda$ we need to consider the following two matrix algebras $A = \begin{pmatrix} S & 0 \\ J^* & S \end{pmatrix}$, and $A' = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}$. We are going to define a coalgebra $V$ over $A$ which is isomorphic to the coalgebra given in Proposition 6.1 of [5]. First consider the morphism of $S$-$S$-bimodules:

$$m : J^* \xrightarrow{\nu^*} (J \otimes_S J)^* \cong J^* \otimes_S J^*$$

where $\nu : J \otimes_S J \to J$ is the multiplication. We have the $k$-vector spaces $W_0 = \begin{pmatrix} 0 & 0 \\ J^* & 0 \end{pmatrix}$, and $W_1 = \begin{pmatrix} J^* & 0 \\ 0 & J^* \end{pmatrix}$, the elements of both vector spaces can be multiplied as matrices by the right and the left by elements of $A'$, thus $W_0$ and $W_1$ are $A'$-$A'$-bimodules.

We have a morphism of $A'$-$A'$-bimodules,

$$\underline{m} : W_1 \to W_1 \otimes_{A'} W_1$$

such that its composition with the isomorphism

$$W_1 \otimes_{A'} W_1 \cong \begin{pmatrix} J^* \otimes_S J^* & 0 \\ 0 & J^* \otimes_S J^* \end{pmatrix},$$

is the map that sends $\begin{pmatrix} h & 0 \\ 0 & g \end{pmatrix}$ to $\begin{pmatrix} m(h) & 0 \\ 0 & m(g) \end{pmatrix}$.

Now, consider the $k$-vector space $\overline{V} = \begin{pmatrix} J^* & 0 \\ M \oplus M & J^* \end{pmatrix}$, with $M = J^* \otimes_S J^*$, this is an $A$-$A$-bimodule with the following actions of $A$ over $\overline{V}$:

$$\begin{pmatrix} s_1 & 0 \\ g & s_2 \end{pmatrix} \begin{pmatrix} h_1 & 0 \\ (w_1, w_2) & h_2 \end{pmatrix} = \begin{pmatrix} s_1 h_1 & 0 \\ (s_2 w_1 + g \otimes h_1, s_2 w_2) & s_2 h_2 \end{pmatrix},$$

$$\begin{pmatrix} h_1 & 0 \\ (w_1, w_2) & h_2 \end{pmatrix} \begin{pmatrix} s_1 & 0 \\ g & s_2 \end{pmatrix} = \begin{pmatrix} h_1 s_1 & 0 \\ (w_1 s_1, w_2 s_1 + h_2 \otimes g) & h_2 s_2 \end{pmatrix}.$$

The $k$-linear map $\delta : A \to \overline{V}$ given by

$$\delta(\begin{pmatrix} s_1 & 0 \\ h & s_2 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ (m(h), -m(h)) & s_2 \end{pmatrix},$$

is a derivation, thus it gives an extension of $A$-$A$-bimodules:

$$0 \to \overline{V} \xrightarrow{i} V \xrightarrow{\epsilon} A \to 0$$

where $V = \overline{V} \oplus A$ as right $A$-modules, and putting $\omega = (0, 1)$, the left action of $A$ over $V$ is given by $a(v + \omega b) = av + \delta(a)b + \omega ab$, for $a, b \in A$, $v \in \overline{V}$. Here $\overline{V}$ is generated by $W_1$ as $A'$-$A'$-bimodule. We have:

(a) $A \cong W_0^{\otimes} = A' \oplus W_0$.

(b) The multiplication map $A \otimes_{A'} W_1 \otimes_{A'} A \to \overline{V}$ is an isomorphism.

We have a morphism of $A$-$A$-bimodules $\mu : V \to V \otimes_A V$, with $\mu(\omega) = \omega \otimes \omega$ and for $v \in W_1$, $\mu(v) = v \otimes \omega + \omega \otimes v + \lambda(v)$, where $\lambda$ is the composition of morphisms:

$$W_1 \xrightarrow{m} W_1 \otimes_{A'} W_1 \to \overline{V} \otimes_A \overline{V} \to V \otimes_A V.$$

The $A$-$A$-bimodule $V$ is a coalgebra over $A$ with counit $\epsilon$ and comultiplication $\mu$.

We have $1 = \sum_{i=1,j=1}^{n,2} f_{i,j}$ a decomposition of the unit of $A$ as a sum of pairwise orthogonal primitive idempotents, where $f_{i,2} = \begin{pmatrix} e_i & 0 \\ 0 & 0 \end{pmatrix}$ and $f_{i,1} = \begin{pmatrix} 0 & 0 \\ 0 & e_i \end{pmatrix}$.

Denote by $D$ the full subcategory of $A$-proj whose objects are all finite direct sums of objects $Af_{i,j}$. By $D'$ we denote the subcategory of $D$ with the same objects as $D$ and such that $D'(X,X) = kid_X$ for all $X \in \text{ind}D$ and $D'(X,Y) = 0$ for $X, Y \in \text{ind}D$ with $X \neq Y$. If $Af$ and $Ag$ are in $\text{ind}D$, and $x \in fAg$ we denote by $\nu_x : Af \to Ag$ the right multiplication by $x$.

Now, if $W$ is an $A$-$A$-bimodule we denote by $\vartheta(W)$ the $D$-$D$ bimodule given by $\vartheta(W)(Af, Ag) = fWg$ and if $\nu_x : Af' \to Af$, $\nu_y : Ag \to Ag'$ are morphisms then $\vartheta(W)(\nu_x, \nu_y) : \vartheta(W)(Af, Ag) \to \vartheta(W)(Af', Ag')$ is given by $\vartheta(W)(\nu_x, \nu_y)(w) = xwy$ for $w \in \vartheta(W)(Af, Ag)$. Similarly, for $L$ a right $A$-module and $M$ a left $A$-module we define functors, $\vartheta(L) : D \to \text{Mod-}k$ and $\vartheta(M) : D^{op} \to \text{Mod-}k$. If $f : W_1 \to W_2$ is a morphism of $A$-$A$-bimodules we have an induced morphism $\vartheta(f) : \vartheta(W_1) \to \vartheta(W_2)$. If $g : W_2 \to W_3$ is a morphism of $A$-$A$-bimodules then $\vartheta(f_2 f_1) = \vartheta(f_2)\vartheta(f_1)$. The morphisms between left $A$-modules and right $A$-modules induce also morphisms between the corresponding functors.

Fixed $L$ a right $A$-module we have $F : A\text{-mod} \to \text{Mod-}k$, given in objects by $F(M) = \vartheta(L) \otimes_D \vartheta(M)$ and if $f : M_1 \to M_2$ is a morphism of left $A$-modules, then $F(f) = 1 \otimes \vartheta(f)$. The functor $F$ is right exact and commutes with direct sums. Consequently, $F \cong W \otimes_A M$, with $W$ the right $A$-module $\vartheta(L)(A) \cong L$, therefore $\vartheta(L) \otimes_D \vartheta(M) \cong L \otimes_A M$ an isomorphism natural in $L$ and $M$.

Now, suppose $V_1$ and $V_2$ are $A$-$A$-bimodules then for $Af, Ag \in \text{ind}D$ we have $(\vartheta(V_1) \otimes_D \vartheta(V_2))(Af, Ag) = \vartheta(V_1)(Af, -) \otimes_D \vartheta(V_2)(-, Ag) \cong \vartheta(fV_1) \otimes_D \vartheta(V_2 g) \cong fV \otimes_A Vg$. Now, it is easy to see that in fact we have :

$$(c) \qquad \vartheta(V_1) \otimes_D \vartheta(V_2) \cong \vartheta(V_1 \otimes_A V_2)$$

The morphism of $A$-bimodules $\mu : V \to V \otimes_A V$ induces a morphism of $D$-$D$-bimodules $\vartheta(\mu) : \vartheta(V) \to \vartheta(V) \otimes_D \vartheta(V)$. In a similar way the morphism of $A$-$A$ bimodules $\epsilon : V \to A$ induces a morphism of $D$-$D$-bimodules $\vartheta(\epsilon) : \vartheta(V) \to \vartheta({}_A A_A) \cong D$. Now it is clear that $\mathcal{D}(\Lambda) = (D, V_D)$ with $V_D = \vartheta(V)$ is a bocs, the Drozd's bocs of $\Lambda$.

