Homogeneous ultrametric spaces

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joint work with

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Ultrametric space

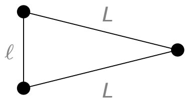
Ultrametric space : definition and examples

Let $R := [0, +\infty[$ and $R_* :=]0, +\infty[$.

Definition : Ultrametric space

An *ultrametric* space is a pair $\mathbf{M} = (X, d)$ with $d : X \times X \to R$ s.t. :

1. $d(x, y) = 0 \iff x = y$ (separation).
2. $d(x, y) = d(y, x)$ (symmetry).
3. $d(x, z) = \max\{d(x, y), d(y, z)\}$ (strong triangular inequality).



A metric space is ultrametric if and only if its triangles are isosceles acute, if and only if any two meeting balls are comparable for inclusion.

Examples

1. $(R, \max)$, given by $d(x, y) = \max\{x, y\}$ if $x \neq y$ and $d(x, x) = 0$.
2. Baire and Cantor spaces, $\mathbb{Q}_p$, *etc.*

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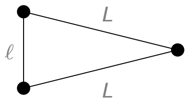

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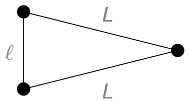

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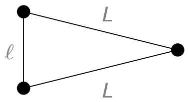

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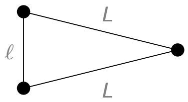

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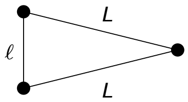

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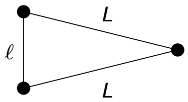

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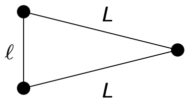

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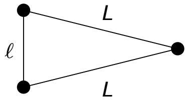

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Ultrametric space : balls

Balls of radius $r \in R$ centered at $x \in X$

- $B_<(x, r) := \{y \in X : d(x, y) < r\}$: *open ball*
- $B_\leqslant(x, r) := \{y \in X : d(x, y) \leqslant r\}$: *closed ball*

Basic properties

- Open balls are clopen, as well as closed balls of non-zero radius. Ultrametric spaces are totally-discontinuous.
- Meeting balls are comparable *w.r.t.* inclusion.
- Each point of a ball is a center : $y \in B(x, r) \Rightarrow B(x, r) = B(y, r)$.
- The diameter of a ball is the least of its radii. In particular : $\operatorname{diam}(B_\leqslant(x, d(x, y))) = d(x, y)$.

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The R-labeled tree Nerve($\mathbf{M}$)

Nerve of an ultrametric space : definition

A ball attains its diameter if and only if it is of the form $B_\leqslant(x, d(x, y))$. The diameter of such a ball is the radius : $\mathrm{diam}(B_\leqslant(x, d(x, y))) = d(x, y)$.

Definition

The *nerve* of $\mathbf{M}$ is the collection of closed balls attaining their diameter :

$$\mathrm{Nerve}(\mathbf{M}) := \{B_\leqslant(x, d(x, y)) : x, y \in X\}$$

This is a *leafy tree* for the relation of inclusion :

- This is a join-semi-lattice, *i.e.*, every pair of nodes has a supremum : $B_\leqslant(x, r) \vee B_\leqslant(x', r') = B_\leqslant(x, \max\{r, r', d(x, x')\})$
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Nerve of an ultrametric space. Sons

Nerve of an ultrametric space : definition

Given an inner node of the nerve, B of diameter $r > 0$,

$$y \sim z :\Longleftrightarrow d(y,z) < r$$

is an equivalence relation on B. Its classes are the open balls of radius r centered in B.

Sons of an inner node of the nerve

$\forall x \neq y : \mathrm{Sons}\, B_{\leqslant}(x, d(x,y)) := \{B_{<}(z, d(x,y)) : z \in B_{\leqslant}(x, d(x,y))\}$
The *degree* of a member of the nerve is the number of its sons.

A son of a node of Nerve($\mathbf{M}$) may fail to belong to Nerve($\mathbf{M}$). It is indeed its son in the tree of all non-empty balls.
In $(R, \max)$, $\mathrm{Sons}([0,1]) = \{[0,1[, \{1\}\}$.

