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On homological classification of pomonoids by regular weak injectivity properties of S-posets

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Abstract: If S is a partially ordered monoid then a right S-poset is a poset A on which S acts from the right in such a way that the action is compatible both with the order of S and A. By regular weak injectivity properties we mean injectivity properties with respect to all regular monomorphisms (not all monomorphisms) from different types of right ideals of S to S. We give an alternative description of such properties which uses systems of equations. Using these properties we prove several so-called homological classification results which generalize the corresponding results for (unordered) acts over (unordered) monoids proved by Victoria Gould in the 1980's.

Keywords: Ordered monoid, S-poset, weak injectivity
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1 Introduction

In the 1980's Victoria Gould characterized several classes of monoids using the injectivity properties of acts (or systems) over them ([4],[5],[6]). Our aim is to prove the analogues of those results in the case of ordered acts (S-posets) over ordered monoids. We make use of regular weak injectivities by which we mean injectivities with respect to regular monomorphisms from different types of ideals to the ordered monoid.

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After giving the necessary preliminaries, in Section 2 we prove, following [4], a result that describes regular weak injectivity properties using systems of equations. In Section 3 we give a construction, that allows for a given S-poset A to construct a regularly divisible, regularly principally weakly injective or regularly fg-weakly injective S-poset that contains A as a regular S-subposet. This construction will be the main tool for obtaining the desired homological classification results in Section 4.

2 Preliminaries

Throughout this paper S will denote a partially ordered monoid (shortly pomonoid), that is, a monoid with a partial order relation $\leq$ such that $s \leq t$ implies $su \leq tu$ and $us \leq ut$ for every $s, t, u \in S$. A poset $(A, \leq)$ together with a mapping $A \times S \to A, (a, s) \mapsto as$, is called a *right S-poset* (and the notation A_S is used) if (1) $a(st) = (as)t$, (2) $a1 = a$, (3) $a \leq b$ implies $as \leq bs$, and (4) $s \leq t$ implies $as \leq at$, for all $a, b \in A$, $s, t \in S$. In this paper we only consider right S-posets, so we usually drop the word 'right'. If A satisfies conditions (1) and (2) then it is called *a right S-act* (see [7]) or *a right S-system* (see, e.g., [4]). Definitions and results about S-acts, used in this paper, can be found in [7]. Morphisms of S-posets are action and order preserving mappings. From [2] we know that in the category of right S-posets monomorphisms are injective morphisms but regular monomorphisms are embeddings, i.e. morphisms $\iota : A_S \to B_S$ such that $a \leq a'$ if and only if $\iota(a) \leq \iota(a')$, $a, a' \in A$. So not every monomorphism of S-posets needs to be regular. For every S-poset A_S and its element a, $\lambda_a : S_S \to A_S$ will denote the right S-poset morphism defined by $\lambda_a(s) = as$ for every $s \in S$.

A poset $(A, \leq_A)$ is called *a (regular) S-subposet* of a right S-poset $(B, \leq_B)$, if A_S is a subact of B_S and $\leq_A \subseteq (\leq_B \cap A^2)$ (resp. $\leq_A = (\leq_B \cap A^2)$). By right ideals of S we mean algebraic ideals, i.e. subsets $I \subseteq S$ such that $IS \subseteq I$. When we consider a right ideal I as a right S-poset, we mean that its order is induced by the order of S.

For a binary relation σ on an S-poset A_S, we write $a \underset{\sigma}{\leq} a'$ if there exist $a_1, \ldots, a_n \in A$ such that

$$a \leq a_1 \sigma a_2 \leq a_3 \sigma \ldots \sigma a_n \leq a'.$$

Such a sequence of elements is called an *σ-chain* connecting a and a'. An *S-poset congruence* (see [3]) on an S-poset A_S is an S-act congruence θ on A, that satisfies the so-called closed chains condition:

$$a \underset{\theta}{\leq} a' \underset{\theta}{\leq} a \Longrightarrow a \theta a'$$

for every $a, a' \in A$. If $H \subseteq A \times A$ is a subset then *the S-poset congruence $\theta(H)$ on A generated by H* (see [1]) is defined by

$$a\theta(H)a' \Longleftrightarrow a \underset{\rho}{\leq} a' \underset{\rho}{\leq} a, \tag{1}$$

$a, a' \in A$, where $\rho = \rho(H)$ is the S-act congruence on A generated by H. The factor

S-poset $A/\theta(H)$ is equipped with the order

$$[a]_{\theta(H)} \leq [a']_{\theta(H)} \Longleftrightarrow a \underset{\rho}{\leq} a'. \tag{2}$$

This makes the canonical epimorphism $A \to A/\theta(H)$ a regular epimorphism (see [2]).

For a set Γ, one can consider *the free right S-poset on* Γ (see [9]) as a set $\Gamma \times S$ with the right S-action defined by $(\gamma, s)t = (\gamma, st)$ and the order relation by $(\gamma, s) \leq (\delta, t)$ if and only if $\gamma = \delta$ and $s \leq t$, $\gamma, \delta \in \Gamma$, $s, t \in S$. We shall write shortly γs instead of $(\gamma, s) \in \Gamma \times S$.

