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ON HOMOGENEOUS KÄHLER MANIFOLDS WITH NON-DEGENERATE CANONICAL HERMITIAN FORM OF SIGNATURE (2, 2(n—1))

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We denote by M a connected homogeneous Kähler manifold of complex dimension n on which a connected Lie group G acts effectively as a group of holomorphic isometries, and by K an isotropy subgroup of G at a point o of M. Let v be the G-invariant volume element corresponding to the Kähler metric. In a local coordinate system $\{z_1, \cdots, z_n\}$, v has an expression $v=i^n F dz_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n$. The G-invariant hermitian form $h=\sum_{i,j}\frac{\partial^2\log F}{\partial z_i\partial\bar{z}_j}dz_i d\bar{z}_j$ is called the canonical hermitian form of $M=G/K$. It is known that the Ricci tensor of the Kähler manifold M is equal to $-h$. The purpose of this paper is to prove the following:

Theorem 1. *Let $M=G/K$ be a simply connected homogeneous Kähler manifold with non-degenerate canonical hermitian form h of signature* $(2, 2(n-1))$. *Then, if either G is semi-simple or G contains a one parameter normal subgroup, $M=G/K$ is a holomorphic fibre bundle whose base space is the unit disk $\{z\in \boldsymbol{C};\ |z|<1\}$, and whose fibre is a homogeneous Kähler manifold of a compact sime-simple Lie group.*

In the case of $\dim_C G/K=2$, the assumption of Theorem 1 is fulfilled and we have

Theorem 2. *Let $M=G/K$ be a complex two dimensional homogeneous Kähler manifold with non-degenerate canonical hermitian form h of signature* (2, 2). *Then G is semi-simple or G contains a one parameter normal subgroup.*

As an application of these Theorems, we obtain a classification of complex two dimensional homogeneous Kähler manifolds with non-degenerate canonical hermitian form.

1. Let (I, g) be the G-invariant Kähler structure on M, i.e., I is the G-invariant complex structure tensor on M and g is the G-invariant Kähler metric on M. Let $\mathfrak{g}$ be the Lie algebraof all left invariant vector fields on G and let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ corresponding to K. We denote by π the canonical projection from G onto $M=G/K$ and denote by π_e the differential of π at the identity e of G. Let X_e, I_o and g_o be the values of X, I and g at e and $\pi(e)=o$ respectively. Then there exist a linear endomorphism J of $\mathfrak{g}$ and a skew symmetric bilinear form ρ on $\mathfrak{g}$ such that

$$\pi_e(JX)_e = I_o(\pi_e X_e), \quad \rho(X, Y) = g_o(\pi_e X_e, \pi_e Y_e),$$

for $X, Y \in \mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{k}, J, \rho)$ satisfies the following properties [2], [3].

(1.1) $J\mathfrak{k} \subset \mathfrak{k}, \quad J^2X \equiv -X \pmod{\mathfrak{k}}$,

(1.2) $[W, JX] \equiv J[W, X] \pmod{\mathfrak{k}}$,

(1.3) $[JX, JY] \equiv J[JX, Y] + J[X, JY] + [X, Y] \pmod{\mathfrak{k}}$,

(1.4) $\rho(W, X) = 0$,

(1.5) $\rho(JX, JY) = \rho(X, Y)$,

(1.6) $\rho(JX, X) > 0, X \notin \mathfrak{k}$,

(1.7) $\rho([X, Y], Z) + \rho([Y, Z], X) + \rho([Z, X], Y) = 0$,

where $X, Y, Z \in \mathfrak{g}$, $W \in \mathfrak{k}$.

Then $(\mathfrak{g}, \mathfrak{k}, J, \rho)$ will be called the Kähler algebra of $M=G/K$.

Koszul proved that the canonical hermitian form h of a homogeneous Kähler manifold G/K has the following expression [3]. Put

$$\eta(X, Y) = h_o(\pi_e X_e, \pi_e Y_e), \quad \text{and}$$

$$\psi(X) = Tr_{\mathfrak{g}/\mathfrak{k}}(\operatorname{ad}(JX) - J\operatorname{ad}(X)), \tag{1.8}$$

it follows then

$$\eta(X, Y) = \frac{1}{2}\psi([JX, Y]), \tag{1.9}$$

for $X, Y \in \mathfrak{g}$. The form ψ satisfies the following properties:

$$\psi([W, X]) = 0, \tag{1.10}$$

$$\psi([JX, JY]) = \psi([X, Y]), \quad \text{for} \quad X, Y \in \mathfrak{g}, W \in \mathfrak{k}. \tag{1.11}$$

Since G acts effectively on G/K, $\mathfrak{k}$ contains no non-zero ideal of $\mathfrak{g}$ and there exists an $\operatorname{ad}(\mathfrak{k})$-invariant inner product (,) on $\mathfrak{g}$. Henceforth, we assume that the canonical hermitian form h of G/K is non-degenerate, which is equivalent to the following condition:

Let $X \in \mathfrak{g}$. If $\eta(X, Y)=0$ for all $Y \in \mathfrak{g}$, then $X \in \mathfrak{k}$.

2. We shall now prepare a few lemmas for later use. The following lemma is due to [2].

Lemma 2.1. *For $E, X, Y \in \mathfrak{g}$,*

$$\frac{d}{dt}\rho(\exp t\,\mathrm{ad}(JE)X, \exp t\,\mathrm{ad}(JE)Y)$$
$$= \rho(JE, \exp t\,\mathrm{ad}(JE)[X, Y]) .$$

Lemma 2.2. *The adjoint representation of* $\mathfrak{g}$ *is faithful.*

Proof. Put $\mathfrak{a} = \{X \in \mathfrak{g}; \mathrm{ad}(X)=0\}$, then $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. We have for $X \in \mathfrak{a}$

$$2\eta(X, Y) = \psi([JX, Y])$$
$$= -\psi([X, JY]) = 0 , \qquad \text{for all} \quad Y \in \mathfrak{g} .$$

Since h is non-degenerate, we have $X \in \mathfrak{k}$, and hence $\mathfrak{a} \subset \mathfrak{k}$. By the effectiveness, we have $\mathfrak{a} = \{0\}$. Q.E.D.

Lemma 2.3. *Let* $\mathfrak{r}$ *be a commutative ideal of* $\mathfrak{g}$. *Then,* $\mathfrak{k} \cap \mathfrak{r} = \{0\}$, $\mathfrak{k} \cap J\mathfrak{r} = \{0\}$.

Proof. Let $A \in \mathfrak{k} \cap \mathfrak{r}$. Since $\mathfrak{r}$ is a commutative ideal, we have $\mathrm{ad}(A)^2=0$. By the effectiveness, it follows that

$$(\mathrm{ad}(A)^2X, X) + (\mathrm{ad}(A)X, \mathrm{ad}(A)X) = 0 ,$$
$$(\mathrm{ad}(A)X, \mathrm{ad}(A)X) = 0 ,$$

for $X \in \mathfrak{g}$, with respect to the $\mathrm{ad}(\mathfrak{k})$-invariant inner product $(\ ,\ )$ on $\mathfrak{g}$. Hence $\mathrm{ad}(A)X=0$ for all $X \in \mathfrak{g}$, and $A=0$ by Lemma 2.2, which proves $\mathfrak{k} \cap \mathfrak{r} = \{0\}$. $\mathfrak{k} \cap J\mathfrak{r} = \{0\}$ follows from $\mathfrak{k} \cap \mathfrak{r} = \{0\}$. Q.E.D.

