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ON HOMEOMORPHISMS PRESERVING PRINCIPAL DIVISORS

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ABSTRACT. Let S_1 and S_2 be compact Riemann surfaces of genus $g > 1$. Let τ: $S_1 \to S_2$ be a continuous map. The map τ induces a group homomorphism from the group of divisors on S_1 into the group of divisors on S_2. This group homomorphism will be denoted by the same letter τ throughout this paper. If $D = \sum_{i=1}^n m_i p_i$ is a divisor on S_1, then $\tau(D) = \sum_{i=1}^n m_i \tau(p_i)$. If τ is a holomorphic or an anti-holomorphic homeomorphism, then $\tau(D)$ is a principal divisor on S_2 if D is a principal divisor on S_1. To what extent is the converse of this statement true?

The answer to this question is provided by Theorem 1 of this paper: If $\tau(D)$ is a principal divisor on S_2 whenever D is a principal divisor on S_1, then τ is either a holomorphic or an anti-holomorphic homeomorphism.

Let τ: $S_1 \to S_2$ be a surjective continuous map between compact Riemann surfaces of genus $g \geqslant 1$. Let $J(S_1)$ and $J(S_2)$ denote the corresponding Jacobi varieties and let j_1: $S_1 \to J(S_1)$ and j_2: $S_2 \to J(S_2)$ be Jacobi maps with basepoints $p_0 \in S_1$ and $\tau(p_0) \in S_2$.

LEMMA 1. (1) *The following statements are equivalent*:

(i) *$\tau(D)$ is a principal divisor on S_2 if D is a principal divisor on S_1.*

(ii) *There is a group homomorphism A: $J(S_1) \to J(S_2)$ such that $A \circ j_1 = j_2 \circ \tau$.*

(2) *A group homomorphism A that satisfies* (ii) *is necessarily continuous, surjective and unique.*

PROOF. (1) Assume that τ satisfies (i). A mapping A can be defined on the image $j_1(S_1) \subset J(S_1)$ as $j_2 \circ \tau \circ j_1^{-1}$. This mapping can be extended upon the group $J(S_1)$ by additivity since points of $j_1(S_1)$ generate $J(S_1)$ as a group. Now we have to prove that A is well defined. Let $\sum_{i=1}^n j_1(p_i) = \sum_{i=1}^m j_1(q_i)$ for p_i, $q_i \in S_1$. Then $p_1 + \cdots + p_n - q_1 - \cdots - q_m + (m-n)p_0$ is a principal divisor on S_1. By our assumption this implies that $\tau(p_1) + \cdots + \tau(p_n) - \tau(q_1) - \cdots - \tau(q_m) + (m-n)\tau(p_0)$ is a principal divisor on S_2. Therefore

$$A\left[\sum_{i=1}^n j_1(p_i)\right] = \sum_{i=1}^n Aj_1(p_i) = \sum_{i=1}^n j_2\tau(p_i)$$
$$= \sum_{i=1}^m j_2\tau(q_i) = \sum_{i=1}^m Aj_1(q_i) = A\left[\sum_{i=1}^m j_1(q_i)\right].$$

Thus A: $J(S_1) \to J(S_2)$ is a well-defined group homomorphism.

Now assume that there exists a group homomorphism A: $J(S_1) \to J(S_2)$ satisfying (ii). Let $D = p_1 + \cdots + p_n - q_1 - \cdots - q_n$ be a principal divisor on S_1.

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Then $\Sigma_{i=1}^{n} j_1(p_i) - \Sigma_{i=1}^{n} j_1(q_i) = 0$. Applying A we obtain the following:

$$0 = A\left[\sum_{i=1}^{n} j_1(p_i) - \sum_{i=1}^{n} j_1(q_i)\right] = \sum_{i=1}^{n} Aj_1(p_i) - \sum_{i=1}^{n} Aj_1(q_i)$$
$$= \sum_{i=1}^{n} j_2\tau(p_i) - \sum_{i=1}^{n} j_2\tau(q_i).$$

This implies that $\tau(p_1) + \cdots + \tau(p_n) - \tau(q_1) - \cdots - \tau(q_n)$ is a principal divisor on S_2.