Nerve of an ultrametric space. Sons

Nerve of an ultrametric space : definition

Given an inner node of the nerve, B of diameter $r > 0$,

$$y \sim z :\Longleftrightarrow d(y,z) < r$$

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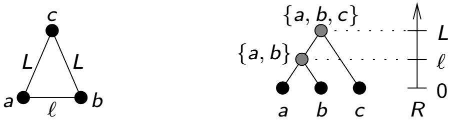

Figure : A triangle and its nerve

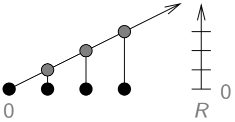

Figure : The nerve of $(R, \max)$

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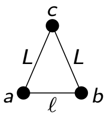

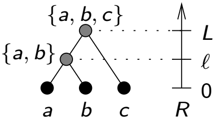

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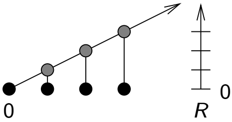

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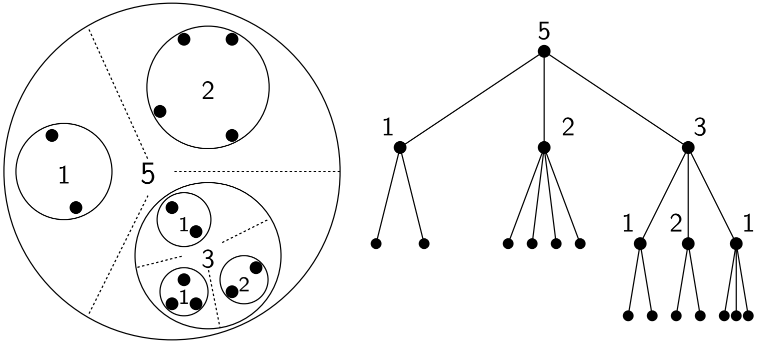

Figure : The nerve of a finite ultrametric space

Ultrametric space

Examples : spaces of sequences and spaces of functions

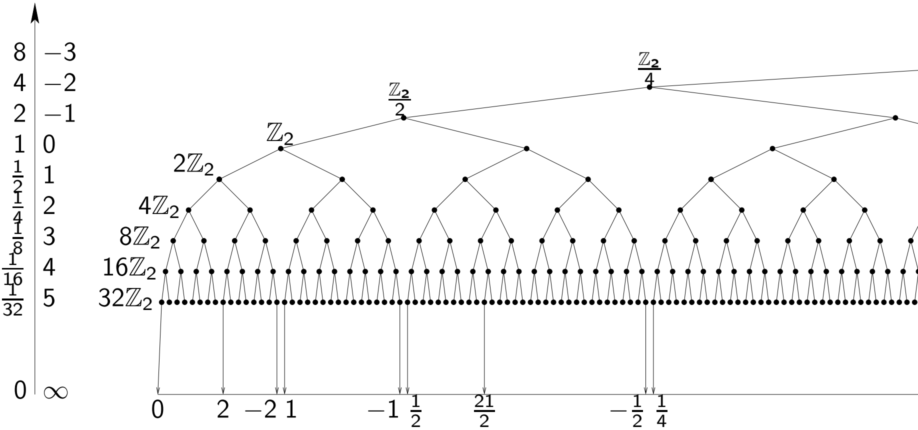

Figure : The nerve of $\mathbb{Q}_2$ has no root

Ultrametric space

Cauchy- and spherical completeness

Spherical completeness

An ultrametric space **M** is *spherically-complete* if every chain of non-empty balls has a non-empty intersection, if and only if every chain of the nerve has a lower bound.

Requiring above that the infimum of the diameters of the members of the chain be 0 yields Cauchy-completeness.

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$(]1,\infty[, \max)$ is Cauchy- but not spherically-complete :
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Motivation : indivisibility of $\mathbf{U}_{\mathbb{Q}\cap[0,1]}$?

Definition (Indivisible metric space)

A metric space $\mathbf{M}$ is *indivisible* if it embeds in some class of each of its finite partitions : $(\mathbf{M} \to [\mathbf{M}]^{\bullet}_{k})$.

Question (Hjorth)

Is the bounded rational Urysohn space indivisible ?