Lemma 2.4. *Let* $\mathfrak{r}$ *be a non-zero commutative ideal of* $\mathfrak{g}$. *Then* $\psi \neq 0$ *on* $\mathfrak{r}$.

Proof, Assume $\psi=0$ on $\mathfrak{r}$. For $X \in \mathfrak{r}$, we have $2\eta(X, Y) = -\psi([JY, X]) = 0$ for all $Y \in \mathfrak{g}$. Since h is non-degenerate, we have $X \in \mathfrak{k}$ and hence $\mathfrak{r} \subset \mathfrak{k}$, which contradicts to Lemma 2.3. Q.E.D.

Lemma 2.5. $Tr_{\mathfrak{g}/\mathfrak{k}}\,\mathrm{ad}(W)=0$, *for* $W \in \mathfrak{k}$.

Proof. Using (1.4), (1.7), we have

$$\rho(W, [X, Y]) + \rho(X, [Y, W]) + \rho(Y, [W, X]) = 0 ,$$
$$\rho(X, [Y, W]) + \rho(Y, [W, X]) = 0 ,$$

for $X, Y \in \mathfrak{g}$. Hence it follows that

$$\rho(JX, [W, Y]) + \rho([W, JX], Y) = 0 ,$$
$$\rho(JX, [W, Y]) + \rho(J[W, X], Y) = 0 ,$$

for $X, Y \in \mathfrak{g}$. This implies that the endomorphism of $\mathfrak{g}/\mathfrak{k}$ which is induced by ad(W) is skew symmetric with respect to the inner product which is defined by $\rho(JX, Y)$. Therefore $Tr_{\mathfrak{g}/\mathfrak{k}}\,\mathrm{ad}(W)=0$. Q.E.D.

Lemma 2.6. *Let $\{E\}$ be a one dimensional ideal of $\mathfrak{g}$. Then $[E, W]=0$, for $W\in\mathfrak{k}$. Moreover, there exists an endomorphism $\tilde{J}$ of $\mathfrak{g}$ such that $\tilde{J}X\equiv JX(mod\ \mathfrak{k})$, $[\tilde{J}E, W]=0$, for $X\in\mathfrak{g}$, $W\in\mathfrak{k}$.*

Proof. Put $[E, W]=\lambda E$, $\lambda\in\boldsymbol{R}$. Using (1.10), we have $0=\psi([E, W])=\lambda\psi(E)$. Since $\psi(E)\neq 0$ by Lemma 2.4, it follows $\lambda=0$ and hence $[E, W]=0$. Put $\mathfrak{h}=\mathfrak{k}+\{JE\}$, then $[\mathfrak{k}, \mathfrak{h}]\subset\mathfrak{h}$. Let $\{L\}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{h}$ with respect to the ad$(\mathfrak{k})$-invariant inner product on $\mathfrak{g}$. Then $[\mathfrak{k}, \{L\}]\subset\{L\}$. We may assume that $L=W_0+JE$ where $W_0\in\mathfrak{k}$. Therefore we can choose a linear endomorphism $\tilde{J}$ on $\mathfrak{g}$ such that $\tilde{J}E=L$, $\tilde{J}X\equiv JX(mod\ \mathfrak{k})$ for $X\in\mathfrak{g}$. Then it follows

$$[\tilde{J}E, \mathfrak{k}] = [L, \mathfrak{k}]\subset\{L\}\ ,$$
$$[\tilde{J}E, \mathfrak{k}]\subset[\mathfrak{h}, \mathfrak{k}]\subset\mathfrak{k}\ .$$

This implies $[\tilde{J}E, W]=0$ for $W\in\mathfrak{k}$. Q.E.D

Therefore, for any one dimensional ideal $\{E\}$ of $\mathfrak{g}$, we may assume that $[JE, \mathfrak{k}]=\{0\}$.

3. We shall prove the following theorem.

Theorem 1′. *Let $(\mathfrak{g}, \mathfrak{k}, J, \rho)$ be the Kähler algebra of a homogeneous Kähler manifold G/K with non-degenerate canonical hermitian form h of signature $(2, 2(n-1))$. If there exists a one dimensional ideal $\mathfrak{r}$ of $\mathfrak{g}$, then we have the following.*

1) *With suitable choice of $E\neq 0\in\mathfrak{r}$, we have $[JE, E]=E$.*

2) *Put $\mathfrak{p}=\{P\in\mathfrak{g};\ [P, E]=[JP, E]=0\}$. Then we have the decomposition $\mathfrak{g}=\{JE\}+\{E\}+\mathfrak{p}$ of $\mathfrak{g}$ into the direct sum of vector spaces. We know also that $\mathfrak{p}$ is a compact semi-simple J-invariant ideal of $\mathfrak{g}$ and that the real parts of the eigenvalues of* ad (JE) *on $\mathfrak{p}$ are equal to* 0.

The first part of the proof of Theorem 1′ is nearly the same as the previous one [6]. But, for the sake of completeness we carry out the proof.

Lemma 3.1. *Let $\{E\}$ be a one dimensional ideal of $\mathfrak{g}$ and put $\mathfrak{p}=\{P\in\mathfrak{g};\ [P, E]=[JP, E]=0\}$. Then we have*

1) $\mathfrak{k}\subset\mathfrak{p}$,

2) $J\mathfrak{p}\subset\mathfrak{p}$, ad$(JE)\mathfrak{p}\subset\mathfrak{p}$,

3) ad$(JE)J\equiv J$ad(JE) *$(mod\ \mathfrak{k})$ on $\mathfrak{p}$.*

Proof. 1) follows from Lemma 2.6. For $P\in\mathfrak{p}$, we have

$$[JE, JP] = J[JE, P]+J[E, JP]+[E, P]+W_0$$
$$= J[JE, P]+W_0$$

for some $W_0 \in \mathfrak{k}$, and hence 3) is proved. For $P \in \mathfrak{p}$, it follows that

$$[[JE, P], E] = [[JE, E], P]+[JE, [P, E]] = 0\,,$$
$$[J[JE, P], E] = [[JE, JP]-W_0, E] = 0\,,$$

where $W_0 \in \mathfrak{k}$. Therefore $\mathrm{ad}(JE)P \in \mathfrak{p}$ for all $P \in \mathfrak{p}$, which proves 2).

Lemma 3.2. *Let $\{E\}$ be a one dimensional ideal of $\mathfrak{g}$. Then $[JE, E] \neq 0$, therefore with suitable choice of $E \neq 0$, we have $[JE, E]=E$.*

Proof. Assume that $[JE, E]=0$. For $X \in \mathfrak{g}$, we have $J[JE, X]= [JE, JX]-J[E, JX]-[E, X]+W_0=[JE, JX]-\lambda JE-\mu E+W_0$, where $\lambda, \mu \in \boldsymbol{R}$, $W_0 \in \mathfrak{k}$. We have

$$[[JE, X], E] = [[JE, E], X]+[JE, [X, E]] = 0\,,$$
$$[J[JE, X], E] = [[JE, JX], E]-\lambda[JE, E]-\mu[E, E] +[W_0, E] = 0\,,$$

which implies that $[JE, X] \in \mathfrak{p}$, and hence we have

$$\mathrm{ad}(JE)\mathfrak{g} \subset \mathfrak{p}\,. \tag{3.1}$$

Let $P \in \mathfrak{p}$. We have

$$\begin{aligned}\rho(JE, [JE, P]) &= \rho(-E, J[JE, P]) \\ &= -\rho(E, [JE, JP]) \\ &= \rho(JE, [JP, E])+\rho(JP, [E, JE]) \\ &= 0\,,\end{aligned}$$

and it follows that for $X \in \mathfrak{g}$

$$\rho(JE, \mathrm{ad}(JE)^2X) = 0\,. \tag{3.2}$$

Applying Lemma 2.1, (3.2), we have for $X, Y \in \mathfrak{g}$

$$\begin{aligned}&\frac{d^3}{dt^3}\rho(\exp t\,\mathrm{ad}(JE)X, \exp t\,\mathrm{ad}(JE)Y) \\ &= \frac{d^2}{dt^2}\rho(JE, \exp t\,\mathrm{ad}(JE)[X, Y]) \\ &= \rho(JE, \mathrm{ad}(JE)^2 \exp t\,\mathrm{ad}(JE)\,[X, Y]) \\ &= 0\,.\end{aligned}$$