(2) Let A satisfy (ii). Then the following is a commutative diagram:

$$\begin{array}{ccc} S_1^g & \xrightarrow{B} & S_2^g \\ \pi_1\downarrow & & \downarrow\pi_2 \\ J(S_1) & \xrightarrow{A} & J(S_2) \end{array}$$

Here

$$S_k^g = \overbrace{S_k \times S_k \times \cdots \times S_k}^{g} \qquad (k = 1, 2).$$

$B(p_1, \ldots, p_g) = (\tau(p_1), \ldots, \tau(p_g))$; π_1 and π_2 are the natural maps induced by the Jacobi maps. Assume that A is discontinuous at $x_0 \in J(S_1)$. Then there exist an open set $V \subset J(S_2)$, $V \ni A(x_0)$, and a sequence $\{x_n\}$ of points in $J(S_1)$ such that $\lim_{n\to\infty} x_n = x_0$ and $A(x_n) \notin V$ for all n. Since π_1 is surjective (by Jacobi Inversion Theorem) we can choose a sequence $\{t_n\}$ of points in S_1^g such that $\pi_1(t_n) = x_n$. Since S_1^g is compact, we can choose a converging subsequence $\{t_{n_i}\}$. Let $t_0 = \lim_{i\to\infty} t_{n_i}$. Then $\pi_1(t_0) = x_0$ since π_1 is continuous. Thus $\pi_2 \circ B(t_0) = A \circ \pi_1(t_0) = A(x_0)$. On the other hand

$$\lim_{i\to\infty} A(x_{n_i}) = \lim_{i\to\infty} A \circ \pi_1(t_{n_i}) = \lim_{i\to\infty} \pi_2 \circ B(t_{n_i}) = \pi_2 \circ B(t_0) = A(x_0)$$

since $\pi_2 \circ B$ is a continuous map. This contradicts our earlier assumption that V does not contain elements of the sequence $A(x_{n_i})$. Hence A is a continuous group homomorphism. Moreover, A maps $j_1(S_1)$ onto $j_2(S_2)$ since j_1 and j_2 are homeomorphisms and τ is a surjective map. Therefore A maps $J(S_1)$ onto $J(S_2)$ since points of $j_2(S_2)$ generate $J(S_2)$ as a group. The same argument (applied to j_1 instead of j_2) proves the uniqueness of A.

COROLLARY 1. *If τ satisfies* (i), *then τ is real-analytic.*

PROOF. If τ satisfies (i), then by Lemma 1 there exists a continuous group homomorphism $A: J(S_1) \to J(S_2)$ such that $A \circ j_1 = j_2 \circ \tau$. This A can be lifted to a real linear map $\tilde{A}: \mathbf{R}^{2g} \to \mathbf{R}^{2g}$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathbf{R}^{2g} & \xrightarrow{\tilde{A}} & \mathbf{R}^{2g} \\ \pi_1\downarrow & & \downarrow\pi_2 \\ J(S_1) & \xrightarrow{A} & J(S_2) \end{array}$$

This is possible since $\mathbf{R}^{2g}$ is the universal covering space for both $J(S_1)$ and $J(S_2)$, and since the only continuous group homomorphisms of $\mathbf{R}^{2g}$ are linear maps.

Therefore A is real-analytic; and τ is real-analytic as well since $\tau = j_2^{-1} \circ A \circ j_1$ where both j_1 and j_2 are complex analytic homeomorphisms.

COROLLARY 2. *Let S_1 and S_2 be compact Riemann surfaces of genus $g = 1$. Then there exists a homeomorphism τ: $S_1 \to S_2$ such that τ and τ^{-1} satisfy* (i).

PROOF. $J(S_1)$ and $J(S_2)$ are one-dimensional complex analytic tori. Let A be any continuous group isomorphism between them. Since $j_1(S_1) = J(S_1)$ and $j_2(S_2) = J(S_2)$, we can define τ: $S_1 \to S_2$ to be $j_2^{-1} \circ A \circ j_1$. Then the pair (A, τ) will satisfy (ii) and, therefore, τ will satisfy (i). The same argument applies to τ^{-1}.

This corollary implies that for any pair of Riemann surfaces of genus $g = 1$, we can find a homeomorphism between them which preserves principal divisors. For $g \geqslant 2$ the situation is quite different. Let τ: $S_1 \to S_2$ be a surjective continuous map between compact Riemann surfces of genus $g \geqslant 2$. The group homomorphism A that was introduced above can be lifted to a real $2g \times 2g$ matrix $\tilde{A}$ as described in the proof of Corollary 1. We will identify $\mathbf{R}^{2g}$ with $\mathbf{C}^g$ in the following way: Let $e_1, \ldots, e_g$ be the standard basis for $\mathbf{C}^g$, then the standard basis for $\mathbf{R}^{2g}$ will be $e_1, ie_1, \ldots, e_g, ie_g$. The matrix $\tilde{A}$ can be considered thus as

$$\begin{bmatrix} A_{11} & \cdots & A_{1g} \\ \vdots & & \vdots \\ A_{g1} & \cdots & A_{gg} \end{bmatrix},$$

where A_{ij}'s are real 2×2 matrices. Let $p \in S_1$ and let z denote a local analytic coordinate in a neighbourhood of p; let $z = x + iy$. Let $w = u + iv$ be a local analytic coordinate in a neighbourhood of $\tau(p)$. The bases for holomorphic differentials $\{\omega_k\}$ and $\{\tilde{\omega}_k\}$ will be considered as column vectors

$$\omega = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_g \end{bmatrix} \quad \text{and} \quad \tilde{\omega} = \begin{bmatrix} \tilde{\omega}_1 \\ \vdots \\ \tilde{\omega}_g \end{bmatrix}$$