We call an element $c \in S$ *left po-cancellable* if $cs \leq ct$ implies $s \leq t$ for all $s, t \in S$. We denote the set of all left po-cancellable elements of S by C.

We write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for the set of nonnegative integers.

3 Regularly (α, R)-injective acts

We say that a subset $R \subseteq S$ is *closed under regular monomorphisms* if $\iota(r) \in R$ for every $r \in R$ and regular monomorphism $\iota : rS \to S$. It is easy to see that S and the set of all left (po-)cancellable elements of S are closed under regular monomorphisms.

Let α be any cardinal greater than 1 and let R be a subset of S that is closed under regular monomorphisms. We call a right ideal I of S a *right* (α, R)*-ideal*, if I has a generating set $G \subseteq R$ of fewer than α elements. If $R = S$ then we speak of just *right* α*-ideals*. So the right $(2, C)$-ideals of S are principal right ideals generated by left po-cancellable elements, right 2-ideals are principal right ideals and right $\aleph_0$-ideals are finitely generated right ideals.

We say that an S-poset A_S *satisfies the* (α, R)*-Baer criterion* (cf. [4]) if every S-poset morphism $f : I \to A$, where I is a right (α, R)-ideal, is given by the left multiplication by some element $a \in A$, i.e. $f = \lambda_a$.

We say that an S-poset A_S is *(regularly)* (α, R)*-injective* if for every right (α, R)-ideal I of S, every (regular) monomorphism $\iota : I \to S$ and every S-poset morphism $f : I \to A$ there exists an S-poset morphism $g : S \to A$ such that the diagram

$$\begin{tikzcd} I \arrow[rr, "\iota"] \arrow[dr, "f"'] & & S \arrow[dl, dashed, "g"] \\ & A & \end{tikzcd}$$

is commutative. If $R = S$, we speak of *(regular)* α*-injectivity*. So (regularly) 2-injective S-posets are (regularly) principally weakly injective S-posets and (regularly) $\aleph_0$-injective S-posets are (regularly) fg-weakly injective S-posets.

We say that an S-poset A_S is *(regularly) divisible* (cf. [6]) if $A = Ac$ for every left (po-)cancellable element $c \in S$. The next lemma shows that regular divisibility can be considered as an injectivity property.

Lemma 3.1. *The following conditions are equivalent for an S-poset A_S:*

(i) A_S *is regularly* $(2, C)$*-injective,*

(ii) A_S *is regularly* $(2, \{1\})$*-injective,*

(iii) A_S *is regularly divisible.*

Proof. $(i) \Rightarrow (ii)$. This is clear, because $1 \in C$.

$(ii) \Rightarrow (iii)$. Let A_S be regularly $(2, \{1\})$-injective, let $c \in S$ be a left po-cancellable element and let $a \in A$. Since, for every $s, t \in S$, $s \leq t$ if and only if $cs \leq ct$, the mapping $\lambda_c : S \to S$ is a regular monomorphism of S-posets.

$$\begin{tikzcd} S_S \arrow[rr, "\lambda_c"] \arrow[dr, "\lambda_a"'] & & S_S \arrow[dl, dashed, "g"] \\ & A_S & \end{tikzcd}$$

By the assumption, there exists an S-poset morphism $g : S \to A$ such that $\lambda_a = g\lambda_c$. Hence

$$a = \lambda_a(1) = g\lambda_c(1) = g(c) = g(1)c \in Ac.$$

$(iii) \Rightarrow (i)$. Suppose that A is regularly divisible. Consider a left po-cancellable element c, a regular monomorphism $\iota : cS \to S$ and an S-poset morphism $f : cS \to A$. Then $c' = \iota(c) \in S$ is also a left po-cancellable element and hence $f(c) = bc'$ for some $b \in A$. Consequently, for every $s \in S$,

$$\lambda_b \iota(cs) = \lambda_b(c's) = bc's = f(c)s = f(cs).$$

So we have the following implications among regular weak injectivity properties of S-posets:

$$\text{regularly weakly injective} \Rightarrow \text{regularly fg-weakly injective} \Rightarrow$$

$$\Rightarrow \text{regularly principally weakly injective} \Rightarrow \text{regularly divisible.}$$

Our next aim is to describe regularly (α, R)-injective S-posets using systems of equations over them. A set Σ of equations with constants from an S-poset A_S is called *consistent* if Σ has a solution in some S-poset B_S that contains A as a regular S-subposet. If α is any cardinal larger than that of Σ, if all equations in Σ are of the form $xs = a$, where $s \in R$ and $a \in A$, and if the same unknown x appears in each equation then we call Σ an (α, R)*-system over* A.

The following two results are analogues of Lemma 3.2 and Proposition 3.3 of [4], respectively.