Theorem (DLPS07 : no)

Every Cantor connected metric space is divisible.
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Two points x and y of a metric space **M** are ε-chainable ($\varepsilon > 0$) if there is finite sequence of points ($x = z_0, z_1, \ldots, z_n = y$) such that $d(x_i, x_{i+1}) \leq \varepsilon$ ($0 \leq i < n$).
They are *chainable* if they are ε-chainable for every ε.
M is *Cantor connected* if any two points are chainable.
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Example : ultrametric spaces are totally Cantor disconnected.

Proposition (Lemin)

A metric space **M** *is totally Cantor disconnected if and only if its distance is* $\geq$ *a ultrametric one. In this case there is a greatest such ultrametric distance :* $d_{\mathrm{Ult}}(x, y)$ *is the supremum of the witnessing* ε*'s.*

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Homogeneous ultrametric spaces

- A *local isometry* of a metric space (X, d) is an isometry form a finite subspace of X into X.
- The metric space is *(ultra)-homogeneous* if every local isometry extends to a bijective isometry of X.
- The metric space is *transitive* if the action of its isomorphy group is transitive on points.

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An ultrametric space is homogeneous as soon as it is transitive.

The countable case is in [DLPS07], and the Polish case is in [Malicki-Solecki09].

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Proposition (Three equivalent formulations)

Consider an ultrametric space $\mathbf{M} = (X, d)$.

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2. $(X, X^2 \xrightarrow{d} R, X \to X/\operatorname{Aut}(\mathbf{M}))$ *is homogeneous.*
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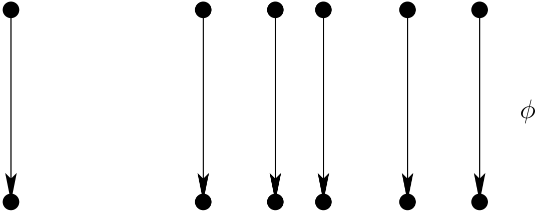

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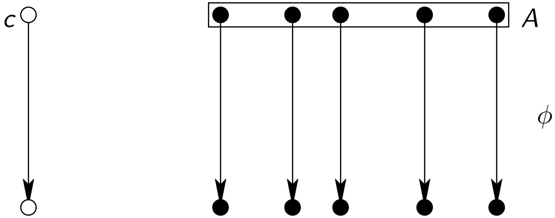

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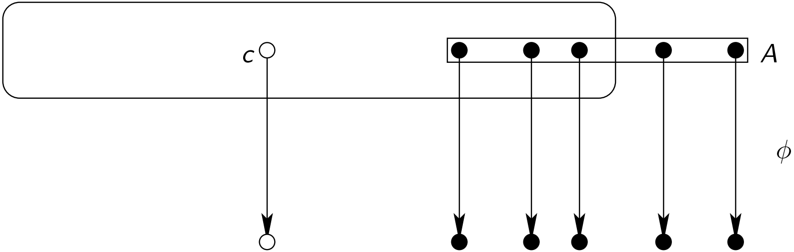

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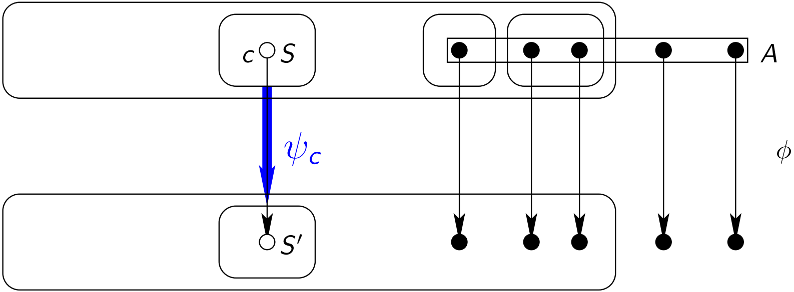