(ω corresponds to S_1 and $\tilde{\omega}$ corresponds to S_2). With respect to the local coordinates z and w,

$$\omega_k = (f_k + ih_k)dz = (f_k dx - h_k dy) + i(h_k dx + f_k dy),$$
$$\tilde{\omega}_k = (\tilde{f}_k + i\tilde{h}_k)dw = (\tilde{f}_k du - \tilde{h}_k dv) + i(\tilde{h}_k du + \tilde{f}_k dv).$$

Here f_k, h_k, $\tilde{f}_k$ and $\tilde{h}_k$ are real-valued local functions. ω_k and $\tilde{\omega}_k$ can be considered thus as pairs of real-valued differential forms on the underlying differentiable structures of S_1 and S_2. Let L_k denote a matrix

$$\begin{pmatrix} f_k & -h_k \\ h_k & f_k \end{pmatrix}$$

and let $\tilde{L}_k$ denote a matrix

$$\begin{bmatrix} \tilde{f}_k & -\tilde{h}_k \\ \tilde{h}_k & \tilde{f}_k \end{bmatrix}.$$

Throughout this paper we will consider **C** as a subalgebra of the algebra of all real 2×2 matrices, and we will identify a complex number $a + ib$ with the real matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Complex-valued functions will be considered as real matrix-valued functions. By Corollary 1 to Lemma 1 τ is real-analytic. Let

$$K = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

be the local Jacobian matrix corresponding to τ and x, y, u and v.

LEMMA 2. *Let* $\tau: S_1 \to S_2$, A, ω *and* $\tilde{\omega}$ *be as above. Then*
(1) *for each* k $(\tilde{L}_k \circ \tau)K = \sum_{s=1}^{g} A_{ks} L_s$.
(2) A *and* τ *are local homeomorphisms.*

PROOF. (1) The Jacobi mappings j_1 and j_2 can be considered as real-analytic maps from the two-dimensional real surfaces S_1 and S_2 into $J(S_1)$ and $J(S_2)$ respectively. The Jacobian matrix of j_1 in terms of the local coordinates (x, y) is

$$\begin{bmatrix} L_1 \\ \vdots \\ L_g \end{bmatrix}.$$

Similarly, the Jacobian matrix of j_2 in terms of the local coordinates (u, v) is

$$\begin{bmatrix} \tilde{L}_1 \\ \vdots \\ \tilde{L}_g \end{bmatrix}.$$

The Jacobian matrix of A is $\tilde{A}$ and since $A \circ j_1 = j_2 \circ \tau$, we obtain

$$(1) \qquad \begin{bmatrix} (\tilde{L}_1 \circ \tau)K \\ \vdots \\ (\tilde{L}_g \circ \tau)K \end{bmatrix} = \tilde{A} \begin{bmatrix} L_1 \\ \vdots \\ L_g \end{bmatrix}.$$

(2) A is surjective by Lemma 1. Therefore $\tilde{A}$ is surjective and hence of rank $2g$. Since

$$\operatorname{rk} \begin{bmatrix} L_1 \\ \vdots \\ L_g \end{bmatrix} = 2$$

at every point of S_1 and

$$\operatorname{rk}\begin{bmatrix} \tilde{L}_1 \\ \vdots \\ \tilde{L}_g \end{bmatrix} = 2$$

at every point of S_2, equation (1) implies that det K is never 0. Now we can prove the main theorem.

THEOREM 1. *Let τ: $S_1 \to S_2$ be a surjective continuous map between compact Riemann surfaces of genus $g \geqslant 2$. If $\tau(D)$ is a principal divisor on S_2 whenever D is a principal divisor on S_1, then*

(1) *τ is either holomorphic or anti-holomorphic.*

(2) *τ is a homeomorphism.*

PROOF. By assertion (1) of Lemma 2 we have g equations of the form $(\tilde{L}_k \circ \tau)K = \sum_{s=1}^{g} A_{ks}L_s$. These equations depend on our choice of the local coordinates x, y, u and v. To get rid of this we will multiply (from the right) $(\tilde{L}_k \circ \tau)K$ by $K^{-1}(\tilde{L}_1 \circ \tau)^{-1}$ and $\sum_{s=1}^{g} A_{ks}L_s$ by $(\sum_{s=1}^{g} A_{1s}L_s)^{-1}$. Then we obtain $g - 1$ equations of the form

$$(\tilde{L}_k\tilde{L}_1^{-1}) \circ \tau = \left(\sum_{s=1}^{g} A_{ks}L_s\right)\left(\sum_{s=1}^{g} A_{1s}L_s\right)^{-1}.$$