Hence we may put

$$(3.3) \qquad \rho(\exp t\,\mathrm{ad}(JE)X, \exp t\,\mathrm{ad}(JE)Y) = at^2+bt+c$$

where a, b and c are real numbers not depending on t. Since $\mathrm{ad}(JE)\mathfrak{p}\subset\mathfrak{p}$, $\mathrm{ad}(JE)\mathfrak{k}=\{0\}$ by Lemma 2.6, (3.1), $\mathrm{ad}(JE)$ induces a linear endomorphism $\widetilde{\mathrm{ad}}(JE)$ on $\mathfrak{p}/\mathfrak{k}$. Let $\alpha+i\beta(\alpha, \beta\in \boldsymbol{R})$ be an eigenvalue of $\widetilde{\mathrm{ad}}(JE)$. As $\mathrm{ad}(JE)J\equiv J\mathrm{ad}(JE)$ (mod $\mathfrak{k}$) on $\mathfrak{p}$, there exists an element $P\in\mathfrak{p}$, $P\notin\mathfrak{k}$ such that $[JE, P]\equiv(\alpha+\beta J)P(\mathrm{mod}\ \mathfrak{k})$, and hence $\exp t\,\mathrm{ad}(JE)P\equiv\exp t(\alpha+\beta J)P(\mathrm{mod}\ \mathfrak{k})$. Therefore we have by Lemma 3.1,

$$\begin{aligned}
&\rho(\exp t\,\mathrm{ad}(JE)JP, \exp t\,\mathrm{ad}(JE)P)\\
&= \rho(J\exp t\,\mathrm{ad}(JE)P, \exp t\,\mathrm{ad}(JE)P)\\
&= \rho(J\exp t(\alpha+\beta J)P, \exp t(\alpha+\beta J)P)\\
&= \rho(\exp t(\alpha+\beta J)JP, \exp t(\alpha+\beta J)P)\\
&= e^{(\alpha+i\beta)t}\overline{e^{(\alpha+i\beta)t}}\rho(JP, P)\\
&= e^{2\alpha t}\rho(JP, P)\,.
\end{aligned}$$

From this and (3.3), we have

$$e^{2\alpha t}\rho(JP, P) = at^2+bt+c\,.$$

Since $P\notin\mathfrak{k}$, $\rho(JP, P)>0$ and hence $\alpha=0$. This fact and $\mathrm{ad}(JE)\mathfrak{k}=\{0\}$ show that the real parts of the eigenvalues of $\mathrm{ad}(JE)$ on $\mathfrak{p}$ are equal to 0. Therefore we have

$$\begin{aligned}
\psi(E) &= Tr_{\mathfrak{g}/\mathfrak{k}}(\mathrm{ad}(JE)-J\mathrm{ad}(E))\\
&= Tr_{\mathfrak{g}}(\mathrm{ad}(JE)-J\mathrm{ad}(E))-Tr_{\mathfrak{k}}(\mathrm{ad}(JE)-J\mathrm{ad}(E))\\
&= Tr_{\mathfrak{g}}\mathrm{ad}(JE)-Tr_{\mathfrak{g}}J\mathrm{ad}(E)\\
&= Tr_{\mathfrak{p}}\mathrm{ad}(JE)-Tr_{\{JE\}}J\mathrm{ad}(E)\\
&= 0\,.
\end{aligned}$$

However this contradicts to $\psi\neq0$ on $\{E\}$ by Lemma 2.4. Q.E.D.

Lemma 3.3 *Let $\{E\}$ be a one dimensional ideal of $\mathfrak{g}$. Then we get the decomposition*

$$\mathfrak{g} = \{JE\}+\{E\}+\mathfrak{p}$$

of $\mathfrak{g}$ into the direct sum of vector spaces with the following properties:

1) $[JE, E]=E$.

2) *The factors of the decomposition are mutually orthogonal with respect to the form η, and η is positive definite on $\{JE\}+\{E\}$.*

3) *The real parts of the eigenvalues of* $\mathrm{ad}(JE)$ *on $\mathfrak{p}$ are equal to* 0 *or* 1/2.

4) $\rho(JE, P)=0$ for $P\in\mathfrak{p}$.

Proof. By lemma 3.2, we may assume that E satisfies the condition $[JE, E]=E$. Since $\{E\}$ is a one dimensional ideal of $\mathfrak{g}$, we get $[X, E]=\alpha(X)E$, $[JX, E]=\beta(X)E$, for $X\in\mathfrak{g}$, where α, β are linear functions on $\mathfrak{g}$. It is easily seen that $P=X-\alpha(X)JE-\beta(X)E$ belongs to $\mathfrak{p}$ for any $X\in\mathfrak{g}$. Therefore we have the decomposition $\mathfrak{g}=\{JE\}+'\{E\}+\mathfrak{p}$. Now, by Lemma 2.1, we have for $P\in\mathfrak{p}$,

$$\begin{aligned}&\frac{d}{dt}\rho(\exp t\,\mathrm{ad}(JE)E, \exp t\,\mathrm{ad}(JE)P)\\&=\rho(JE, \exp t\,\mathrm{ad}(JE)[E, P])\\&=0\,.\end{aligned}$$

Since $\exp t\,\mathrm{ad}(JE)E=e^tE$, we have

$$\rho(E, \exp t\,\mathrm{ad}(JE)P)=a'e^{-t}$$

where a' is a constant determined by P and independent of t. We have then

$$\begin{aligned}\rho(JE, \exp t\,\mathrm{ad}(JE)P)&=-\rho(E, J\exp t\,\mathrm{ad}(JE)P)\\&=-\rho(E, \exp t\,\mathrm{ad}(JE)JP)\\&=ae^{-t}\end{aligned}$$

where a is a constant determined by JP. Let $X=\lambda JE+\mu E+P\in\mathfrak{g}$, where λ, $\mu\in\boldsymbol{R}$, $P\in\mathfrak{p}$. Then we have

$$\begin{aligned}\rho(JE, \exp t\,\mathrm{ad}(JE)X)&=\rho(JE, \lambda JE+\mu e^tE+\exp t\,\mathrm{ad}(JE)P)\\&=\mu\rho(JE, E)e^t+\rho(JE, \exp t\,\mathrm{ad}(JE)P)\\&=ae^{-t}+be^t\end{aligned}$$

where a, b are constants independent of t. This fact and Lemma 2.1 show that for X, $Y\in\mathfrak{g}$

$$\begin{aligned}&\frac{d}{dt}\rho(\exp t\,\mathrm{ad}(JE)X, \exp t\,\mathrm{ad}(JE)Y)\\&=\rho(JE, \exp t\,\mathrm{ad}(JE)[X, Y])\\&=ae^{-t}+be^t\,.\end{aligned}$$