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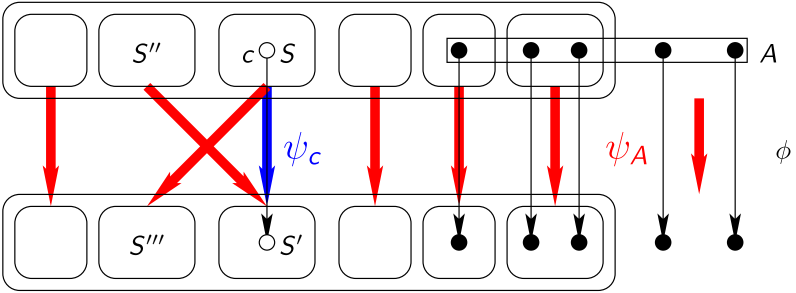

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$S'' := \psi_A^{-1}[S']$

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Extending a local isometry $\phi : A \dot\cup \{c\} \to X$ from extensions $\psi_A \supset \phi \restriction A$ and $\psi_c \supset \phi \restriction \{c\}$

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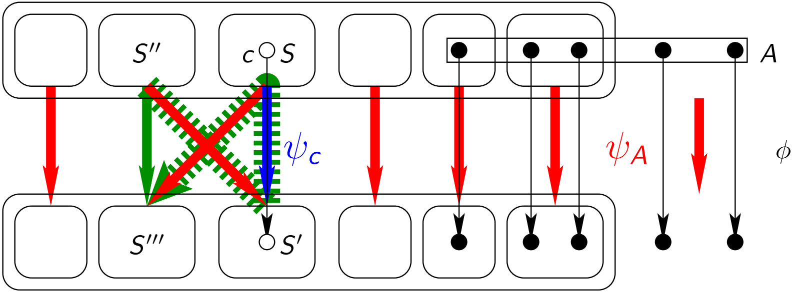

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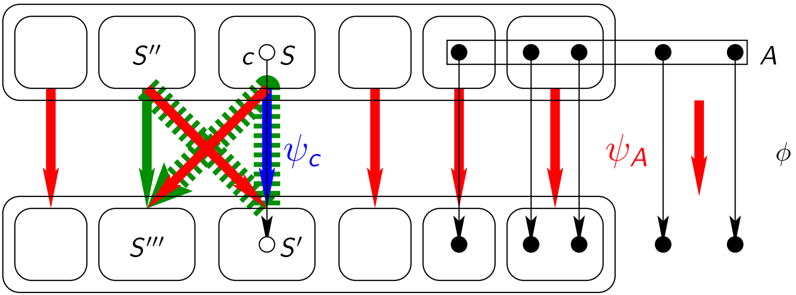

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$$\psi : \begin{pmatrix} S & S'' & \text{elsewhere} \\ \psi_c & \psi_A \psi_c^{-1} \psi_A & \psi_A \end{pmatrix}$$

Hereditarily decomposable symmetric binary structure

Remark

The homogeneity of $(\mathbf{M}, X \to X/\operatorname{Aut}(\mathbf{M}))$ holds more generally for every *hereditarily decomposable symmetric binary relational structure* $\mathbf{M}$.

A *module* of an irreflexive binary relational structure $(X, R_i : i \in I)$ is a set M of vertices of which the elements all look alike for each vertex outside.

The empty set, the singletons, and the vertex set are trivially modules.

A structure is *prime* if it has at least three vertices and all its modules are trivial.

A structure is *hereditarily decomposable* if it embeds no prime structure.

- Notice that a linearly ordered set is hereditarily decomposable but not symmetric.
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Each binary relational structure has a modular decomposition tree from which it is recoverable. For an ultrametric space, this is its nerve.

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Hereditary decomposable 2-structure

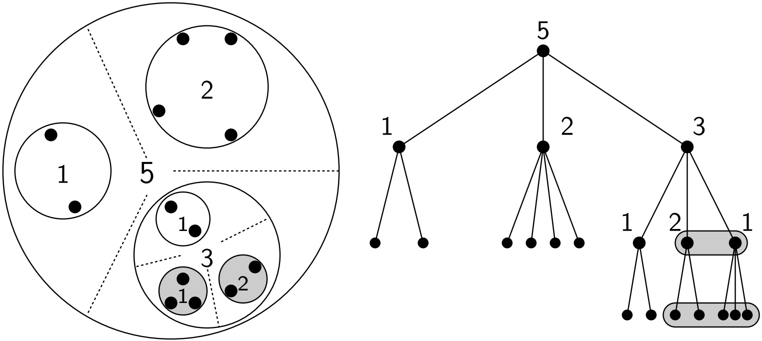

Figure : Besides the balls, the modules of an ultrametric space look like this

Spectrum and degree sequence of an ultrametric space $\mathbf{M} = (X, d)$

Spectrum

The *spectrum* of a point $x \in X$ is the set of the distances that it realizes :