Perhaps some explanations are necessary at this point. First, $\tilde{L}_1 \circ \tau$ is a local map from S_1 into the algebra of real 2×2 matrices. $(\tilde{L}_1 \circ \tau)^{-1}$ means taking the inverse of a value of this matrix-valued function: $(\tilde{L}_1 \circ \tau)^{-1} = (\tilde{L}_1^{-1}) \circ \tau$. Second, the matrix $\tilde{L}_1$ is either invertible or 0; and we take the inverse at all points except those for which $\tilde{L}_1$ is 0 (such points form the divisor of $\tilde{\omega}_1$). Third, by assertion (2) of Lemma 2 det K is never 0 and K^{-1} always exists.

Returning to our proof, let us consider any one of the equations obtained above. The left part of it has the form $(\tilde{L}_k\tilde{L}_1^{-1}) \circ \tau$. But $\tilde{L}_k\tilde{L}_1^{-1} = \tilde{\omega}_k/\tilde{\omega}_1$ which is a global meromorphic function on S_2. Let us denote this function by $\tilde{F}_k$. $(\tilde{L}_k\tilde{L}_1^{-1}) \circ \tau$ is, therefore, the pull-back of this function by τ. The right part of our equation is a matrix of the form $(\sum_{s=1}^{g} A_{ks}L_s)(\sum_{s=1}^{g} A_{1s}L_s)^{-1}$. But this matrix can be written as $(\sum_{s=1}^{g} A_{ks}L_sL_1^{-1})(\sum_{s=1}^{g} A_{1s}L_sL_1^{-1})^{-1}$. $L_sL_1^{-1}$ is a global meromorphic function on S_1. Let us denote this function by F_s. Thus we have $g - 1$ equations of the form

$$\tilde{F}_k \circ \tau = \left(\sum_{s=1}^{g} A_{ks}F_s\right)\left(\sum_{s=1}^{g} A_{1s}F_s\right)^{-1} \quad \text{or} \quad (\tilde{F}_k \circ \tau)\sum_{s=1}^{g} A_{1s}F_s = \sum_{s=1}^{g} A_{ks}F_s. \tag{2}$$

It should be mentioned one more time that we consider complex numbers as a subalgebra of the algebra of real 2×2 matrices. These equations are considered in the point-wise sense; and they are valid for all points of S_1 that are not poles for any of F_k's and such that their images in S_2 are not poles for any of $\tilde{F}_k$'s. Since τ is a local homeomorphism, there is only a finite set of such exceptional points.

Let us consider now a complex-valued real-linear functional T that can be defined on the algebra of real 2×2 matrices as follows:

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a - d + i(b + c).$$

This functional has the following property:

$$T\left[\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = (\alpha + i\beta)T\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$T\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\right] = (\alpha - i\beta)T\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Furthermore, the kernel of T consists of those matrices which correspond to complex numbers. Let us apply T to the equation (2). We obtain the following:

$$(3) \qquad (\tilde{F}_k \circ \tau) \sum_{s=1}^{g} T(A_{1s})\bar{F}_s = \sum_{s=1}^{g} T(A_{ks})\bar{F}_s.$$

Suppose that $T(A_{ks}) = 0$ for all indices k and s. Then all matrices A_{ks} are merely complex numbers c_{ks} and the equation (2) implies that

$$\tilde{F}_k \circ \tau = \frac{\sum_{s=1}^{g} c_{ks} F_s}{\sum_{s=1}^{g} c_{1s} F_s}.$$

This means that $\tilde{F}_k \circ \tau$ is a meromorphic function on S_1 and, therefore, τ is holomorphic. If not all of $T(A_{ks})$ are equal to 0, then the equation (3) implies that

$$\tilde{F}_k \circ \tau = \frac{\sum_{s=1}^{g} T(A_{ks})\bar{F}_s}{\sum_{s=1}^{g} T(A_{1s})\bar{F}_s}$$

($T(A_{ks})$ is a complex number). This means that τ is anti-holomorphic. Finally, since τ is either a holomorphic or an anti-holomorphic local homeomorphism it must be a global homeomorphism (since it has no branch points and $g > 1$).

COROLLARY. *Let S_1 and S_2 be compact Riemann surfaces of genus $g > 2$ with Jacobi mappings j_1 and j_2. Let A: $J(S_1) \to J(S_2)$ be a continuous group homomorphism such that $Aj(S_1) = j_2(S_2)$. Then*

(1) *A is either holomorphic or anti-holomorphic.*

(2) *A is a group isomorphism.*

The proof is obvious.

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