The bocs $\mathcal{D}(\Lambda)$ is isomorphic to the one given in Theorem 4.1 of [8] (see also the bocs given in the proof of Theorem 11 in [7]). We have now a grouplike $\omega_D$ relative to $D'$, given by $\omega_{Af} = f\omega f \in \vartheta(V)(Af, Af)$. Observe that we have $\vartheta(\mu)(\omega_{Af}) = \omega_{Af} \otimes \omega_{Af}$. The set of elements $\omega_{Af}$ is called a *normal section* in [8].

We are now going to construct a layer for $\mathcal{D}(\Lambda)$, with this purpose for each $i = 1, ..., n$, consider the following elements of $D$ and $V_D = \vartheta(V)$,

$b_i = \nu_{x(i)} \in D(Af_{t(i),1}, Af_{s(i),2}) = \text{Hom}_A(Af_{t(i),1}, Af_{s(i),2})$, $x(i) = \begin{pmatrix} 0 & 0 \\ \alpha_i^* & 0 \end{pmatrix}$; $v_{i,1} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha_i^* \end{pmatrix} \in \vartheta(V)(Af_{t(i),1}, Af_{s(i),1}) = f_{t(i),1} V f_{s(i),1}$, $v_{i,2} = \begin{pmatrix} \alpha_i^* & 0 \\ 0 & 0 \end{pmatrix}$, an element in $\vartheta(V)(Af_{t(i),2}, Af_{s(i),2}) = f_{t(i),2} V f_{s(i),2}$.

Consider the set $L = (D'; \omega_D; b_1, ..., b_n; v_{1,1}, ..., v_{n,1}, v_{1,2}, ..., v_{n,2})$. We will see that $L$ is a layer for $\mathcal{D}(\Lambda)$. Here $D'$ is a minimal category, so $L.1$ is satisfied. Properties $(a)$, $(b)$ and $(c)$ imply $L.2$ and $L.4$. By (1) of Proposition 3.1 of [8], we have $L.3$.

For proving $L.5$ observe that $m(\alpha_i^*) = \sum_{s,t} \alpha_i^*(\alpha_s\alpha_t)\alpha_t^* \otimes \alpha_s^*$, then

$$\delta_1(b_i) = V(1, b_i)\omega_{X_{t(i),1}} - V(b_i, 1)\omega_{X_{s(i),2}} = -\delta(x_i) =$$

$$\sum_{s,t} \alpha_i^*(\alpha_s\alpha_t)(v_{t,1}x_s - x_t v_{s,2}) = \sum_{s,t} \alpha_i^*(\alpha_s\alpha_t)(b_s v_{t,1} - v_{s,2}b_t).$$

Then by our choice of the $\alpha_i$, we have $\alpha_i^*(\alpha_s\alpha_t) = 0$ for $s \geq i$ or $t \geq i$. This proves $L.5$, therefore $L$ is a layer for $\mathcal{D}(\Lambda)$.

In the following we put $\mathcal{D}(\Lambda) = \mathcal{D}$ and $X_{i,j} = Af_{i,j}$ for $i = 1, ..., n; j = 1, 2$.

There is an equivalence of categories $\Xi : \mathcal{D}\text{-Mod} \to P^1(\Lambda)$. If $M \in \mathcal{D}$- Mod then,

$$\Xi(M) : \bigoplus_{i=1}^{n} \Lambda e_i \otimes_k M(X_{1,i}) \to \bigoplus_{i=1}^{n} \Lambda e_i \otimes_k M(X_{2,i}),$$

such that for $m_i \in M(X_{1,i})$, and $c_i \in \Lambda e_i$,

$$\Xi(M)(\sum_{i=1}^{n} c_i \otimes m_i) = \sum_{j=1}^{n} c_{s(j)}\alpha_j \otimes M(b_j)(m_{s(j)}).$$

For a morphism of the form $f = (f^0, f^1) : M \to N$ in $\mathcal{D}$-Mod, $\Xi(f)$ is given by the pair of morphisms:

$$\Xi(f)_u : \bigoplus_{i=1}^{n} \Lambda e_i \otimes_k M(X_{u,i}) \to \bigoplus_{i=1}^{n} \Lambda e_i \otimes_k N(X_{u,i}), \quad u = 1, 2$$

such that for $m_i \in M(X_{i,u})$ and $c_i \in \Lambda e_i$ we have

$$\Xi(f)_u(\sum_{i=1}^{n} c_i \otimes m_i) = \sum_{i=1}^{n} c_i \otimes f^0_{X_{i,u}}(m_i) + \sum_{j=1}^{n} c_{s(j)}\alpha_j \otimes f^1(v_{j,u})(m_{s(j)}).$$

Observe that if $M$ is a proper $\mathcal{D}$-$k(x)$-bimodule then $\Xi(M)$ is an object in $P^1(\Lambda^{k(x)})$, and if $f : M \to N$ is a morphism between proper $\mathcal{D}$-$k(x)$-bimodules then $\Xi(f)$ is a morphism in $P^1(\Lambda^{k(x)})$. Therefore $\Xi$ induces an equivalence:

$$\Xi^{k(x)} : \mathcal{D}\text{-}k(x)\text{-Mod}^p \to P^1(\Lambda^{k(x)}).$$

**Lemma 9.1.** *There are constants $l_1$ and $l_2$ such that if we have an almost split sequence in $\mathcal{D}(\Lambda)$-mod starting in $H'$ and ending in $H$ such that $\Xi H$ is not $\mathcal{E}$-injective, then $|H'| \leq l_1|H|$ and $|H| \leq l_2|H'|$.*

**Proof.** We put $l = \dim_k\Lambda$. Suppose $\Xi H = f : P_1 \to P_2$, here $\Xi H$ is indecomposable and it is not $\mathcal{E}$-injective. Therefore, $\Xi H$ has not direct summands of the form $P \to 0$, this implies that $\ker f$ is contained in $\mathrm{rad}P_1$, then $f$ induces a monomorphism $P_1/\mathrm{rad}P_1 \to \mathrm{Im}f/\mathrm{radIm}f$, consequently $\dim_k(P_1/\mathrm{rad}P_1) \leq \dim_k \mathrm{Im}f \leq \dim_k P_2$. Then we have:

$$\dim_k Cok(\Xi H) \leq \dim_k P_2 \leq \dim_k P_1 + \dim_k P_2 \leq |H|l.$$

Moreover:

$$\dim_k P_2 \leq l \dim_k(P_2/\mathrm{rad}P_2) \leq l \dim_k Cok(\Xi H)$$

and $|H| = \dim_k(P_1/\mathrm{rad}P_1) + \dim_k(P_2/\mathrm{rad}P_2) \leq \dim_k P_2 + \dim_k Cok(\Xi H)$
$\leq (1+l)\dim_k Cok(\Xi H)$.