Lemma 3.2. *Let* A_S *be an* S*-poset,* $R \subseteq S$ *a subset that is closed under regular monomorphisms,* α *a cardinal,* J *a set with* $|J| < \alpha$ *and*

$$\Sigma = \{xs_j = a_j \mid j \in J, s_j \in R, a_j \in A\}$$

an (α, R)-system over A. Then Σ is consistent if and only if for all $u, v \in S$ and $i, j \in J$,

$$s_i u \leq s_j v \Longrightarrow a_i u \leq a_j v.$$

Proof. Necessity. If Σ is consistent then there is an S-poset $(B_S, \leq_B)$ and an element $b \in B$ such that $(A_S, \leq_A)$ is a regular S-subposet of B_S and b is a solution of Σ. If now $s_i u \leq s_j v$, $u, v \in S$, $i, j \in J$, then $a_i u = b s_i u \leq_B b s_j v = a_j v$. Since A is a regular S-subposet of B, we have $a_i u \leq_A a_j v$.

Sufficiency. Let z be a symbol which is not in A or S and consider the S-poset $B_S = A_S \amalg F_S$, where $F_S = (zS)_S$ is the free S-poset on $\{z\}$ and the S-action and order on disjoint union are defined componentwise. Let θ be the S-poset congruence on B generated by the set

$$H = \{(a_j, z s_j) \mid j \in J\} \subseteq B^2,$$

that is, for $b, b' \in B$,

$$b \theta b' \Longleftrightarrow b \underset{\rho}{\leq} b' \underset{\rho}{\leq} b,$$

where $\rho = \rho(H)$ is the S-act congruence on B_S generated by H. Using the assumption, one can show that $b \rho b'$ if and only if one of the following four cases is true:

(1) $b, b' \in A \cup F$ and $b = b'$,

(2) $b = z s_i u, b' = z s_j v \in F$ and $a_i u = a_j v$ for some $u, v \in S$ and $i, j \in J$,

(3) $b = a_j u \in A, b' = z s_j u \in F$ for some $u \in S$ and $j \in J$,

(4) $b = z s_j u \in F, b' = a_j u \in A$ for some $u \in S$ and $j \in J$.

Suppose that $b \underset{\rho}{\leq} b'$ where $b, b' \in A$. Using the above description of ρ we have either $b \leq b'$ or

$$b \leq d'_1 \rho y'_1 \underset{\rho|_F}{\leq} y_1 \rho d_2 \leq d'_2 \rho y'_2 \underset{\rho|_F}{\leq} y_2 \rho d_3 \ldots d'_n \rho y'_n \underset{\rho|_F}{\leq} y_n \rho d_{n+1} \leq b',$$

where $\rho|_F = \rho \cap F^2$, for some $n \in \mathbb{N}$ and elements $d'_1, \ldots, d'_n, d_2, \ldots, d_{n+1} \in A$, $y'_1, \ldots, y'_n$, $y_1, \ldots, y_n \in F$. Since $d'_r \rho y'_r$ and $y_r \rho d_{r+1}$, for every $r \in \{1, \ldots, n\}$ there exist $k_r, l_r \in J$ and $u_{k_r}, v_{k_r} \in S$ such that $d'_r = a_{k_r} u_{k_r}$, $y'_r = z s_{k_r} u_{k_r}$, $y_r = z s_{l_r} v_{l_r}$ and $d_{r+1} = a_{l_r} v_{l_r}$.

Now $y'_r \underset{\rho|_F}{\leq} y_r$ implies

$$z s_{k_r} u_{k_r} = y'_r \leq g_1 \rho h_1 \leq g_2 \rho h_2 \leq \ldots \leq g_p \rho h_p \leq y_r = z s_{l_r} v_{l_r}$$

for some $p \in \mathbb{N}$ and $g_m, h_m \in F$, $m \in \{1, \ldots, p\}$. From the description of ρ we obtain $i_m, j_m \in J$, $u_{i_m}, v_{j_m} \in S$, $m \in \{1, \ldots, p\}$, such that $g_m = z s_{i_m} u_{i_m}$, $h_m = z s_{j_m} v_{j_m}$ and $a_{i_m} u_{i_m} = a_{j_m} v_{j_m}$. Since $h_m \leq g_{m+1}$, we have $s_{j_m} v_{j_m} \leq s_{i_{m+1}} u_{i_{m+1}}$ for every $m \in \{1, \ldots, p-1\}$. Also $y'_r \leq g_1$ implies $s_{k_r} u_{k_r} \leq s_{i_1} u_{i_1}$ and $h_p \leq y_r$ implies $s_{j_p} v_{j_p} \leq s_{l_r} v_{l_r}$. By assumption, $a_{k_r} u_{k_r} \leq a_{i_1} u_{i_1}$, $a_{j_p} v_{j_p} \leq a_{l_r} v_{l_r}$ and $a_{j_m} v_{j_m} \leq a_{i_{m+1}} u_{i_{m+1}}$ for every $m \in \{1, \ldots, p-1\}$. Hence

$$d'_r = a_{k_r} u_{k_r} \leq a_{i_1} u_{i_1} = a_{j_1} v_{j_1} \leq a_{i_2} u_{i_2} = a_{j_2} v_{j_2} \leq \ldots \leq a_{j_p} v_{j_p} \leq a_{l_r} v_{l_r} = d_{r+1}$$