Hence we obtain

$$\begin{aligned}&\rho(\exp t\,\mathrm{ad}(JE)X, \exp t\,\mathrm{ad}(JE)Y)\\&=ae^{-t}+be^t+c\,,\end{aligned}$$

where a, b and c are constants independent of t. Let $\alpha+i\beta$ be an eigenvalue of $\widetilde{\mathrm{ad}}(JE)$ on $\mathfrak{p}/\mathfrak{k}$. As $\mathrm{ad}(JE)J\equiv J\mathrm{ad}(JE)$ (mod $\mathfrak{k}$) on $\mathfrak{p}$, there exists an element $P\in\mathfrak{p}$, $P\notin\mathfrak{k}$ such that $\mathrm{ad}(JE)P\equiv(\alpha+\beta J)P$ (mod $\mathfrak{k}$). Hence we have

$$\begin{aligned}
&\rho(\exp t\,\mathrm{ad}(JE)JP, \exp t\,\mathrm{ad}(JE)P)\\
&= \rho(J \exp t\,\mathrm{ad}(JE)P, \exp t\,\mathrm{ad}(JE)P)\\
&= \rho(J \exp t(\alpha+\beta J)P, \exp t(\alpha+\beta J)P)\\
&= \rho(\exp t(\alpha+\beta J)JP, \exp t(\alpha+\beta J)P)\\
&= e^{(\alpha+i\beta)t}\overline{e^{(\alpha+i\beta)t}}\rho(JP, P)\\
&= e^{2\alpha t}\rho(JP, P)\,.
\end{aligned}$$

Therefore

$$e^{2\alpha t}\rho(JP, P) = ae^{-t}+be^{t}+c\,. \tag{3.4}$$

Since $P\notin\mathfrak{k}$ and $\rho(JP, P)>0$, we have $\alpha=0$ or $1/2$ or $-1/2$. Let $\tilde{J}$ be the linear endomorphism of $\tilde{\mathfrak{p}}=\mathfrak{p}/\mathfrak{k}$ which is induced by J and put for $\alpha, \beta\in\boldsymbol{R}$;

$$\begin{aligned}
\tilde{\mathfrak{p}}_{(\alpha+i\beta)} &= \{\tilde{P}\in\tilde{\mathfrak{p}};\ (\widetilde{\mathrm{ad}}(JE)-(\alpha+\beta\tilde{J}))^m\tilde{P}=0\}\,,\\
\tilde{\mathfrak{p}}_{\alpha} &= \sum_{\beta}\tilde{\mathfrak{p}}_{(\alpha+i\beta)}\,.
\end{aligned}$$

Then we have

$$\tilde{\mathfrak{p}} = \sum_{\alpha+i\beta}\tilde{\mathfrak{p}}_{(\alpha+i\beta)}$$

where $\alpha=0$ or $1/2$ or $-1/2$.

Let $\tilde{P}\neq 0\in\tilde{\mathfrak{p}}_{(\alpha+i\beta)}$ and let $P\in\mathfrak{p}$ be a representation of $\tilde{P}$. Then there exists a positive integer m such that $(\widetilde{\mathrm{ad}}(JE)-(\alpha+\beta\tilde{J}))^m\tilde{P}=0$. Therefore we have

$$\begin{aligned}
\exp t\widetilde{\mathrm{ad}}(JE)\tilde{P} &= \exp t(\alpha+\beta\tilde{J})\sum_{l=0}^{m-1}\frac{t^l}{l!}(\widetilde{\mathrm{ad}}(JE)-(\alpha+\beta\tilde{J}))^l\tilde{P}\\
&= e^{\alpha t}\{\cos\beta t\sum_{l=0}^{m-1}\frac{t^l}{l!}(\widetilde{\mathrm{ad}}(JE)-(\alpha+\beta\tilde{J}))^l\tilde{P}\\
&\quad +\sin\beta t\sum_{l=0}^{m-1}\frac{t^l}{l!}(\widetilde{\mathrm{ad}}(JE)-(\alpha+\beta\tilde{J}))^l\tilde{J}\tilde{P}\}\,.
\end{aligned}$$

This shows that

$$\begin{aligned}
\exp t\mathrm{ad}(JE)P \equiv e^{\alpha t}\{&\cos\beta t\sum_{l=0}^{m-1}\frac{t^l}{l!}(\mathrm{ad}(JE)-(\alpha+\beta J))^lP\\
&+\sin\beta t\sum_{l=0}^{m-1}\frac{t^l}{l!}(\mathrm{ad}(JE)-(\alpha+\beta J))^lJP\}\pmod{\mathfrak{k}}\,.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\rho(JE, \exp t\,\mathrm{ad}(JE)P)\\
&= e^{\alpha t}\{\cos\beta t\sum_{l=0}^{m-1}\frac{1}{l!}\rho(JE, (\mathrm{ad}(JE)-(\alpha+\beta J))^lP)t^l\\
&\quad +\sin\beta t\sum_{l=0}^{m-1}\frac{1}{l!}\rho(JE, (\mathrm{ad}(JE)-(\alpha+\beta J))^lJP)t^l\}\,.
\end{aligned}$$

Put

$$h(t) = \sum_{l=0}^{m-1} \frac{1}{l!} \rho(JE, (\mathrm{ad}(JE)-(\alpha+\beta J))^l P)t^l ,$$

$$k(t) = \sum_{l=0}^{m-1} \frac{1}{l!} \rho(JE, (\mathrm{ad}(JE)-(\alpha+\beta J))^l JP)t^l .$$

Then $h(t)$ and $k(t)$ are polynomials of degree $\leqq m-1$. We have then

$$h(t)\cos\beta t + k(t)\sin\beta t = a e^{-(1+\alpha)t} ,$$

$$\left|\frac{h(t)}{t^m}\cos\beta t + \frac{k(t)}{t^m}\sin\beta t\right| = \left|a\frac{t^{-(1+\alpha)t}}{t^m}\right| .$$

Assume that $a\neq 0$. Since $1+\alpha>0$ and since $h(t)$ and $k(t)$ are polynomials of degree $\leqq m-1$, the left side of the above formula approaches to 0 and the right side to ∞, when $t\to-\infty$. This is a contradiction, and we get $a=0$, which implies that

$$\rho(JE, \exp t\mathrm{ad}(JE)P) = 0$$

where P is a representative of $\tilde{P}\in\tilde{\mathfrak{p}}_{(\alpha+i\beta)}$. Thus we have

$$\rho(JE, \exp t\mathrm{ad}(JE)P) = 0 , \qquad \text{for all} \quad P\in\mathfrak{p} ,$$

and hence

$$\rho(JE, P) = 0 , \qquad \text{for all} \quad P\in\mathfrak{p} .$$

Therefore 4) is proved. Moreover the formula (3.4) is reduced to

$$(3.4)' \qquad e^{2\alpha t}\rho(JP, P) = b e^t + c .$$

This implies that $\alpha=0$ or $1/2$. Therefore we know that the real parts of the eigenvalues of $\mathrm{ad}(JE)$ on $\mathfrak{p}$ are equal to 0 or $1/2$. Thus the assertion 3) is proved. Now we shall show 2). The assertion that the decomposition $\mathfrak{g}=\{JE\}+\{E\}+\mathfrak{p}$ is an orthogonal decomposition is clear. Put $f=\mathrm{ad}(JE)-J\mathrm{ad}(E)$. Then we have $f(W)=0$ for $W\in\mathfrak{k}$, $f(JE)=JE$, $f(E)=E$ and $f(P)=[JE, P]$ for $P\in\mathfrak{p}$. Hence it follows that

$$\begin{aligned}\psi(E) &= Tr_{\mathfrak{g}/\mathfrak{k}}(\mathrm{ad}(JE)-J\mathrm{ad}(E))\\ &= Tr_{\mathfrak{g}}(\mathrm{ad}(JE)-J\mathrm{ad}(E))\\ &= 2+Tr_{\mathfrak{p}}\mathrm{ad}(JE)\\ &>0 .\end{aligned}$$

Therefore $2\eta(JE, JE)=2\eta(E, E)=\psi(E)>0$. Q.E.D.