On the other hand, there is a constant $l_0$ such that for all non projective indecomposable $M \in \Lambda$-mod, $\dim_k M \leq l_0 \dim_k DtrM$ (see proof of Theorem D in [5]). By Propositions 3.10 and 3.13, $Cok(\Xi H') \cong DtrCok(\Xi H)$. Then $\dim_k Cok(\Xi H') \leq l_0 \dim_k Cok(\Xi H)$. Therefore :

$$|H'| \leq \dim_k(Cok(\Xi H'))(1+l) \leq$$

$$l_0 \dim_k(Cok(\Xi H))(1+l) \leq l_0|H|l(1+l) = l_1|H|.$$

The second part of our statement is proved in a similar way. □

**Theorem 9.2.** *Let $\mathcal{D} = (D, V)$ be the Drozd's bocs of a tame algebra $\Lambda$. Then ($\mathcal{D}$-Mod, $\mathcal{E}_\mathcal{D}$) is an exact category, with $\mathcal{E}_\mathcal{D}$ the class of proper exact sequences. This exact category restricted to $\mathcal{D}$-mod has almost split sequences in the sense of Definition 2.5. Given a positive integer $r$, there is a composition of reduction functors $\theta : D \to B$ with $\mathcal{B} = (B, V_B) = \mathcal{D}^B$ a minimal layered bocs having the following properties.*

*(i) For any indecomposable $M \in \mathcal{D}$-mod with $|M| \leq r$ there is a $N \in \mathcal{B}$-mod with $M \cong \theta^*(N)$. Moreover any proper almost split sequence in $\mathcal{D}$-mod starting or ending in an indecomposable $M$ with $|M| \leq r$ is the image under $\theta^*$ of an almost split sequence (in the sense of Definition 2.1) in $\mathcal{B}$-mod.*

*(ii) The image under $\theta^*$ of a proper exact sequence in $\mathcal{B}$-mod is a proper exact sequence in $\mathcal{D}$-mod.*

*(iii) The image under $\theta^*$ of a proper almost split sequence in $\mathcal{B}$-mod is an almost split sequence in $\mathcal{D}$-mod.*

*(iv) Let $Z_1, ..., Z_s$ be all the marked objects of $\mathrm{ind}B$ with*

$$R_i = B(Z_i, Z_i) = k[x, h_i(x)^{-1}], \quad h_i(x) \in k[x],$$

*and $M(Z_i, p, m)$, $Q_{Z_i}$, the indecomposable objects in $\mathcal{B}$-Mod defined in section 5 and 6 respectively. Then $B_i = \mathrm{Hom}_B(Z_i, -)$ is a $B$-$R_i$-bimodule such that $Q_{Z_i} \cong B_i \otimes_{R_i} k(x)$ and $M(Z_i, p, m) \cong B_i \otimes_{R_i} R_i/(p^m)$.*

*Take the $D$-$R_i$-bimodule $D_i = \theta^*(B_i)$, then*

$$\theta^*(Q_{Z_i}) \cong D_i \otimes_{R_i} k(x), \quad \textit{and} \quad \theta^*(M(Z_i, p, n)) \cong D_i \otimes_{R_i} R_i/(p^m).$$

*Moreover, $\mathbf{dim}(D_i \otimes_{R_i} R_i/(p^m)) = m\,\mathbf{dim}_{k(x)}(D_i \otimes_{R_i} k(x))$.*

**Proof.** There is an equivalence $\Xi : \mathcal{D}\text{-Mod} \to P^1(\Lambda)$, observe that if $(a)$ is a pair of composable morphisms $X \to E \to Y$ in $\mathcal{D}$-Mod, $\Xi(a)$ is a sequence in the class $\mathcal{E}$ in $P^1(\Lambda)$ if and only if $(a)$ is a proper exact sequence. Therefore if $\mathcal{E}_1$ is the class of proper exact sequences in $\mathcal{D}$-mod, the pair $(\mathcal{D}\text{-mod}, \mathcal{E}_1)$ is an exact category with almost split sequences, moreover if $(a)$ is a pair of composable morphisms in $\mathcal{D}$-mod, $\Xi(a)$ is an almost split $\mathcal{E}$-sequence if and only if $(a)$ is an almost split $\mathcal{E}_1$-sequence.

Take the number $r(1+l)$, with $l = max\{l_1, l_2\}$, $l_1, l_2$ the constants of Lemma 9.1. Then by Theorem 8.5 there is a composition of reduction functors $\theta_1 : D \to C$ with $\mathcal{C} = (C, V_C) = \mathcal{D}^C$ a minimal bocs with layer $(C'; \omega; w_1, ..., w_s)$ such that the full and faithful functor $\theta_1^* : \mathcal{C}\text{-Mod} \to \mathcal{D}\text{-Mod}$ has the property that for all $M \in \mathcal{D}$-Mod with $|M| \leq r$, there is a $N \in \mathcal{C}$-Mod with $(\theta_1)^*(N) \cong M$. Take now $\theta_2 : C \to B$ the deletion of all marked indecomposable objects $Z \in \text{ind}C$ with $|t^{\theta_1}(\mathbf{d}_Z)| > r$, where $\mathbf{d}_Z \in \text{Dim}(\mathcal{C})$ with $\mathbf{d}_Z(Z) = 1$, and $\mathbf{d}_Z(Z') = 0$ for $Z' \neq Z$, $Z' \in \text{ind}C$. Then we have $\theta = \theta_2\theta_1 : D \to B$ and $\mathcal{B} = (B, V_B) = ((\mathcal{D})^C)^B = \mathcal{D}^B$ is a minimal layered bocs.

(i) Take an indecomposable object $M \in \mathcal{D}$-mod with $|M| \leq r$, then there is a $N_1 \in \mathcal{C}$-mod with $(\theta_1)^*(N_1) \cong M$. Since $N_1$ is an indecomposable object in the minimal bocs $\mathcal{C}$, then either $M \cong M(Z, p, m)$ for some marked $Z \in \text{ind}C$ or $M \cong S_Z$ for some non-marked $Z \in \text{ind}C$. In the first case $|t^{\theta_1}(\mathbf{dim}N_1)| = m|t^{\theta_1}(\mathbf{d}_Z)| = |\mathbf{dim}M| \leq r$. Thus, $|t^{\theta_1}(\mathbf{d}_Z)| \leq r$. Consequently, in both cases $N_1(W) = 0$ for $W$ a marked object in ind$C$ with $|t^{\theta_1}(\mathbf{d}_W)| > r$, then there is a $N \in \mathcal{B}$-mod with $N_1 \cong (\theta_2)^*(N)$. Therefore $M \cong \theta^*(N)$ proving the first part of (i). For the second part take $M \to E \to L$ a proper almost split sequence in $\mathcal{D}$-mod, then if either $M$ or $L$ have dimension equal or smaller than $r$, all indecomposable summands of the other terms of the sequence have dimension equal or smaller than $(l+1)r$, consequently our proper almost split sequence is isomorphic to the image under $(\theta_1)^*$ of an almost split sequence (in the sense of Definition 2.1) $(a_1)$ : $M_1 \to E_1 \to L_1$ in $\mathcal{C}$-mod. Then if $M_1$ or $L_1$ is an object of the form $M(Z, p, m)$, with $Z$ a marked object in ind$C$, we have $M_1 \cong L_1$ and $E_1 = M(Z, p, m-1) \oplus M(Z, p, m+1)$. Here $|M(Z, p, m)| \leq r$ implies $|t^{\theta_1}(\mathbf{d}_Z)| \leq r$, then the sequence $(a_1)$ is the image under $(\theta_2)^*$ of an almost split sequence in $\mathcal{B}$-mod. In case that $M_1$ or $L_1$ is an object of the form $S_Z$ for a non marked object in ind$C$, then all other terms of $(a_1)$ are sums of objects of the form $S_W$ with $W$ a non-marked object in ind$C$. Therefore, again $(a_1)$ is the image under $(\theta_2)^*$ of an almost split sequence in $\mathcal{B}$-mod. This proves the second part of (i).

(ii) Follows from Lemma 7.7.