for every $r \in \{1, \ldots, n\}$. So $b \leq d'_1 \leq d_2 \leq d'_2 \leq \ldots \leq d_{n+1} \leq b'$, and we have proved that, for every $b, b' \in A$,

$$b \underset{\rho}{\leq} b' \Longleftrightarrow b \leq b'.$$

It follows that if $\pi : B \to B/\theta$, $b \mapsto [b]_\theta$, is the natural S-poset morphism then $\pi|_A$ is an embedding, thus we may identify the S-posets A and $\pi|_A(A) = \pi(A)$, and, moreover, $\pi(A)$ is a regular S-subposet of B. Since

$$a_j \equiv [a_j]_\theta = [zs_j]_\theta = [z]_\theta s_j$$

for every $j \in J$, $[z]_\theta$ is a solution of Σ in B/θ, so Σ is consistent.

Proposition 3.3. *The following conditions are equivalent for an S-poset A_S, a subset $R \subseteq S$ that is closed under regular monomorphisms, and a cardinal α:*

(i) every consistent (α, R)-system over A has a solution in A,

(ii) A satisfies the (α, R)-Baer criterion,

(iii) A is regularly (α, R)-injective.

Proof. $(i) \Rightarrow (ii)$. Let I be a right (α, R)-ideal of S, that is, $I = \bigcup_{j \in J} t_j S$, where $|J| < \alpha$ and $t_j \in R$ for every $j \in J$. Consider an S-poset morphism $f : I \to A$. Then

$$t_i u \leq t_j v \Longrightarrow f(t_i)u \leq f(t_j)v$$

for every $i, j \in J$ and $u, v \in S$. By Lemma 3.2,

$$\Sigma = \{xt_j = f(t_j) \mid j \in J\}$$

is a consistent (α, R)-system over A. By assumption, Σ has a solution a in A, which means that f is given by left multiplication by a.

$(ii) \Rightarrow (iii)$. Let I be a right (α, R)-ideal of S, that is, $I = \bigcup_{j \in J} t_j S$, where $|J| < \alpha$ and $t_j \in R$ for every $j \in J$, let $\iota : I \to S$ be a regular monomorphism and let $f : I \to A$ be an S-poset morphism. By assumption, there exists $a \in A$ such that $f(t_j) = at_j$ for every $j \in J$. Now $\iota(I) = \bigcup_{j \in J} \iota(t_j)S$ is also a right (α, R)-ideal of S. We define a mapping $h : \iota(I) \to A$ by

$$h(\iota(t_j)s) = at_j s,$$

for all $j \in J$, $s \in S$. Since, for every $i, j \in J$ and $u, v \in S$,

$$\iota(t_i)u \leq \iota(t_j)v \Longrightarrow \iota(t_i u) \leq \iota(t_j v) \Longrightarrow t_i u \leq t_j v \Longrightarrow f(t_i u) \leq f(t_j v) \Longrightarrow at_i u \leq at_j v,$$

h is an order preserving and well-defined S-act morphism. By assumption, there exists $b \in A$ such that $h(\iota(t_j)s) = b\iota(t_j)s$ for every $j \in J$ and $s \in S$. Hence

$$(\lambda_b \iota)(t_j s) = b\iota(t_j)s = h(\iota(t_j)s) = at_j s = f(t_j s)$$

for every $j \in J$, $s \in S$, i.e. $\lambda_b \iota = f$.

$$\begin{tikzcd} I \arrow[rr, "\iota"] \arrow[dr, "f"'] & & S \arrow[dl, dashed, "\lambda_b"] \\ & A & \end{tikzcd}$$

$(iii) \Rightarrow (i)$. Consider a consistent (α, R)-system

$$\Sigma = \{xs_j = a_j \mid j \in J, s_j \in R, a_j \in A\}$$

where $|J| < \alpha$ and the right (α, R)-ideal $I = \bigcup_{j\in J} s_j S$ of S. By Lemma 3.2,

$$s_i u \leq s_j v \Longrightarrow a_i u \leq a_j v$$

for every $i, j \in J$ and $u, v \in S$. Hence the mapping $f : I \to A, s_j s \mapsto a_j s$, is an S-poset morphism. By assumption, there exists an S-poset morphism $g : S \to A$ such that $g\iota = f$ where $\iota : I \to S$ is the inclusion. Therefore

$$a_j = f(s_j) = g\iota(s_j) = g(s_j) = g(1)s_j$$

for every $j \in J$, and so $g(1)$ is a solution of Σ in A.

Denote the *directed kernel* $\{(a, a') \in A^2 \mid f(a) \leq f(a')\}$ of an S-poset morphism $f : A_S \to B_S$ by $\overrightarrow{\mathrm{Ker}}\, f$ (see [3]). Taking $\alpha = 2$ and $R = S$, from Lemma 3.2 and Proposition 3.3 we obtain the following result.