For α, $\beta\in\boldsymbol{R}$, put

$$\mathfrak{p}_{(\alpha+i\beta)} = \{P \in \mathfrak{p};\ (\mathrm{ad}(JE)-(\alpha+i\beta))^m P = 0\}\ ,$$
$$\mathfrak{p}_\alpha = \sum_\beta \mathfrak{p}_{(\alpha+i\beta)}\ ,$$

and let π' be the canonical projection from $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{k}$. Then we have

$$\tilde{\mathfrak{p}}_{(\alpha+i\beta)} = \pi'(\mathfrak{p}_{(\alpha+i\beta)})\ , \quad \tilde{\mathfrak{p}}_\alpha = \pi'(\mathfrak{p}_\alpha)\ ,$$
$$\mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_{\frac{1}{2}}\ ,$$
$$J\mathfrak{p}_\alpha \subset \mathfrak{k} + \mathfrak{p}_\alpha\ ,$$
$$\mathrm{ad}(JE)\mathfrak{p}_\alpha \subset \mathfrak{p}_\alpha\ .$$

Lemma 3.4. *The form η is positive definite on $\mathfrak{p}_{\frac{1}{2}}$.*

Proof. We shall first prove that the decomposition $\mathfrak{p}_{\frac{1}{2}} = \sum_\beta \mathfrak{p}_{(\frac{1}{2}+i\beta)}$ is an orthogonal decomposition with respect to η. Let $P \neq 0 \in \mathfrak{p}_{(\frac{1}{2}+i\beta)}$, $Q \neq 0 \in \mathfrak{p}_{(\frac{1}{2}+i\beta')}$, and assume $\beta \neq \beta'$. Then we have

$$\exp t\,\mathrm{ad}(JE)P \equiv \exp t\,(1/2+\beta J)\sum_{l=0}^{r-1} \frac{t^l}{l!}(\mathrm{ad}(JE)-(1/2+\beta J))^l P \ (\mathrm{mod}\ \mathfrak{k})\ ,$$
$$\exp t\,\mathrm{ad}(JE)Q \equiv \exp\,(1/2+\beta' J)\sum_{l=0}^{s-1} \frac{t^l}{l!}(\mathrm{ad}(JE)-(1/2+\beta' J))^l Q \ (\mathrm{mod}\ \mathfrak{k})\ .$$

By Lemma 2.1, we have

$$\frac{d}{dt}\rho(\exp t\,\mathrm{ad}(JE)JP,\quad \exp t\,\mathrm{ad}(JE)Q) = \rho(JE,\ \exp t\,\mathrm{ad}(JE)[JP,\ Q])\ . \tag{3.5}$$

The left side of this equation is equal to

$$\frac{d}{dt}\rho(J\exp t\,\mathrm{ad}(JE)P,\quad \exp t\,\mathrm{ad}(JE)Q)$$
$$= \frac{d}{dt}\rho(J\exp t(1/2+\beta J)\sum_{l=0}^{r-1}\frac{t^l}{l!}(\mathrm{ad}(JE)-(1/2+\beta J))^l P,$$
$$\exp t(1/2+\beta' J)\sum_{l=0}^{s-1}\frac{t^l}{l!}(\mathrm{ad}(JE)-(1/2+\beta' J))^l Q)$$
$$= \frac{d}{dt}e^t\rho(\exp \beta t J\sum_{l=0}^{r-1}\frac{t^l}{l!}(\mathrm{ad}(JE)-(1/2+\beta J))^l JP,$$
$$\exp \beta' t J\sum_{l=0}^{s-1}\frac{t^l}{l!}(\mathrm{ad}(JE)-(1/2+\beta' J))^l Q)$$
$$= \frac{d}{dt}e^t\rho(\{\cos\beta t+(\sin\beta t)J\}u(t),\quad \{\cos\beta' t+(\sin\beta' t)J\}v(t))$$

$$= \frac{d}{dt} e^t \{(\cos \beta t \cos \beta' t + \sin \beta t \sin \beta' t)\rho(u(t), v(t))$$
$$+(\sin \beta t \cos \beta' t - \cos \beta t \sin \beta' t)\rho(Ju(t), v(t))\}$$
$$= \frac{d}{dt} e^t \{h(t) \cos (\beta - \beta')t + k(t) \sin (\beta - \beta')t\}$$
$$= e^t \{a(t) \cos (\beta - \beta')t + b(t) \sin (\beta - \beta')t\} ,$$

where

$$u(t) = \sum_{l=0}^{r-1} \frac{t^l}{l!} (\mathrm{ad}(JE) - (1/2 + \beta J)^l) JP ,$$
$$v(t) = \sum_{l=0}^{s-1} \frac{t^l}{l!} (\mathrm{ad}(JE) - (1/2 + \beta' J))^l Q ,$$
$$h(t) = \rho(u(t), v(t)) , \; k(t) = \rho(Ju(t), v(t)) ,$$
$$a(t) = h(t) + h'(t) + (\beta - \beta')k(t) ,$$
$$b(t) = k(t) + k'(t) - (\beta - \beta')\mathrm{h}(t) .$$

Hence $a(t)$ and $b(t)$ are polynomials. Since $[JP, Q] \in [\mathfrak{k} + \mathfrak{p}_{\frac{1}{2}}, \mathfrak{p}_{\frac{1}{2}}] \subset \{E\} + \mathfrak{p}_{\frac{1}{2}}$, we put $[JP, Q] = \lambda E + P'$, where $\lambda \in \boldsymbol{R}, P' \in \mathfrak{p}_{\frac{1}{2}}$. Using Lemma 3.3; 4), the right side of the equation (3.5) is equal to

$$\rho(JE, \exp t \mathrm{ad}(JE)[JP, Q])$$
$$= \rho(JE, \exp t \mathrm{ad}(JE)(\lambda E + P'))$$
$$= e^t \lambda \rho(JE, E) + \rho(JE, \exp t \mathrm{ad}(JE) P')$$
$$= e^t \lambda \rho(JE, E) .$$

Therefore we have

$$a(t) \cos (\beta - \beta')t + b(t) \sin (\beta - \beta')t = \lambda \rho(JE, E) .$$

Since $a(t) - \lambda \rho(JE, E)$ is a polynomial and since $a(t_n) - \lambda \rho(JE, E) = 0$ for $t_n = \frac{2n\pi}{\beta - \beta'}$, where n integer, it follows that $a(t)$ is a constant a. Similarly $b(t)$ is a constant b. Hence we have

$$a \cos (\beta - \beta')t + b \sin (\beta - \beta')t = \lambda \rho(JE, E) .$$

By this formula, we have $(\beta - \beta')^2 \lambda \rho(JE, E) = 0$. Since $\beta - \beta' \neq 0$ and $\rho(JE, E) > 0$, we get $\lambda = 0$. Moreover $\mathrm{ad}(JE)$ is non-singular on $\mathfrak{p}_{\frac{1}{2}}$ and so there exists an element $P'' \in \mathfrak{p}_{\frac{1}{2}}$ such that $P' = [JE, P'']$. Thus we have

$$\begin{aligned} 2\eta(P, Q) &= \psi([JP, Q]) \\ &= \psi(P') \\ &= \psi([JE, P'']) \\ &= -\psi([E, JP'']) \\ &= 0 . \end{aligned}$$