(iii) Take now $Z$ a marked indecomposable in $B$ and $M(Z, p, 1) \in \mathcal{B}$-mod with $p$ a fixed prime element in $R_Z = B(Z, Z)$. By definition of $B$ we have $|t^{\theta}(\mathbf{d}_Z)| \leq r$ and $\theta_2(Z) = Z \in C$. There is a non-trivial proper sequence ending and starting in $M(Z, p, 1)$, since $\theta^*$ is a full and faithful functor, there is a non-trivial proper exact sequence ending and starting in $\theta^*(M(Z, p, 1))$. Then $H = \theta^*(M(Z, p, 1))$ is not $\mathcal{E}_1$-projective. Therefore, there is an almost split sequence $(a) : H' \to H_0 \to H$. By the second part of (i) the sequence $(a)$ is the image under $\theta^*$ of an almost split sequence $(b)$ in $\mathcal{B}$-mod. Then using Proposition 2.6 we obtain (iii).

(iv) The first part follows from the definition of $\theta^*$. For proving the second part take

$X$ an indecomposable object in $D$ and assume $\theta(X) = \bigoplus_{j=1}^{t} n_j Z_j$, where $Z_1, ..., Z_j$ are all indecomposable objects of $B$. Then for each $i \in \{1, ..., s\}$:

$$\mathbf{dim}_{k(x)}(\theta^* B_i \otimes_{R_i} k(x))(X) = \dim_{k(x)}(B(Z_i, \theta(X)) \otimes_{R_i} k(x)) =$$

$$\dim_{k(x)}(R_i^{n_i} \otimes_{R_i} k(x)) = n_i.$$

On the other hand:

$$t^{\theta}(\mathbf{d}_{Z_i})(X) = \mathbf{d}_{Z_i}(\theta(X)) = n_i.$$

Therefore $t^{\theta}(\mathbf{d}_{Z_i}) = \mathbf{dim}(\theta^* B_i \otimes_{R_i} k(x))$. Then

$$\mathbf{dim}(D_i \otimes_{R_i} R_i/(p^m)) = \mathbf{dim}(\theta^*(M(Z_i, p, m)) = mt^{\theta}(\mathbf{d}_{Z_i}),$$

proving (iv). □

In the following we put $\Lambda^{k(x)} = \Lambda \otimes_k k(x)$.

**Definition 9.3.** If $R$ is a $k$-algebra a $P(\Lambda)$-$R$-bimodule is a morphism $X = f_X : P_X \to Q_X$, where $P_X$ and $Q_X$ are $\Lambda$-$R$-bimodules which are projectives as left $\Lambda$-modules and $f_X$ is a morphism of $\Lambda$-$R$-bimodules. If $Z$ is a left $R$-module, $X \otimes_R Z = f \otimes 1 : P_X \otimes_R Z \to Q_X \otimes_R Z$.

We recall from section 3 that if $X : P_X \to Q_X$ is an object in $p^1(\Lambda)$, then $\mathbf{dim}X = (\mathbf{dim}(\mathrm{top}P_X), \mathbf{dim}(\mathrm{top}Q_X))$. Then if $H' \in \mathcal{D}$-mod, $\mathbf{dim}(\Xi H') = \mathbf{dim}H'$. In case $X \in p^1(\Lambda^{k(x)})$ we put $\mathbf{dim}_{k(x)}X = (\mathbf{dim}_{k(x)}(\mathrm{top}P_X), \mathbf{dim}_{k(x)}(\mathrm{top}Q_X))$, then if $H' \in \mathcal{D}$-$k(x)$-mod, we have $\mathbf{dim}_{k(x)}(\Xi H') = \mathbf{dim}_{k(x)}H'$.

An indecomposable object $H = f_H : P_H \to Q_H$ in $P(\Lambda)$ which is not in $p(\Lambda)$ is called generic if $P_H$ and $Q_H$ have finite length as $\mathrm{End}_{P(\Lambda)}(H)$-modules. A structure of $P(\Lambda)$-$k(x)$-bimodule for $H$ is called admissible in case $\mathrm{End}_{P(\Lambda)}(H) = k(x)_m \oplus \mathcal{R}$, where $\mathcal{R} = \mathrm{radEnd}_{P(\Lambda)}(H)$ and $k(x)_m$ denotes the set of morphisms $h : H \to H$ of the form $h = (m(x)id_{P_H}, m(x)id_{Q_H})$ with $m(x) \in k(x)$.

**Definition 9.4.** Suppose $\hat{T} = f_{\hat{T}} : P_{\hat{T}} \to Q_{\hat{T}}$ is a $P(\Lambda)$-$R$-bimodule with $R$ a finitely generated localization of $k[x]$ and $P_{\hat{T}}, Q_{\hat{T}}$ finitely generated as right $R$-modules. We say that $\hat{T}$ is a realization of $H$ if $\hat{T} \otimes_R k(x) \cong H$. The realization $\hat{T}$ of $H$ over $R$ is called good if:

(i) $P_{\hat{T}}$ and $Q_{\hat{T}}$ are free as right $R$-modules;

(ii) the functor $\hat{T} \otimes_R - : R\text{-Mod} \to P(\Lambda)$ preserves isomorphism classes and indecomposable objects;

(iii) for $p$ a prime in $R$, and $n$ a positive integer $\hat{T} \otimes_R S_{p,n}$ is an almost split sequence, where $S_{p,n}$ is the sequence given in (iii) of Definition 1.1.

We are now ready for giving a version of Theorem 1.2 for $P(\Lambda)$.

**Theorem 9.5.** *Let $\Lambda$ be a finite-dimensional algebra over an algebraically closed field $k$ of tame representation type. Let $r$ be a positive integer. Then there are indecomposable objects in $p^1(\Lambda)$, $\hat{L}_1, ..., \hat{L}_t$ with $|\hat{L}_j| \leq r$ for $j = 1, ..., t$ and generic objects in $P^1(\Lambda)$ with admissible structure of $P(\Lambda)$-$k(x)$-bimodules, $H_1, ..., H_s$ such that for $j = 1, ..., s$, $H_j$ has a good realization $\hat{T}_j$ over $R_j$, a finitely generated localization of $k[x]$, with the following properties:*

*(i) If $X$ is an indecomposable object in $p^1(\Lambda)$ with $|X| \leq r$, then either $X \cong \hat{L}_j$ for some $j \in \{1, ..., t\}$ or $X \cong \hat{T}_i \otimes_{R_i} R_i/(p^m)$ for some $i \in \{1, ..., s\}$, some prime element $p \in R_i$ and some natural number $m$.*

*(ii) If $X = \hat{T}_i \otimes_{R_i} R_i/(p^m), Y = \hat{T}_j \otimes_{R_j} R_j/(q^n)$, with $i, j \in \{1, ..., s\}$, $p$ a prime in $R_i$, $q$ a prime in $R_j$, and $\hat{L}_u$ with $u \in \{1, ..., t\}$, then*

$$\dim_k \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X, Y) = mn\dim_{k(x)}\mathrm{rad}_{p^1(\Lambda^{k(x)})}(H_i, H_j),$$

$$\dim_k \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X, \hat{L}_u) = m\dim_{k(x)}\mathrm{rad}_{p^1(\Lambda^{k(x)})}(H_i, \hat{L}_u^{k(x)}),$$

$$\dim_k \mathrm{rad}^{\infty}_{p^1(\Lambda)}(\hat{L}_u, X) = m\dim_{k(x)}\mathrm{rad}_{p^1(\Lambda^{k(x)})}(\hat{L}_u^{k(x)}, H_i).$$

*(iii) If $X = \hat{T}_i \otimes_{R_i} R_i/(p^m), Y = \hat{T}_j \otimes_{R_j} R_j/(q^n)$, then if $i = j$ and $p = q$,*

$$\mathrm{Hom}_{p^1(\Lambda)}(X, Y) \cong \mathrm{Hom}_{R_i}(R_i/(p^n), R_i/(p^m)) \oplus \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X, Y).$$