Corollary 3.4. *For an S-poset A_S, the following conditions are equivalent:*

(i) A_S is regularly principally weakly injective,

(ii) for every $s \in S$ and S-poset morphism $f : sS \to A_S$, there exists an element $z \in A_S$ such that $f(x) = zx$ for every $x \in sS$,

(iii) for every $s \in S, a \in A$ with $\overrightarrow{\mathrm{Ker}}\, \lambda_s \subseteq \overrightarrow{\mathrm{Ker}}\, \lambda_a$, one has that $a = zs$ for some $z \in A$.

4 Regularly (α, R)-injective extension of an S-poset

Construction 4.1. Let A_S be an arbitrary S-poset, let $R \subseteq S$ be any subset that is closed under regular monomorphisms, and let α be any cardinal with $1 < \alpha \leq \aleph_0$. Our aim is to give a construction of a regularly (α, R)-injective S-poset $A^{(\alpha,R)}$ containing A as a regular S-subposet. The first step in this direction is to define $\Gamma, H, U(\alpha, R, A)$ as follows.

For every natural number n, where $1 \leq n < \alpha$, set

$$\begin{aligned}\Gamma^n := \{&((s_1, a_1), \ldots, (s_n, a_n)) \in (R \times A)^n \mid \\ &\text{for all } u, v \in S, \text{ and } i, j \in \{1, \ldots, n\} \quad s_i u \leq s_j v \text{ implies } a_i u \leq a_j v\}.\end{aligned}$$

If $\gamma \in \Gamma^n$, we write γ_j for the j-th component of the n-tuple γ. Further we put

$$\Gamma := \bigcup_{1\leq n<\alpha} \Gamma^n,$$
$$F_S := (\Gamma \times S)_S,$$

that is, F is the free right S-poset on Γ (we again write γs for the element (γ, s) of F), and

$$H := \{(\gamma s_j, a_j) \mid \gamma \in \Gamma^n, 1 \leq n < \alpha, (s_j, a_j) = \gamma_j, j \in \{1, \ldots, n\}\} \subseteq (F \amalg A)^2.$$

Let $\theta(H)$ be the S-poset congruence on $F_S \amalg A_S$ generated by H (see (1)) and define a right S-poset

$$U(\alpha, R, A)_S := (F_S \amalg A_S)/\theta(H).$$

First we need to examine the properties of the S-act congruence $\rho(H)$ on $F_S \amalg A_S$ generated by H.

Lemma 4.2. *If $y\rho(H)y'$ for $y, y' \in F$ then either $y = y'$ or there exist $1 \leq n, n' < \alpha$, $j \in \{1, \dots, n\}$, $j' \in \{1, \dots, n'\}$, $\gamma \in \Gamma^n$, $\gamma' \in \Gamma^{n'}$, $s, s' \in R$, $t, t' \in S$, $a, a' \in A$ such that*

$$\begin{aligned} y = \gamma st \qquad \gamma' s' t' &= y', \\ at &= a't' \end{aligned}$$

$\gamma_j = (s, a)$ and $\gamma'_{j'} = (s', a')$.

Proof. Suppose that $y, y' \in F$ and $y\rho(H)y'$. Then by Lemma 1.4.37 of [7] either $y = y'$ or there exist elements $x_1, \dots, x_m, x'_1, \dots, x'_m \in F \amalg A, t_1, \dots, t_m \in S$ such that $(x_i, x'_i) \in H$ or $(x'_i, x_i) \in H$ for each $i \in \{1, \dots, m\}$ and

$$\begin{array}{cccc} y = x_1 t_1 \quad & x'_2 t_2 = x_3 t_3 \;\dots & & x'_m t_m = y', \\ & x'_1 t_1 = x_2 t_2 & x'_{m-1} t_{m-1} = x_m t_m & \end{array}$$

where $m \in \mathbb{N}$ is minimal. From $y = x_1 t_1 \in F$ we get that $x_1 \in F$. Hence $(x_1, x'_1) \in H$ and therefore $x_1 = \gamma s_{j_1}$ and $x'_1 = a_{j_1}$ for some $n_1 < \alpha$, $j_1 \in \{1, \dots, n_1\}$ and $\gamma \in \Gamma^{n_1}$ with $\gamma_{j_1} = (s_{j_1}, a_{j_1})$.

If $m > 2$ then $(x'_2, x_2), (x_3, x'_3) \in H$, so there exist $n_2, n_3 < \alpha$, $j_2 \in \{1, \dots, n_2\}$, $j_3 \in \{1, \dots, n_3\}$, $\delta \in \Gamma^{n_2}$ and $\nu \in \Gamma^{n_3}$ such that $\delta_{j_2} = (s_{j_2}, a_{j_2})$, $\nu_{j_3} = (s_{j_3}, a_{j_3})$, $x'_2 = \delta s_{j_2}$, $x_2 = a_{j_2}$, $x_3 = \nu s_{j_3}$ and $x'_3 = a_{j_3}$. Now the equality $\delta s_{j_2} t_2 = x'_2 t_2 = x_3 t_3 = \nu s_{j_3} t_3$ implies $\delta = \nu$ (hence $n_2 = n_3$) and $s_{j_2} t_2 = s_{j_3} t_3$. By the definition of Γ^{n_2}, $a_{j_2} t_2 = a_{j_3} t_3$. It follows that $x'_1 t_1 = x_2 t_2 = a_{j_2} t_2 = a_{j_3} t_3 = x'_3 t_3$, but this contradicts the minimality of m.