This shows that $\mathfrak{p}_{(\frac{1}{2}+i\beta)}$ and $\mathfrak{p}_{(\frac{1}{2}+i\beta')}$ are mutually orthogonal with respect to η. Now, let $P \neq 0 \in \mathfrak{p}_{(\frac{1}{2}+i\beta)}$. Then we have

$$\exp t\,\mathrm{ad}(JE)P \equiv \exp t(1/2+\beta J)u(t) \pmod{\mathfrak{k}},$$

where $u(t)=\sum_{l=0}^{m-1}\frac{t^l}{l!}(\mathrm{ad}(JE)-(1/2+\beta J))^l P$. By Lemma 2.1, it follows that

$$\begin{aligned} \text{(3.6)} \qquad & \frac{d}{dt}\rho(\exp t\,\mathrm{ad}(JE)JP, \exp t\,\mathrm{ad}(JE)P) \\ &= \rho(JE, \exp t\,\mathrm{ad}(JE)[JP, P]) \end{aligned}$$

The left side of the equation (3.6) is equal to

$$\begin{aligned} & \frac{d}{dt}\rho(J \exp t\,\mathrm{ad}(JE)P, \exp t\,\mathrm{ad}(JE)P) \\ &= \frac{d}{dt}\rho(J \exp t(1/2+\beta J)u(t), \exp t(1/2+\beta J)u(t)) \\ &= \frac{d}{dt}\rho(\exp t(1/2+\beta J)Ju(t), \exp t(1/2+\beta J)u(t)) \\ &= \frac{d}{dt}e^{(\frac{1}{2}+i\beta)t}e^{(\frac{1}{2}+i\beta)t}\rho(Ju(t), u(t)) \\ &= \frac{d}{dt}e^t\rho(Ju(t), u(t)) \\ &= e^t(h'(t)+h(t)) \end{aligned}$$

where $h(t)=\rho(Ju(t), u(t))$, and $h(t)$ is a polynomial of degree $\leqq 2m-2$. Since $[JP, P]=\lambda E+P'$, where $\lambda \in \boldsymbol{R}$, $P' \in \mathfrak{p}_{\frac{1}{2}}$, the right side of the equation (3.6) is equal to

$$\begin{aligned} & \rho(JE, \exp t\,\mathrm{ad}(JE)(\lambda E+P')) \\ &= e^t\lambda\rho(JE, E)+\rho(JE, \exp t\,\mathrm{ad}(JE)P') \\ &= e^t\lambda\rho(JE, E) . \end{aligned}$$

Hence we have

$$h'(t)+h(t) = \lambda\rho(JE, E).$$

The solution of this equation is $h(t)=ce^{-t}+\lambda\rho(JE, E)$, where c is an arbitrary constant. However, $h(t)$ is a polynomial and so $c=0$. Hence we have

$$h(t)=\lambda\rho(JE, E),$$

and hence it follows that

$$\lambda=\frac{h(t)}{\rho(JE, E)}=\frac{h(0)}{\rho(JE, E)}=\frac{\rho(JP, P)}{\rho(JE, E)}>0.$$

Therefore we have

$$\begin{aligned}2\eta(P, P) &= \psi([JP, P]) \\ &= \lambda\psi(E)+\psi(P') \\ &= \lambda\psi(E)>0.\end{aligned}$$

This shows that η is positive definite on $\mathfrak{p}_{(\frac{1}{2}+i\beta)}$ and hence on $\mathfrak{p}_{\frac{1}{2}}=\sum_{\beta}\mathfrak{p}_{(\frac{1}{2}+i\beta)}$.

Q.E.D.

Proof of Theorem 1′. Since η is positive definite on $\{JE\}+\{E\}+\mathfrak{p}_{\frac{1}{2}}$ and since the signature of h is $(2, 2(n-1))$, we have $\mathfrak{p}_{\frac{1}{2}}=\{0\}$, and hence $\mathfrak{p}=\mathfrak{p}_0$. Let $P, Q\in\mathfrak{p}$. Since $[P, Q]\in[\mathfrak{g}_0, \mathfrak{g}_0]\subset\mathfrak{g}_0$, where $\mathfrak{g}_0=\{JE\}+\mathfrak{p}$, we put $[P, Q]=\lambda JE+P'$, where $\lambda\in\boldsymbol{R}$, $P'\in\mathfrak{p}$. It follows that $[E, [P, Q]]=[E, \lambda JE+P']=-\lambda E$ and $[E, [P, Q]]=[[E, P], Q]+[P, [E, Q]]=0$. This implies that $\lambda=0$ and $[P, Q]\in\mathfrak{p}$. Therefore $\mathfrak{p}$ is a subalgebra of $\mathfrak{g}$ and also an ideal of $\mathfrak{g}$. Moreover we see easily that $(\mathfrak{p}, \mathfrak{k}, J, \rho)$ is an effective Kähler algebra. Since the decomposition $\mathfrak{g}=\{JE\}+\{E\}+\mathfrak{p}$ is orthogonal with respect to η and η is positive definite on $\{JE\}+\{E\}$ and since the signature of h is $(2, 2(n-1))$, we know that $\eta(P, P)<0$, for $P\in\mathfrak{p}$, $P\notin\mathfrak{k}$. Now, for $P, Q\in\mathfrak{p}$, put

$$\begin{aligned}&\psi'(P)=Tr_{\mathfrak{p}/\mathfrak{k}}(\operatorname{ad}(JP)-J\operatorname{ad}(P)), \\ &2\eta'(P, Q)=\psi'([JP, Q]).\end{aligned}$$

For $P\in\mathfrak{p}$, $P\notin\mathfrak{k}$, we have $(\operatorname{ad}(JP)-J\operatorname{ad}(P))E=0$, $(\operatorname{ad}(JP)-J\operatorname{ad}(P))JE\equiv0$ (mod $\mathfrak{k}$) and hence $\psi(P)=\psi'(P)$. This implies that

$$\begin{aligned}2\eta'(P, P) &= \psi'([JP, P]) \\ &= \psi([JP, P]) \\ &= 2\eta(P, P)<0,\end{aligned}$$

which proves that the canonical hermitian form of $(\mathfrak{p}, \mathfrak{k}, J, \rho)$ is negative definite. Therefore we know that $\mathfrak{p}$ is a compact semi-simple subalgebra of $\mathfrak{g}$ [5].

Q.E.D.