*If $i \neq j$ or $i = j$ and $(p) \neq (q)$:*

$$\mathrm{Hom}_{p^1(\Lambda)}(X, Y) = \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X, Y).$$

*Moreover:*

$$\mathrm{Hom}_{p^1(\Lambda)}(\hat{L}_u, X) = \mathrm{rad}^{\infty}_{p^1(\Lambda)}(\hat{L}_u, X), \quad \mathrm{Hom}_{p^1(\Lambda)}(X, \hat{L}_u) = \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X, \hat{L}_u).$$

**Proof.** We apply Theorem 8.5 for the Drozd's bocs $\mathcal{D} = (D, V_D)$ of $\Lambda$ and the positive integer $r(l+1)$ with $l = max\{l_1, l_2\}$ where $l_1, l_2$ are the integers given in Lemma 9.1. Then we obtain a minimal layered bocs $\mathcal{B} = (B, V_B)$ having properties $(i)$-$(iv)$ of Theorem 9.2. We have the reduction functor $\theta : D \to B$, suppose $\theta(X_{j,i}) = \bigoplus_l n^l_{j,i} Z_l$ with $j = 1, 2$ and $i = 1, ..., n$ given in the beginning of this section.

Let $Z_1, ..., Z_s$ be the marked objects of ind$B$ and $Z_{s+1}, ..., Z_{s+t}$ be the non-marked objects. We have $B_i, R_i$ and $D_i$ given in $(iv)$ of Theorem 9.2.

Consider $\hat{T}_i = \Xi D_i$. $\hat{T}_i = g_i : P_i \to Q_i$, then:

$$P_i = \bigoplus_v \Lambda e_v \otimes D_i(X_{1,v}) = \bigoplus_v \Lambda e_v \otimes_k \mathrm{Hom}_B(Z_i, \theta(X_{1,v})) \cong \bigoplus_v \Lambda e_v \otimes_k n^i_{1,v} R_i.$$

Similarly $Q_i \cong \bigoplus_v \Lambda e_v \otimes_k n^i_{2,v} R_i$. If $\lambda \in \Lambda e_v$, and $m \in D_i(X_{1,v})$, then:

$$g_i(\lambda \otimes m) = \sum_{d_j : X_{1,s(j)} \to X_{2,t(j)}, s(j) = v} \lambda \alpha_j \otimes \mathrm{Hom}_B(1, \theta(b_j))(m)$$

We have

$$H_i = \Xi D_i \otimes_{R_i} k(x) = f_i : P_{H_i} \to Q_{H_i}, P_{H_i} = P_i \otimes_{R_i} k(x), Q_{H_i} = Q_i \otimes_{R_i} k(x),$$

with $f_i = g_i \otimes 1_{k(x)}$, therefore $H_i = \hat{T}_i \otimes_{R_i} k(x)$.

Moreover, $P_{H_i} \cong \bigoplus_v n^i_{1,v} \Lambda^{k(x)}(e_v \otimes 1)$ and $Q_{H_i} \cong \bigoplus_v n^i_{2,v} \Lambda^{k(x)}(e_v \otimes 1)$.

For $i = 1, ..., s$ consider the objects $H_i \in P^1(\Lambda)$. For all $i = 1, ..., s$ we have an isomorphism induced by the functor $\Xi\theta^*$:

$$\mathrm{End}_{\mathcal{B}}(Q_{Z_i}) = \mathrm{End}_{\mathcal{B}}(Q_{Z_i})^0 \oplus \mathrm{End}_{\mathcal{B}}(Q_{Z_i})^1 \to \mathrm{End}_{P^1(\Lambda)}(H_i),$$

where $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})^0$ denotes the morphisms of the form $(f^0, 0)$ and $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})^1$ denotes the morphisms of the form $(0, f^1)$. Here $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})^0 \cong \mathrm{End}_{R_i}(k(x)) = k(x)_m$, where $k(x)_m$ denotes the right multiplication by elements of $k(x)$. Here $\mathcal{B}$ is a layered bocs, therefore a morphism $(f^0, f^1)$ is an isomorphism if and only if $f^0$ is an isomorphism, thus the elements in $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})^1$ are the non-units in $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})$. Thus since the sum of non-units is again non-unit, $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})$ is a local ring and its radical is $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})^1$. The image under $\Xi\theta^*$ of an element in $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})^0$ is of the form $(id_{P_{H_i}} m(x), id_{Q_{H_i}} m(x))$, with $m(x) \in k(x)$. From here we obtain that the $P(\Lambda)$-$k(x)$-structure of $H_i$ is admissible. Clearly, $\hat{T}_i$ is a realization of $H_i$.

In order to prove that $\hat{T}_i$ is a good realization of $H_i$, we must prove conditions (i), (ii) and (iii) of Definition 9.4. Condition (i) is clear. For proving condition (ii) take $\epsilon_{\mathcal{B}} : V_{\mathcal{B}} \to B$ the counit of the bocs $\mathcal{B}$. By Lemma 5.3 the functor $(id_B, \epsilon_{\mathcal{B}})^* : B$-Mod $\to \mathcal{B}$-Mod preserves indecomposables and isomorphism classes. Consider $\hat{B}_i$ the full subcategory of $B$ whose unique indecomposable object is $Z_i$, then we have the composition $\eta_i$ of full and faithful functors:

$$R_i\text{-Mod} \to \hat{B}_i\text{-Mod} \to B\text{-Mod}.$$

The composition:

$$R_i\text{-Mod} \xrightarrow{\eta_i} B\text{-Mod} \xrightarrow{(id_B, \epsilon_{\mathcal{B}})^*} \mathcal{B}\text{-Mod} \xrightarrow{\theta^*} \mathcal{D}\text{-Mod} \xrightarrow{\Xi} P^1(\Lambda)$$

is isomorphic to $\hat{T}_i \otimes_{R_i} -$. Therefore the functor $\hat{T}_i \otimes_{R_i} -$ preserves isomorphism classes and indecomposable modules. The condition (iii) of Definition 9.4 is a consequence of (iii) of Theorem 9.2.

Now, we may assume that $\hat{L}_j = \Xi\theta^*(S_{Z_{s+j}})$ for $j = 1, ..., t$ is such that $|\hat{L}_j| \leq r$.

(i) Take $X$ an indecomposable object in $p^1(\Lambda)$ with $|X| \leq r$, then by (i) of Theorem 9.2 there is an indecomposable object $N$ in $\mathcal{B}$-mod with $\Xi\theta^*(N) \cong X$. Since $N$ is indecomposable, then $N \cong S_{Z_{s+j}}$ for some $j = 1, ..., t$ and then either $X \cong \hat{L}_j$, or $N \cong M(Z_i, p, n)$ for some $i = 1, ..., s$, some prime element $p \in R_i$ and some positive integer $n$, in this case by (iv) of Theorem 9.2 we have $M(Z_i, p, n) \cong B_i \otimes_{R_i} R_i/(p^n)$. Then $X \cong \Xi\theta^* B_i \otimes_{R_i} R_i/(p^n) \cong \hat{T}_i \otimes_{R_i} R_i/(p^n)$. Thus we have proved i).

(ii) Consider $\mathcal{C}$ the full subcategory of $p^1(\Lambda)$ whose objects are the objects of the form $\hat{T}_i \otimes_{R_i} R_i/(p^m)$. We have already proved that $\hat{T}_i$ is a good realization of $H_i$, then by property (iii) of Definition 9.4 the category $\mathcal{C}$ consists of whole Auslander-Reiten components of $p^1(\Lambda)$, thus $\mathcal{C}$ has property $(A)$ of section 2, then by Corollary 2.4 for $X = \hat{T}_i \otimes_{R_i} R_i/(p^m)$, $Y = \hat{T}_i \otimes_{R_i} R_i/(q^n)$, $\dim_k \mathrm{rad}^\infty_{p^1(\Lambda)}(X,Y) = \dim_k \mathrm{rad}^\infty_{\mathcal{C}}(X,Y) = \dim_k \mathrm{rad}^\infty_{\mathcal{B}}(M(Z,p,m), M(Z',q,n))$.