$$\mathrm{spec}(x) := \{d(x, y) : y \in X\} \subseteq R$$

The *spectrum* of $\mathbf{M}$ is the set of all realized distances :

$$\mathrm{spec}(\mathbf{M}) := \{d(x, y) : x, y \in X\} = \cup\{\mathrm{Spec}(x) : x \in X\}$$

Degree sequence

$$\mathrm{Dgr} : \begin{cases} \mathrm{spec}(\mathbf{M}) \to \mathbf{CARD}_* \\ r \mapsto \sup\{\mathrm{card}(\mathrm{Sons}(B)) : B \in \mathrm{Nerve}(\mathbf{M}), \mathrm{diam}(B) = r\} \end{cases}$$

maps each element r of the spectrum to the supremum of the degrees of nodes of Nerve($\mathbf{M}$) of diameter r.

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If an ultrametric space $\mathbf{M} = (X, d)$ is homogeneous, then :

- its points have the same spectrum and
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Say of such a space that it is *sub-homogeneous*.

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Ultrametric space $\mathbf{M}_\nu^{\mathcal{I}}$

Definitions

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For x and y in $\mathbf{ON}^{V_*}$, let :

- $\Delta(x, y) := \{r \in V_* : x(r) \neq y(r)\}$,
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Proposition

Letting $X_\nu^{\mathcal{I}} := \{x \in \mathbf{ON}^{V_*} : x < \nu$ *and* $\mathrm{supp}(x) \in \mathcal{I}\}$ *:*

- $\mathbf{M}_\nu^{\mathcal{I}} := (X_\nu^{\mathcal{I}}, \sup \Delta)$ *is a an ultrametric space ;*
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- $\mathbf{M}$ is spherically complete and homogeneous if and only if it is isomorphic to some $\mathbf{M}_\nu^{\mathrm{coWell}(V_*)}$.

- (Delon) $\mathbf{M}_\nu^{\mathrm{Fin}(V_*)}$ embeds in every sub-homogeneous ultrametric space $\mathbf{M}$ with $V \subseteq \mathrm{spec}(\mathbf{M})$ and $\nu \leq \mathrm{Dgr}(\mathbf{M})$.
- (Lemin) If $\mathbf{M}$ has density at most κ and spectrum inlcuded in V, then it embeds in $\mathbf{M}_\kappa^{\mathrm{coWell}(V_*)}$.

For each ordinal ξ, let $\mathrm{coWell}(V_*, \xi)$ denote the ideal of those W of type less than ω^ξ. If $V \supseteq \mathbb{Q}_+$ and $\nu \geq 2$, then the family $(\mathbf{M}_\nu^{\mathrm{coWell}(V_*,\xi)} : \xi < \omega_1)$ is strictly increasing *w.r.t.* embeddability.

Consequences

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Spec-homogeneity

Spec-homogeneity : definition

Definition (Local spec-isometry)

A *local spec-isometry* of $\mathbf{M}$ is a local automorphism of the enriched structure $(X, X^2 \xrightarrow{d} R, X \xrightarrow{\text{spec}} \wp(R))$.

Thus a local spec-isometry is a local isometry that maps each point to a point with the same spectrum.

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The ultrametric space $\mathbf{M} = (X, d)$ is *spec-homogeneous* if every local spec-isometry of $\mathbf{M}$ extends to a bijective isometry.

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Spectral homogeneity

Characterization

Definition (Similar balls)

Two balls of the same radius and type ("open" or "closed") B_1 and B_2 are *similar* if they have points with the same spectrum :
$\exists x_1 \in B_1, x_2 \in B_2 : \mathrm{spec}(x_1) = \mathrm{spec}(x_2)$.