Obviously $m \neq 1$ because $y, y' \in F$. So $m = 2$, i.e. $x'_1, x_2 \in A$ and there exist $n, n' < \alpha$, $j \in \{1, \dots, n\}$, $j' \in \{1, \dots, n'\}$, $\gamma \in \Gamma^n$, $\gamma' \in \Gamma^{n'}$ such that $x_1 = \gamma s$ and $x'_2 = \gamma' s'$ where $\gamma_j = (s, x'_1)$ and $\gamma'_{j'} = (s', x_2)$. Thus we have $y = \gamma s t_1$, $x'_1 t_1 = x_2 t_2$ and $\gamma' s' t_2 = y'$.

The following lemma can be proved by an argument similar to that of [5], p. 76.

Lemma 4.3. *If $a\rho(H)a'$ for $a, a' \in A$ then $a = a'$.*

Lemma 4.4. *If $a\rho(H)y$ for $a \in A, y \in F$ then there exist $1 \leq n < \alpha$, $j \in \{1, \dots, n\}$, $\gamma \in \Gamma^n$, $s \in R$, $t \in S$, $b \in A$ such that $a = bt$, $\gamma st = y$ and $\gamma_j = (s, b)$.*

Proof. By using a proof, similar to that of Lemma 4.2, one has that $a = x_1 t_1$ and $x'_1 t_1 = y$

for some $t_1 \in S$ and $(x_1', x_1) \in H$. So $x_1' = \gamma s_j$ for some $n < \alpha$, $\gamma \in \Gamma^n$ and $j \in \{1, \dots, n\}$ such that $\gamma_j = (s_j, x_1)$.

Lemma 4.5. *Suppose that*

$$y_1' \leq y_2 \rho(H) y_2' \leq \dots \leq y_m \rho(H) y_m' \leq y_{m+1} \tag{3}$$

where $y_{k+1}, y_k' \in F$, $y_k \neq y_k'$ for every $k \in \{1, \dots, m\}$, and $y_1' = \gamma s' t'$, $y_{m+1} = \delta v$ for some $t', v \in S$, $n, n' < \alpha$, $j' \in \{1, \dots, n'\}$, $\gamma \in \Gamma^{n'}$, $\delta \in \Gamma^n$ such that $\gamma_{j'} = (s', a')$. Then

$$a' t' \leq bs, \quad zs \leq v \quad \textit{and} \quad \delta_l = (z, b)$$

for some $s \in S$, $z \in R$, $b \in A$ and $l \in \{1, \dots, n\}$. Moreover, if $v = st$ for some $t \in S$, $j \in \{1, \dots, n\}$ such that $\delta_j = (s, a)$ then $a' t' \leq at$.

Proof. If $m = 1$, that is, (3) has the form $\gamma s' t' = y_1' \leq y_2 = \delta v$ then $\gamma = \delta$, $s' t' \leq v$, $a' t' \leq a' t'$ and $\delta_{j'} = \gamma_{j'} = (s', a')$.

Suppose that $m > 1$. By Lemma 4.2, for every $k \in \{2, \dots, m\}$ there exist $n_k, p_k < \alpha$, $i_k \in \{1, \dots, n_k\}$, $j_k \in \{1, \dots, p_k\}$, $\gamma^k \in \Gamma^{n_k}$, $\delta^k \in \Gamma^{p_k}$, $u_k, v_k \in S$ such that

$$y_k = \gamma^k s_k u_k, \quad a_k u_k = b_k v_k, \quad \delta^k z_k v_k = y_k' \quad \text{where } \gamma^k_{i_k} = (s_k, a_k), \delta^k_{j_k} = (z_k, b_k).$$

Since $y_k' \leq y_{k+1}$ in F, we conclude that $\delta^k = \gamma^{k+1}$, $p_k = n_{k+1}$ and $z_k v_k \leq s_{k+1} u_{k+1}$ for every $k \in \{2, \dots, m-1\}$. By the definition of Γ^{p_k}, $b_k v_k \leq a_{k+1} u_{k+1}$ for every $k \in \{2, \dots, m-1\}$. Moreover, $\gamma s' t' = y_1' \leq y_2 = \gamma^2 s_2 u_2$ and $\delta^m z_m v_m = y_m' \leq y_{m+1} = \delta v$ imply $\gamma = \gamma^2$, $n' = n_2$, $s' t' \leq s_2 u_2$, $\delta^m = \delta$, $p_m = n$, $z_m v_m \leq v$. The inequality $s' t' \leq s_2 u_2$ implies $a' t' \leq a_2 u_2$ by the definition of $\Gamma^{n'}$. Now

$$a' t' \leq a_2 u_2 = b_2 v_2 \leq a_3 u_3 = b_3 v_3 \leq \dots \leq b_m v_m,$$

where $(z_m, b_m) = \delta^m_{j_m} = \delta_{j_m}$. If $v = st$ for some $t \in S$ and $j \in \{1, \dots, n\}$ such that $\delta_j = (s, a)$ then $z_m v_m \leq st$ implies $b_m v_m \leq at$ and hence $a' t' \leq at$.