We recall from the discussion at the beginning of section 6 that the full and faithful functor $\theta^* : \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$ restricts to a full and faithful functor $(\theta^*)^{k(x)} : \mathcal{B}\text{-}k(x)\text{-Mod}^p \to \mathcal{D}\text{-}k(x)\text{-Mod}^{op}$. Then the first equality of (ii) follows from that of Proposition 6.5.

Observe that $\hat{L}_u^{k(x)} = \Xi\theta^*(S_{Z_{s+u}})^{k(x)} \cong \Xi\theta^*(S^{k(x)}_{Z_{s+u}})$. The second and third equality of (ii) follow from those of Proposition 6.5.

(iii) Follows from Corollary 5.11 and from Corollary 2.4. □

## 10 Hom-spaces in $\Lambda$-Mod

In this section we discuss the Hom-spaces in $\Lambda$-Mod for a tame algebra $\Lambda$ and prove our main result, Theorem 1.2. For $X = f_X : P_X \to Q_X \in p(\Lambda)$ we define $|X| = |\mathbf{dim}X| = \dim_k(P_X/\mathrm{rad}P_X) + \dim_k(Q_X/\mathrm{rad}Q_X)$.

There is an integer $l_0$ such that for any indecomposable non-injective $\Lambda$-module $M$, $\dim_k trDM \leq l_0 \dim_k M$. Let $d$ be any positive integer greater than $\dim_k\Lambda$, consider $d_0 = d(1+l_0)$ take $s(d_0) = (\dim_k(\Lambda)+1)d_0$. If $M \in \Lambda$-mod with $\dim_k M \leq d_0$ and $X = f_X : P_X \to Q_X$ is a minimal projective presentation of $M$, we have $\dim_k(Q_X/\mathrm{rad}Q_X) \leq d_0$ and $\dim_k(P_X/\mathrm{rad}P_X) \leq \dim_k(\mathrm{Im}f_X) \leq \dim_k Q_X \leq \dim_k(M/\mathrm{rad}M)\dim_k\Lambda \leq d_0\dim_k\Lambda$, so $|X| \leq s(d_0)$. Taking the number $r = s(d_0)(1+l)$ in Theorem 9.5 with $l = max\{l_1, l_2\}$, where $l_1$ and $l_2$ are the constants of Lemma 9.1, we obtain the generic objects in $P(\Lambda)$, $H_1, ..., H_s$ with admissible $\Lambda$-$k(x)$ structures and the indecomposables in $p^1(\Lambda)$, $\hat{L}_1, ..., \hat{L}_t$. For each $i = 1, ..., s$ we have the realizations $\hat{T}_i$ over $R_i$ of $H_i$. We have the generic $\Lambda$-modules $G_i = Cok(H_i)$ and the following isomorphism of $\Lambda$-$k(x)$-bimodules, $G_i = Cok(H_i) \cong Cok(\hat{T}_i \otimes_{R_i} k(x)) \cong Cok(\hat{T}_i) \otimes_{R_i} k(x)$, with $T_i = Cok(\hat{T}_i)$ a $\Lambda$-$R_i$-bimodule finitely generated as right $R_i$-module. The $\Lambda$-$k(x)$ structure of $H_i$ is admissible, then $\mathrm{End}_{P(\Lambda)}(H_i) = k(x)_m \oplus \mathcal{R}_i$ with $\mathcal{R}_i$ a nilpotent ideal. Then, $\mathrm{End}_\Lambda(G_i) = k(x)id_{G_i} \oplus \mathrm{radEnd}_\Lambda(G_i)$, therefore, the endolength of $G_i$ coincides with $\dim_{k(x)}G_i$. Consequently, $T_i$ is a realization of $G_i$.

**Lemma 10.1.** *$G_i$ and $T_i$ satisfy the conditions (ii) and (iii) of Definition 1.1.*

**Proof.** Take $W \in R_i$-Mod, we claim that $\hat{T}_i \otimes_{R_i} W$ has not indecomposable direct summands of the form $Z(P) = P \to 0$. Suppose some indecomposable $Z(P)$ is a direct summand of $\hat{T}_i \otimes_{R_i} W = \Xi\theta^*(W')$, with $W' = (id_B, \epsilon_{\mathcal{B}})^*\eta_i(W)$. Here $Z(P)$ is injective in $P^1(\Lambda)$, then $Z(P) = \Xi\theta^*(S_{Z_u})$ for some non-marked indecomposable object $Z_u \in B$. Since the functor $\Xi\theta^*$ is full and faithful, we have that $S_{Z_u}$ is direct summand of $W'$, but this is impossible because $W'(Z_u) = 0$. The above proves that $\hat{T}_i \otimes_{R_i} W$ is in $P^2(\Lambda)$, the full subcategory of $P^1(\Lambda)$ whose objects have not direct summands of the form $Z(P)$.

Now the functor $Cok : P^2(\Lambda) \to \Lambda$-Mod preserves indecomposables and isomorphism classes (see (2) of Lemma 3.2 of [6]). Consequently, the functor $Cok(\hat{T}_i \otimes_{R_i} -) \cong T_i \otimes_{R_i} -$ preserves indecomposables and isomorphism classes. This proves that $T_i$ has property (ii) of Definition 1.1.

For proving condition (iii) of Definition 1.1 take $p$ a prime element in $R_i$. There is an almost split sequence in $p^1(\Lambda)$ starting in $\hat{T}_i \otimes_{R_i} R_i/(p^m)$, therefore this object is not injective in $p^1(\Lambda)$ and therefore its cokernel is not zero. By Proposition 3.13 the image under the functor $Cok$ of the almost split sequence starting in $\hat{T}_i \otimes_{R_i} R_i/(p^m)$ is an almost split sequence in $\Lambda$-mod. This proves that the $\Lambda$-$R_i$-bimodule $T_i$ satisfies condition iii) for all $i \in \{1, ..., s\}$. □

**Lemma 10.2.** *Let $L_j = Cok(\hat{L}_j)$ with $j = 1, ..., t$. If $M$ is an indecomposable $\Lambda$-module with $\dim_k M \leq d$, then $M$ has the form given in (i) of Theorem 1.2.*

**Proof.** There is an indecomposable object $X \in p^1(\Lambda)$ with $M \cong Cok(X)$, since $|X| \leq s(d) \leq r$, $X \cong \hat{T}_i \otimes_{R_i} R_i/(p^m)$ or $X \cong \hat{L}_j$. But then either $M \cong Cok(\hat{T}_i \otimes_{R_i} R_i/(p^m)) \cong T_i \otimes_{R_i} R_i/(p^m)$, or $M \cong L_j$. This proves the first part of (i). For the second part of (i), by Proposition 5.9 of [1] we have that if $X$ is an indecomposable object in $p^1(\Lambda)$ with $Cok(X)$ non-simple injective, then there is an almost split sequence in $p(\Lambda)$ starting in $X$ and ending in an injective object with all its terms in $p^1(\Lambda)$, so this is an almost split sequence in $p^1(\Lambda)$. If $Cok(X)$ is simple then $X$ is injective in $p^1(\Lambda)$, if $Cok(X)$ is projective, then $X$ is projective in $p^1(\Lambda)$. Now if $X \cong \hat{T}_i \otimes_{R_i} R_i/(p^m)$, since $\hat{T}_i$ is a good realization of $H_i$, there is an almost split sequence starting and ending in $X$. Therefore, if $M$ is an injective, projective or simple $\Lambda$-module, then $M \cong L_j$ for some $j = 1, ..., t$. □