Theorem

An ultrametric space is spectral-homogenous if and only if any two similar balls are isomorphic.

Corollary

An ultrametric space is homogenous if and only if all points have the same spectrum and any two balls of the same type ("open" or "closed") and the same radius are isomorphic.

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The Cauchy-completion of a homogeneous ultrametric space is homogeneous.

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A countable ultrametric space is spectral-homogenous if and only if any two similar members of the nerve are isomorphic.

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In particular a countable ultrametric space is spectral-homogenous if any two members of the nerve with the same diameter are isomorphic.

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Can the countabitliy condition in the above statements be lifted ?

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Therefore $(C_1, C_2) \mapsto \mathrm{dist}(C_1, C_2)$ is an ultrametric distance on $X/\,\mathrm{Aut}(\mathbf{M})$.

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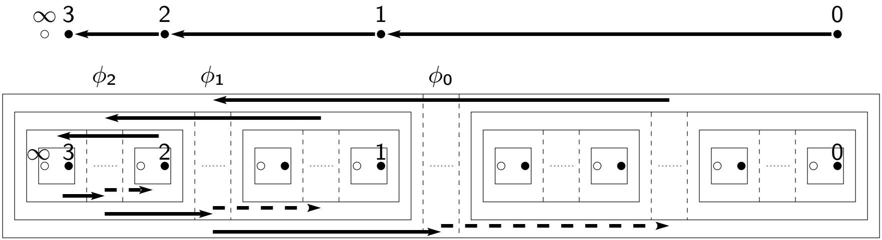
The Aut(**M**)-orbits are closed

$$0 \leftarrow \cdots < d(\infty, x_n) = r_n < \cdots < d(\infty, x_0) = r_0$$



$$\phi_n : \begin{cases} x_n \xrightarrow{\phi_n} x_{n+1} \\ \phi_n \restriction X \setminus B_{\leqslant}(x_n, r_n) = \mathrm{id} \end{cases}$$

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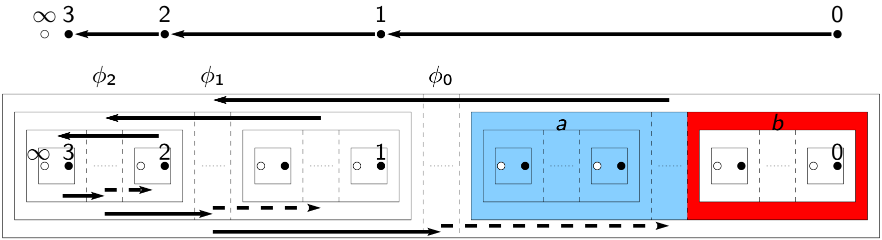

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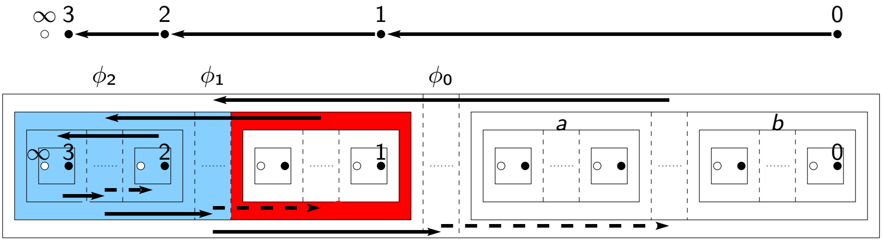

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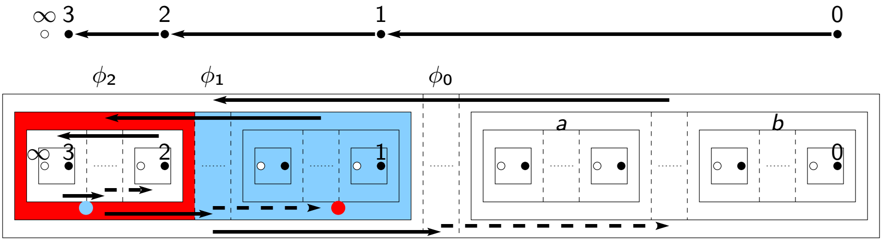

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