Proof of Theorem 1. When G is a semi-simple Lie group, our assertion follows from the results of Borel [1] and Koszul [3]. We shall show the case where G contains a one parameter normal subgroup of G. Let $\{E\}$ be the ideal of $\mathfrak{g}$ corresponding to the one parameter subgroup. With appropriate choice of J, we may assume that $J^2E=-E$. Put $\mathfrak{g}'=\{JE\}+\{E\}$. Then $(\mathfrak{g}', J, \rho)$ is a Kähler algebra of the unit disk $\{z\in \boldsymbol{C};\ |z|<1\}$. Now, for X, $Y\in\mathfrak{g}$, we define

$$\tilde{\rho}(X, Y)=\rho(X', Y')$$

where X', Y' are the $\mathfrak{g}'$-components of X, Y with respect to the decomposition $\mathfrak{g}=\mathfrak{g}'+\mathfrak{p}$ respectively. Then $(\mathfrak{g}, \mathfrak{p}, J, \tilde{\rho})$ is a Kähler algebra. We denote by G' (resp. P) the connected subgroup of G corresponding to $\mathfrak{g}'$ (resp. $\mathfrak{p}$). Since $(\mathfrak{g}, \mathfrak{p}, J, \tilde{\rho})$ is a Kähler algebra, G/P admits an invariant Kähler structure and is holomorphically isomorphic to the G'-orbit passing through the origin o. We know by Theorem 1′ that G/K is a holomorphic fibre bundle whose base space is $G/P\simeq\{z\in \boldsymbol{C};\ |z|<1\}$, and whose fibre is P/K. Q.E.D.

4. Proof of Theorem 2

Let $(\mathfrak{g}, \mathfrak{k}, J, \rho)$ be the Kähler algebra of G/K. We show that, if $\mathfrak{g}$ is not semi-simple, then there exists a one dimensional ideal of $\mathfrak{g}$. Assume that $\mathfrak{g}$ is not semi-simple. Then there exists a non-zero commutative ideal $\mathfrak{r}$ of $\mathfrak{g}$. Consider a J-invariant subalgebra $\mathfrak{g}'=\mathfrak{k}+J\mathfrak{r}+\mathfrak{r}$. Then we have

Lemma 4.1. *$\dim_C \mathfrak{g}'/\mathfrak{k}=1$.*

Since $\mathfrak{k}\cap\mathfrak{r}=\{0\}$ by Lemma 2.3 and $\dim_C \mathfrak{g}/\mathfrak{k}=2$, $\dim_C \mathfrak{g}'/\mathfrak{k}=1$ or 2. Suppose that $\dim_C \mathfrak{g}'/\mathfrak{k}=2$. Then $\dim_C \mathfrak{g}/\mathfrak{k}=\dim_C \mathfrak{g}'/\mathfrak{k}$, $\dim\mathfrak{g}=\dim\mathfrak{g}'$ and hence $\mathfrak{g}=\mathfrak{k}+J\mathfrak{r}+\mathfrak{r}$. Since $\dim\mathfrak{g}'/\mathfrak{k}=4$ and $\mathfrak{k}\cap\mathfrak{r}=\{0\}$, we have $2\leqq\dim\mathfrak{r}\leqq 4$. Let π' be the projection from $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{k}$. Then it follows that

$$\begin{aligned}\dim\pi'(J\mathfrak{r})\cap\pi'(\mathfrak{r}) &= \dim\pi'(J\mathfrak{r})+\dim\pi'(\mathfrak{r})-\dim\pi'(\mathfrak{g})\\ &= 2\dim\mathfrak{r}-4\,.\end{aligned}$$

First, we shall show $\dim\mathfrak{r}\neq 3, 4$. Suppose $\dim\mathfrak{r}=3$ or 4. Then $\dim\pi'(J\mathfrak{r})\cap\pi'(\mathfrak{r})>0$, and so there exist $A\neq 0$, $B\neq 0\in\mathfrak{r}$ and $W\in\mathfrak{k}$ such that $JA=B+W$. Therefore we have $2\eta(A, C)=\psi([JA, C])=\psi([B+W. C])=0$ and $2\eta(A, JC)=\psi([JA, JC])=\psi([A, C])=0$ for all $C\in\mathfrak{r}$. Since $\mathfrak{g}=\mathfrak{k}+J\mathfrak{r}+\mathfrak{r}$, we know $\eta(A, X)=0$ for all $X\in\mathfrak{g}$. This implies $A\in\mathfrak{k}$, which is a contradiction to Lemma 2.3. Next, we shall prove $\dim\mathfrak{r}\neq 2$. Suppose $\dim\mathfrak{r}=2$. Then $\dim\pi'(J\mathfrak{r})\cap\pi'(\mathfrak{r})=0$, and hence $\mathfrak{g}=\mathfrak{k}+J\mathfrak{r}+\mathfrak{r}$ is a direct sum as vector spaces. Let A be an element in $\mathfrak{r}$ such that $\eta(A, B)=0$ for all $B\in\mathfrak{r}$. Since $\mathfrak{g}=\mathfrak{k}+J\mathfrak{r}+\mathfrak{r}$ and $2\eta(A, JB)=\psi([JA, JB])=\psi([A, B])=0$, we have $\eta(A, X)=0$ for any $X\in\mathfrak{g}$, which implies $A\in\mathfrak{k}$, and hence $A=0$ by Lemma 2.3. This shows that η is non-degenerate on $\mathfrak{r}$. Therefore there exists a unique non-zero element $E\in\mathfrak{r}$ such that $2\eta(E, A)=\psi(A)$ for all $A\in\mathfrak{r}$. We have then

$$(4.1)\qquad \begin{aligned}&[JE, E] = E,\\ &\psi(E) \neq 0.\end{aligned}$$

Indeed, for $A \in \mathfrak{r}$ we have

$$\begin{aligned}2\eta([JE, E], A) &= \psi([J[JE, E], A])\\ &= -\psi([[JE, E], JA])\\ &= \psi([[E, JA], JE]) + \psi([[JA, JE], E])\\ &= -\psi([E, JA]) + \psi([J[JA, E] + J[A, JE], E])\\ &= \psi(A) + \psi([JA, E]) + \psi([A, JE])\\ &= \psi(A).\end{aligned}$$

This shows that $[JE, E]=E$. Let F be an element in $\mathfrak{r}$ independent of E. Put $[JE, F]=\lambda E+\mu F$, where $\lambda, \mu \in \boldsymbol{R}$. Then $\psi(E)=Tr_{\mathfrak{g}/\mathfrak{k}}(\mathrm{ad}(JE)-J\mathrm{ad}(E))=2(1+\mu)$. We shall show $\psi(E)\neq 0$. Suppose $\psi(E)=0$. Then $\mu=-1$ and $\psi(F)=2\eta(E, F)=\psi([JE, F])=\lambda\psi(E)-\psi(F)=-\psi(F)$. Therefore $\psi(F)=0$, and hence $\psi=0$ on $\mathfrak{r}$, which is a contradiction to Lemma 2.4.

(4.2) There exists an element F in $\mathfrak{r}$ independent of E such that

$$\begin{aligned}&[JE, E] = E, && [JE, F] = \alpha F,\\ &[JF, E] = \beta F, && [JF, F] = -E,\\ &\psi(F) = 0,\end{aligned}$$

where $\alpha, \beta \in \boldsymbol{R}$.