**Lemma 10.3.** *Let $X = \hat{T}_i \otimes_{R_i} R_i/(p^n), Y = \hat{T}_i \otimes_{R_i} R_i/(p^m)$, $M = CokX$, $N = CokY$, then the functor $Cok$ induces an isomorphism:*

$$\underline{Cok} : \mathrm{Hom}_{P^1(\Lambda)}(X, Y)/\mathrm{rad}^\infty(X, Y) \to \mathrm{Hom}_\Lambda(M, N)/\mathrm{rad}^\infty(M, N).$$

**Proof.** In fact, take a morphism $u : X \to Y$ such that $Cok(u) = 0$. Then by Proposition 3.3, $u$ is a morphism which is a sum of compositions of the form $u_2 u_1$ with $u_1 : X \to W$, $u_2 : W \to Y$ and $W$ an indecomposable injective in $P(\Lambda)$. Then either $W = Z(P) = (P \xrightarrow{0} 0)$ or $W = J(P) = (P \xrightarrow{id_P} P)$ for some indecomposable projective $\Lambda$-module $P$. In the first case $W$ is also an injective object in $p^1(\Lambda)$, then $W$ is not in the Auslander-Reiten component containing $X$, therefore $u_2 u_1 \in \mathrm{rad}^\infty(X, Y)$. Now, if $W = J(P)$, we recall (see Lemma 3.6) that there is a right minimal almost split morphism $\sigma(P) : R(P) \to J(P)$, then $u_1 = \sigma(P) u_1'$, with $u_1' : X \to R(P)$. Here $R(P)$ is injective in $p^1(\Lambda)$, then $u_2 u_1 = u_2 \sigma(P) u_1'$ is in $\mathrm{rad}^\infty(X, Y)$, therefore, $u \in \mathrm{rad}^\infty(X, Y)$, proving our Lemma. □

**Lemma 10.4.** *If $M = T_i \otimes_{R_i} R_i/(p^m)$, $N = T_j \otimes_{R_j} R_j/(q^n)$, $L_u^{k(x)} = L_u^{k(x)}$ with $i, j \in \{1, ..., s\}$, $u \in \{1, ..., t\}$, $p$ a prime element of $R_i$, $q$ a prime element of $R_j$, then $M, N, L_u$ satisfy (iii) of Theorem 1.2.*

**Proof.** Let $M = CokX, N = CokY$, $X, Y \in p^1(\Lambda)$. If $i = j$ and $p = q$ by the first formula in (iii) of Theorem 9.5 and Lemma 10.3 we obtain our result. If $i \neq j$ or $(p) \neq (q)$ we have $\mathrm{Hom}_{p^1(\Lambda)}(X, Y) = \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X, Y)$, thus $\mathrm{Hom}_{\Lambda}(M, N) = \mathrm{rad}^{\infty}_{\Lambda}(M, N)$. Moreover, the third and fourth formula of (iii) of Theorem 9.5 gives $\mathrm{Hom}_{\Lambda}(L_u, M) = \mathrm{rad}^{\infty}_{\Lambda}(L_u, M)$ and $\mathrm{Hom}_{\Lambda}(M, L_u) = \mathrm{rad}^{\infty}_{\Lambda}(M, L_u)$ respectively. □

**Lemma 10.5.** *Let $M = T_i \otimes_{R_i} R_i/(p^m), N = T_j \otimes_{R_j} R_j/(q^n)$, for $i, j \in \{1, ..., s\}$, $p$ a prime in $R_i$, $q$ a prime in $R_j$. Then*

$$\dim_k \mathrm{rad}^{\infty}_{\Lambda}(M, N) = mn \dim_{k(x)} \mathrm{rad}_{\Lambda^{k(x)}}(G_i, G_j).$$

**Proof.** Suppose $X = \hat{T}_i \otimes_{R_i} R_i/(p^m)$ and $Y = \hat{T}_j \otimes_{R_j} R_j/(q^n)$ are minimal projective presentations of $M$ and $N$ respectively. Then if $\mathbf{z}_u = \mathbf{dim}_{k(x)} H_u$ for $u = 1, ..., s$, by (iv) of Theorem 9.2 we have $\mathbf{dim}_k X = m\mathbf{z}_i$, $\mathbf{dim}_k Y = n\mathbf{z}_j$.

Suppose now $i \neq j$ or $i = j$ and $(p) \neq (q)$. In this case $\mathrm{Hom}_{\Lambda}(M, N) = \mathrm{rad}^{\infty}_{\Lambda}(M, N)$ and $\mathrm{Hom}_{p^1(\Lambda)}(Y, X) = \mathrm{rad}^{\infty}_{p^1(\Lambda)}(Y, X)$. Here $DtrN \cong N$, then by (3) of Proposition 3.14 and the first equality in (ii) of Theorem 9.5 we obtain

$$\dim_k \mathrm{Hom}_{\Lambda}(M, N) = mn(\dim_{k(x)} \mathrm{rad}_{p^1(\Lambda^{k(x)})}(H_j, H_i) - g_{\Lambda}(\mathbf{z}_j, \mathbf{z}_i)).$$

On the other hand, since $Dtr_{\Lambda^{k(x)}} G_j \cong G_j$ (see Proposition 6.5 of [2]) we have

$$\dim_{k(x)} \mathrm{Hom}_{\Lambda^{k(x)}}(G_i, G_j) = \dim_{k(x)} \mathrm{Hom}_{p^1(\Lambda^{k(x)})}(H_j, H_i) - g_{\Lambda^{k(x)}}(\mathbf{z}_j, \mathbf{z}_i).$$

We know from Corollary 2.3 of [2], that the indecomposable projective $\Lambda^{k(x)}$-modules are of the form $P \otimes_k k(x)$, with $P$ indecomposable projective $\Lambda$-module, then $g_{\Lambda} = g_{\Lambda^{k(x)}}$. Observe that if $i \neq j$, $\mathrm{rad}_{p^1(\Lambda^{k(x)})}(H_j, H_i) = \mathrm{Hom}_{p^1(\Lambda^{k(x)})}(H_j, H_i)$ and $\mathrm{rad}_{\Lambda^{k(x)}}(G_i, G_j) = \mathrm{Hom}_{\Lambda^{k(x)}}(G_i, G_j)$, moreover for $i = j$,
$\dim_{k(x)} \mathrm{End}_{p^1(\Lambda^{k(x)})}(H_i) = 1 + \dim_{k(x)} \mathrm{radEnd}_{p^1(\Lambda^{k(x)})}(H_i)$ and
$\dim_{k(x)} \mathrm{End}_{\Lambda^{k(x)}}(G_i) = 1 + \dim_{k(x)} \mathrm{radEnd}_{\Lambda^{k(x)}}(G_i)$. Thus we obtain:

$$\dim_{k(x)} \mathrm{rad}_{\Lambda^{k(x)}}(G_i, G_j) = \dim_{k(x)} \mathrm{rad}_{p^1(\Lambda^{k(x)})}(H_j, H_i) - g_{\Lambda}(\mathbf{z}_j, \mathbf{z}_i).$$

From here we obtain our equality for $i \neq j$ or $i = j$ and $(p) \neq (q)$.