Lemma 4.6. *If $a \underset{\rho(H)}{\leq} a'$, where $a, a' \in A$, then $a \leq a'$.*

Proof. Let $a \underset{\rho(H)}{\leq} a'$ where $a, a' \in A$. Since the elements of A are incomparable to elements of F and also having Lemma 4.3 in mind, there exist elements $a_k' \in A$ and $y_k, y_k' \in F$, $k \in \{1, \dots, m\}$ such that

$$a \leq a_1' \rho(H) y_1' \underset{\rho(H)}{\leq} y_1 \rho(H) a_2 \leq a_2' \rho(H) y_2' \underset{\rho(H)}{\leq} y_2 \rho(H) a_3 \dots y_{m-1} \rho(H) a_m \leq a',$$

and for every $k \in \{1, \dots, m-1\}$, y_k' and y_k are connected by a $\rho(H)$-chain of the form (3). By Lemma 4.4, for every $k \in \{1, \dots, m-1\}$, $a_k' \rho(H) y_k'$ and $y_k \rho(H) a_{k+1}$ imply that there exist $n_k, p_k < \alpha$, $i_k \in \{1, \dots, n_k\}$, $j_k \in \{1, \dots, p_k\}$, $\gamma^k \in \Gamma^{n_k}$, $\delta^k \in \Gamma^{p_k}$, $u_k, v_k \in S$ such that

$$a_k' = b_k' u_k, \gamma^k s_k u_k = y_k', \gamma^k_{i_k} = (s_k, b_k') \quad \text{and} \quad a_{k+1} = b_k v_k, \delta^k z_k v_k = y_k, \delta^k_{j_k} = (z_k, b_k).$$

By Lemma 4.5, $y'_k \underset{\rho(H)}{\leq} y_k$ implies $b'_k u_k \leq b_k v_k$ for every $k \in \{1, \ldots, m-1\}$. Hence

$$a \leq a'_1 = b'_1 u_1 \leq b_1 v_1 = a_2 \leq a'_2 = b'_2 u_2 \leq \ldots \leq b_{m-1} v_{m-1} = a_m \leq a'.$$

From (1) and Lemma 4.6 we obtain the following result.

Corollary 4.7. *If $a\theta(H)a'$ for $a, a' \in A_S$ then $a = a'$.*

Lemma 4.8. *1. If $a \underset{\rho(H)}{\leq} y$, where $a \in A, y = \delta v \in F$, then $a \leq bs$ and $zs \leq v$ for some $s \in S$, $z \in R$, $b \in A$, $n < \alpha$ and $l \in \{1, \ldots, n\}$ such that $\delta_l = (z, b)$;*

2. if $y \underset{\rho(H)}{\leq} a$, where $y = \delta v \in F, a \in A$, then $v \leq zs$ and $bs \leq a$ for some $s \in S$, $z \in R$, $b \in A$, $n < \alpha$ and $l \in \{1, \ldots, n\}$ such that $\delta_l = (z, b)$.

Proof. 1. If $a \underset{\rho(H)}{\leq} y$ where $a \in A, y = \delta v \in F$, $\delta \in \Gamma^n$ and $n < \alpha$, then using Lemma 4.6 we have a $\rho(H)$-chain

$$a \leq a'\rho(H)y' \underset{\rho(H)}{\leq} y$$

where $a' \in A$ and the $\rho(H)$-chain connecting y' and y is of the form (3). By Lemma 4.4, there exist $n' < \alpha$, $j' \in \{1, \ldots, n'\}$, $\gamma \in \Gamma^{n'}$, $t' \in S$ such that $a' = b't'$, $\gamma s't' = y'$ and $\gamma_{j'} = (s', b')$. By Lemma 4.5, $b't' \leq bs$, $zs \leq v$ and $\delta_l = (z, b)$ for some $s \in S$, $z \in R$, $b \in A$ and $l \in \{1, \ldots, n\}$. Hence $a \leq a' = b't' \leq bs$.

2. The proof is symmetric to the case 1.

Proposition 4.9. *Preserving the notations of Construction 4.1, let*

$$\pi : F_S \amalg A_S \to U(\alpha, R, A)_S$$

be the canonical surjection. Then $\pi|_A : A_S \to U(\alpha, R, A)_S$ is a regular monomorphism, that is, $U(\alpha, R, A)_S$ is an extension of A_S.

Proof. Note that π is obviously an S-poset morphism and the fact that $\pi|_A : A_S \to U(\alpha, R, A)_S$ is a regular monomorphism follows from (2) and Lemma 4.6.

In what follows, we shall identify A_S with the regular S-subposet $\pi|_A(A)$ of $U(\alpha, R, A)$.