Proof. By $2\eta(E, E)=\psi(E)\neq 0$, there exists $\tilde{F}\neq 0$ such that $2\eta(E, \tilde{F})=\psi(\tilde{F})=0$. Since η is non-degenerate on $\mathfrak{r}$ and the signature is (1,1), we have $\eta(E, E)\eta(\tilde{F}, \tilde{F})<0$. Put $[JE, \tilde{F}]=\alpha\tilde{F}+\alpha' E$, where $\alpha, \alpha' \in \boldsymbol{R}$. We have then $0=\psi(\tilde{F})=2\eta(E, \tilde{F})=\psi([JE, \tilde{F}])=\alpha\psi(\tilde{F})+\alpha'\psi(E)=\alpha'\psi(E)$, and hence $\alpha'=0$, and $[JE, \tilde{F}]=\alpha\tilde{F}$. Similarly we have $[J\tilde{F}, E]=\beta\tilde{F}$. Now, we put $[J\tilde{F}, \tilde{F}]=\gamma E+\delta\tilde{F}$, where $\gamma, \delta \in \boldsymbol{R}$. Then we have $0=\psi(\tilde{F})=Tr_{\mathfrak{g}/\mathfrak{k}}(\mathrm{ad}(J\tilde{F})-J\mathrm{ad}(\tilde{F}))=2\delta$ and so $[J\tilde{F}, \tilde{F}]=\gamma E$. Since $2\eta(\tilde{F}, \tilde{F})=\psi([J\tilde{F}, \tilde{F}])=\gamma\psi(E)=2\gamma\eta(E, E)$, it follows $\gamma=\dfrac{\eta(\tilde{F}, \tilde{F})}{\eta(E, E)}<0$. Putting $F=\dfrac{1}{\sqrt{-\gamma}}\tilde{F}$, we have $[JF, F]=-E$.

Q.E.D.

$$(4.3)\qquad \mathfrak{k} = \{0\}.$$

Proof. For $W\in\mathfrak{k}$, put $[W, E]=\lambda E+\mu F$. Since $0=\psi([W, E])=\lambda\psi(E)+\mu\psi(F)=\lambda\psi(E)$, $\lambda=0$ and hence $[W, E]=\mu F$. We have $\psi([JF, [W, E]])=-\mu\psi(E)$ and $\psi([JF, [W, E]])=\psi([[JF, W], E])+\psi([W, [JF, E]])=\psi([J[F, W], E])=\psi([JE, [F, W]])=\psi([F, W])=0$. Thus $\mu=0$ and $[W, E]=0$. Now, put $[W, F]=\lambda E+\mu F$. By $0=\psi([W, F])=\lambda\psi(E)+\mu\psi(F)=\lambda\psi(E)$,

we have $\lambda=0$ and $[W, F]=\mu F$. Hence it follows $\psi([JF, [W, F]])=-\mu\psi(E)$. On the other hand we have $\psi([JF, [W, F]])=\psi([[JF, W], F])+\psi([W, [JF, F]])=\psi([J[F, W], F])=-\mu\psi([JF, F])=\mu\psi(E)$. Therefore $2\mu\psi(E)=0$, and hence $\mu=0$, $[W, F]=0$. Thus $[\mathfrak{k}, \mathfrak{r}]=0$. Since $[\mathfrak{k}, J\mathfrak{r}]\subset\mathfrak{k}$, $[\mathfrak{k}, \mathfrak{k}]\subset\mathfrak{k}$ and $\mathfrak{g}=\mathfrak{k}+J\mathfrak{r}+\mathfrak{r}$, we know that $\mathfrak{k}$ is an ideal of $\mathfrak{g}$. By the effectiveness, we have $\mathfrak{k}=\{0\}$.
Q.E.D.

(4.4) $$2\alpha=\beta+1\,.$$

Proof. Using Jacobi identity and (1.3), we have

$$\begin{aligned}0 &= [[JE, JF], F]+[[JF, F], JE]+[[F, JE], JF]\\ &= [[J[JE, F]+J[E, JF], F]+[[JF, F], JE]+[[F, JE], JE]\\ &= (\alpha-\beta)[JF, F]-[E, JE]-\alpha[F, JF]\\ &= (-2\alpha+\beta+1)E\,.\end{aligned}$$

Hence it follows $2\alpha=\beta+1$. Q.E.D.

By (1.7), (4.2) and (4.4), we have

$$\begin{aligned}0 &= \rho([JE, F], JF)+\rho([F, JF], JE)+\rho([JF, JE], F)\\ &= \alpha\rho(F, JF)+\rho(E, JE)-(\alpha-\beta)\rho(JF, F)\\ &= (-2\alpha+\beta)\rho(JF, F)-\rho(JE, E)\\ &= -\rho(JF, F)-\rho(JE, E)\,.\end{aligned}$$

This contradicts to $\rho(JE, E)>0$, $\rho(JF, F)>0$. Therefore $\dim\mathfrak{r}\neq 2$ and hence $\dim_C \mathfrak{g}'/\mathfrak{k}\neq 2$. Thus we have proved $\dim_C \mathfrak{g}'/\mathfrak{k}=1$, this completes the proof of Lemma 4.1. Q.E.D.

Let $\mathfrak{r}\neq\{0\}$ be a commutative ideal of $\mathfrak{g}$. Since $\dim \mathfrak{g}'/\mathfrak{k}=2$ by Lemma 4.1 and $\mathfrak{k}\cap\mathfrak{r}=\{0\}$, it follows that $\dim\mathfrak{r}=1$ or 2. Assume $\dim\mathfrak{r}=2$. Then we have

$$\begin{aligned}\dim \pi'(J\mathfrak{r})\cap\pi'(\mathfrak{r}) &= \dim \pi'(J\mathfrak{r})+\dim \pi'(\mathfrak{r})-\dim(\pi'(J\mathfrak{r})+\pi'(\mathfrak{r}))\\ &= 2\dim\mathfrak{r}-2\\ &= 2\,.\end{aligned}$$

This implies $\pi'(J\mathfrak{r})=\pi'(\mathfrak{r})$ and hence $J\mathfrak{r}\subset\mathfrak{k}+\mathfrak{r}$. For any $A\in\mathfrak{r}$, we have $JA=A'+W$, where $A'\in\mathfrak{r}$, $W\in\mathfrak{k}$. It follows then

$$\begin{aligned}\psi(A) &= Tr_{\mathfrak{g}/\mathfrak{k}}(\mathrm{ad}(JA)-J\mathrm{ad}(A))\\ &= Tr_{\mathfrak{g}/\mathfrak{k}}(\mathrm{ad}(A')-J\mathrm{ad}(A))+Tr_{\mathfrak{g}/\mathfrak{k}}\mathrm{ad}(W)\\ &= Tr_{\mathfrak{k}+\mathfrak{r}/\mathfrak{k}}(\mathrm{ad}(A')-J\mathrm{ad}(A))\\ &= 0\,.\end{aligned}$$

Hence $\psi=0$ on $\mathfrak{r}$, which is a contradiction to Lemma 2.4. Thus $\mathfrak{r}$ is a one dimensional ideal of $\mathfrak{g}$. Therefore Theorem 2 is proved. Q.E.D.

5. We shall classify two dimensional connected simply connected homogeneous Kähler manifolds with non-degenerate canonical hermitian form h. The signature of h is (4, 0) or (2, 2) or (0, 4).

(i) The case (4, 0). Since h is positive definite, G/K is isomorphic to a homogeneous bounded domain. Hence G/K is either $\{z\in \boldsymbol{C};\ |z|<1\}\times\{z\in \boldsymbol{C};\ |z|<1\}$ or $\{(z_1, z_2)\in \boldsymbol{C}^2;\ |z_1|^2+|z_2|^2<1\}$.

(ii) The case (0, 4). Since h is negative definite, G is a compact semisimple Lie group by [5]. By a theorem in [4], G/K is a hermitian symmetric space. Hence G/K is either $P_1(\boldsymbol{C})\times P_1(\boldsymbol{C})$ or $P_2(\boldsymbol{C})$, where $P_n(\boldsymbol{C})$ is a complex n-dimensional projective space.

(iii) The case (2, 2). Applying Theorem 1 and 2, we obtain that G/K is a holomorphic fibre bundle whose base space is the unit disk $\{z\in \boldsymbol{C};\ |z|<1\}$, and whose fibre is $P_1(\boldsymbol{C})$.

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