For $i = j$ and $p = q$ and the first equality of (iii) of Theorem 9.5 we obtain

$$\dim_k \mathrm{Hom}_{p^1(\Lambda)}(X, Y) = min\{m, n\} + mn \dim_{k(x)} \mathrm{radHom}_{p^1(\Lambda^{k(x)})}(H_i, H_i),$$

therefore

$$\dim_k \mathrm{Hom}_{\Lambda}(M, N) = min\{m, n\} + mn \dim_{k(x)} \mathrm{radHom}_{\Lambda^{k(x)}}(G_i, G_i).$$

By Lemma 10.4 the first equality of (iii) Theorem 1.2 holds, then we have $\dim_k \mathrm{rad}^{\infty}_{\Lambda}(M, N) = mn \dim_{k(x)} \mathrm{radEnd}_{\Lambda^{k(x)}}(G_i)$, obtaining our result. □

**Lemma 10.6.** *Let $M = T_i \otimes_{R_i} R_i/(p^m)$ for $i \in \{1, ..., s\}$, $p$ a prime element in $R_i$, $L_u = Cok(\hat{L}_u)$, for some $u \in \{1, ..., t\}$. Then*

$$\dim_k \mathrm{rad}^{\infty}_{\Lambda}(L_u, M) = m \dim_{k(x)} \mathrm{rad}_{\Lambda^{k(x)}}(G_i, L_u^{k(x)}).$$

*In particular for $\Lambda e$ an indecomposable projective $\Lambda$-module there is a $u \in \{1, ..., t\}$ such that $\Lambda e \cong L_u$, then* $\dim_k eM = m\dim_{k(x)} eG_i$.

**Proof.** Consider $\mathbf{l}_u = \mathbf{dim}_k \hat{L}_u = \mathbf{dim}_{k(x)} \hat{L}_u^{k(x)}$. We have $DtrM \cong M$, then by (3) of Proposition 3.14 and the second equality of (ii) of Theorem 9.5 we have:

$$\dim_k \mathrm{Hom}_\Lambda(L_u, M) = m\dim_{k(x)} \mathrm{Hom}_{p^1(\Lambda^{k(x)})}(H_i, \hat{L}_u^{k(x)}) - mg_\Lambda(\mathbf{z}_i, \mathbf{l}_u).$$

We have $Cok\hat{L}_u^{k(x)} \cong (Cok\hat{L}_u)^{k(x)} = L_u^{k(x)}$, thus again by 3) of Proposition 3.14, recalling that $Dtr_{\Lambda^{k(x)}} G_i \cong G_i$, we obtain:

$$\dim_{k(x)} \mathrm{Hom}_{\Lambda^{k(x)}}(L_u^{k(x)}, G_i) = \dim_{k(x)} \mathrm{Hom}_{p^1(\Lambda^{k(x)})}(H_i, \hat{L}_u^{k(x)}) - g_\Lambda(\mathbf{z}_i, \mathbf{l}_u).$$

From here we obtain the first part of our Lemma. For the second part of the Lemma, observe that by assumption, $\dim_k \Lambda \leq d$, then by Lemma 10.4 we obtain our result. □

**Lemma 10.7.** *Let $M = T_i \otimes_{R_i} R_i/(p^m)$ for $i \in \{1, ..., s\}$, $p$ a prime in $R_i$, $L_u = Cok(\hat{L}_u)$ for $u \in \{1, ..., t\}$. Then*

$$\dim_k \mathrm{rad}_\Lambda^\infty(M, L_u) = m\dim_{k(x)} \mathrm{rad}_{\Lambda^{k(x)}}(G_i, L_u^{k(x)}).$$

**Proof.** Assume first $L_u$ is injective, then we may suppose $L_u = D(e\Lambda)$. We have:

$$\dim_k \mathrm{Hom}_\Lambda(M, D(e\Lambda)) = \dim_k \mathrm{Hom}_{\Lambda^{op}}(e\Lambda, D(M)) = \dim_k D(M)e = \dim_k(eM)$$

$$= m\dim_{k(x)} \mathrm{Hom}_{\Lambda^{k(x)}}(G_i, D_x((e \otimes 1)\Lambda^{k(x)})) = m\dim_{k(x)} \mathrm{Hom}_{\Lambda^{k(x)}}(G_i, (D(e\Lambda)^{k(x)})).$$

Where $D_x(-) = \mathrm{Hom}_{k(x)}(-, k(x))$.

Now assume $L$ is not injective. Consider an almost split sequence starting in $L$:

$$0 \to L \xrightarrow{f} \oplus_{s=1}^m E_s \xrightarrow{g} L' \to 0,$$

with $E_s$ indecomposable for $s = 1, ..., m$.

By the choice of the integer $d_0$, the objects $E_s$ and $L'$ are isomorphic to objects $L_v$ or $T_j \otimes_{R_j} R_j/(p^m)$, but in this latter case $L$ is in the component of an object of the form $T_j \otimes_{R_j} R_j/(p^m)$, which implies that $L \cong T_j \otimes_{R_j} R_j/(p^n)$ for some $n$, which is not the case therefore $L' \cong L_v$ for some $v = 1, ..., t$. Then $L' \cong Cok\hat{L}_v$. Take $\mathbf{l}_v = \mathbf{dim}\hat{L}_v = \mathbf{dim}_{k(x)} \hat{L}_v^{k(x)}$.

By (3) of Proposition 3.14 and the third equality of (iii) of Theorem 9.5 we obtain

$$\dim_k \mathrm{Hom}_\Lambda(M, L) = m(\dim_{k(x)} \mathrm{Hom}_{p^1(\Lambda^{k(x)})}(\hat{L}_v^{k(x)}, H_i) - g_\Lambda(\mathbf{l}_v, \mathbf{z}_i)).$$

On the other hand, by Corollary 2.2 of [2] we have

$$Dtr_{\Lambda^{k(x)}}(L_v^{k(x)}) \cong (DtrL_v)^{k(x)} \cong L^{k(x)}.$$

Then:

$$\dim_{k(x)} \mathrm{Hom}_{\Lambda^{k(x)}}(G_i, L^{k(x)}) = \dim_{k(x)} \mathrm{Hom}_{p^1(\Lambda^{k(x)})}(\hat{L}_v^{k(x)}, H_i) - g_\Lambda(\mathbf{l}_v, \mathbf{z}_i).$$

From here we obtain our Lemma. □

**Lemma 10.8.** *$T_i$ is a free right $R_i$-module, for $i = 1, ...s$.*

**Proof.** Since $T_i$ is a finitely generated right $R_i$-module if it is not a free right $R_i$-module there is a primitive idempotent $e$ of $\Lambda$ such that $eT_i = C_0 \oplus C_1$ with $C_0$ free and $C_1$ a torsion $R_i$-module, then we may assume $C_1 = (\oplus_{j=1}^{a} R_i/(p^{m_j})) \oplus C_2$ with a prime element $p \in R_i$, positive integers $m_j$, and $C_2 \cong \oplus_b R_i/(q_b^{n_b})$, where $p, q_b$ are coprime in $R_i$. Suppose $m = min\{m_1, ..., m_a\}$, $C_0 \cong R_i^l$. Take $M = T_i \otimes_{R_i} R_i/(p^m)$, then by the second part of Lemma 10.6, $\dim_k eM = m\dim_{k(x)} eG_i = m\dim_{k(x)} eT_i \otimes_{k(x)} k(x) = m\dim C_0 \otimes_{k(x)} k(x) = ml$. But $\dim_k eM = \dim_k eT_i \otimes_{R_i} R_i/(p^m) = \dim_k C_0 \otimes_{R_i} R_i/(p^m) + \dim_k (R_i/(p^m))^a = ml + am$, a contradiction. Therefore, $T_i$ is free as right $R_i$-module proving our result. □

**Proof** (of Theorem 1.2)**.** The $\Lambda$-$R_i$-bimodule $T_i$ is a good realization of $G_i$ over $R_i$ for $i = 1, ..., s$ by Lemma 10.8 and Lemma 10.1.

(i) of Theorem 1.2 follows from Lemma 10.2, (ii) follows from Lemma 10.5, Lemma 10.6 and Lemma 10.7. Finally (iii) follows from Lemma 10.4. □

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