Theorem 4.10. *Let A_S be an S-poset, $R \subseteq S$ a subset that is closed under regular monomorphisms and α a cardinal with $1 < \alpha \leq \aleph_0$. Set $A_0 = A_S$ and $A_i = U(\alpha, R, A_{i-1})_S$ for every $i \in \mathbb{N}$. Let*

$$A^{(\alpha,R)} := \bigcup_{i \in \mathbb{N}_0} A_i$$

and define a relation $\leq$ on $A^{(\alpha,R)}$ by

$$a \leq b \Longleftrightarrow a \leq_n b$$

where $n \in \mathbb{N}_0$ is any number such that $a, b \in A_n$, and $\leq_n$ is the partial order in A_n. Then $A^{(\alpha,R)}$ is a regularly (α, R)-injective S-poset that contains A as a regular S-subposet.

Proof. For every $i \in \mathbb{N}$, denote by $F_i := \Gamma_i \times S$ the free S-poset, by $H_i \subseteq (F_i \amalg A_i)^2$ the set, by $\rho_i := \rho(H_i)$ and $\theta_i := \theta(H_i)$ the relations on $F_i \amalg A_i$ defined using A_i as in Construction 4.1. So $A_{i+1} = (F_i \amalg A_i)/\theta_i$ and the order relation $\leq_{i+1}$ on A_{i+1} is defined by

$$[x]_i \leq_{i+1} [x']_i \Longleftrightarrow x \underset{\rho_i}{\leq} x',$$

$x, x' \in F_i \amalg A_i$, where $[x]_i$ is the θ_i-class of x. It is easy to understand that $A^{(\alpha,R)}$ is an S-poset and contains A as a regular S-subposet. Consider a consistent (α, R)-system

$$\Sigma = \{xs_j = a_j \mid j \in J, s_j \in R, a_j \in A^{(\alpha,R)}\},$$

where $|J| < \alpha$. Since $\alpha \leq \aleph_0$, J is a finite set and we may assume that $J = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ with $n < \alpha$. Hence there exists $m \in \mathbb{N}_0$ such that $a_j \in A_m$ for every $j \in J$. By Lemma 3.2,

$$\gamma = ((s_1, a_1), \ldots, (s_n, a_n)) \in \Gamma_m^n \subseteq \Gamma_m,$$

so $\gamma 1 \in F_m$ and $[\gamma 1]_m \in A_{m+1} \subseteq A^{(\alpha,R)}$. Moreover, $(\gamma s_j, a_j) \in H_m$ for every $j \in J$, and thus

$$[\gamma 1]_m s_j = [(\gamma 1) s_j]_m = [\gamma s_j]_m = [a_j]_m \equiv a_j,$$

i.e. $[\gamma 1]_m$ is a solution of Σ in A_{m+1} and hence in $A^{(\alpha,R)}$. By Proposition 3.3, $A^{(\alpha,R)}$ is (α, R)-injective.

We call the S-poset $A^{(\alpha,R)}$ (defined as in Theorem 4.10) *the regularly (α, R)-injective extension of A*. We also write $A^{(2)} = A^{(2,S)}$ and $A^{(\aleph_0)} = A^{(\aleph_0,S)}$ and call them *the regularly principally weakly injective extension of A* and *the regularly fg-weakly injective extension of A*, respectively. Since regular $(2, C)$-injectivity is by Lemma 3.1 the same as regular divisibility, we call $A^{(2,C)}$ *the regularly divisible extension of A*.

5 Homological classification

In this section we give descriptions of pomonoids over which all right S-posets with some weaker regular weak injectivity property have some stronger regular weak injectivity property.

5.1 When all S-posets are regularly divisible

Proposition 5.1. *The following conditions are equivalent:*

(i) All right S-posets are regularly divisible,

(ii) all right ideals of S are regularly divisible,

(iii) S_S is regularly divisible,

(iv) every left po-cancellable element of S is left invertible.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$. These are obvious.

$(iii) \Rightarrow (iv)$. Suppose that S_S is regularly divisible and $c \in S$ is a left po-cancellable element. Then $S = Sc$ implies that there exists $s \in S$ such that $sc = 1$, so c is left invertible.

$(iv) \Rightarrow (i)$. Let $c \in S$ be a left po-cancellable element and A_S a right S-poset. By (iv) there is an $s \in S$ satisfying $sc = 1$. So $A = Asc = Ac$.

5.2 When regularly divisible S-posets are regularly principally weakly injective

In [6], Victoria Gould introduced the notion of a right almost regular monoid and proved that these are precisely the monoids over which all divisible acts are principally weakly injective. We shall prove an analogue of this result for S-posets.

Theorem 5.2. *The following conditions are equivalent for a pomonoid S:*

(i) all regularly divisible right S-posets are regularly principally weakly injective,

(ii) for every element $s \in S$ there exist $r, r_1, \ldots, r_n, s_1, \ldots, s_n, s'_1, \ldots, s'_n \in S$ and left po-cancellable elements $c_1, \ldots, c_n \in S$ such that

$$\begin{gathered} c_1 s_1 \leq r_1 s \leq c_1 s'_1 \\ c_2 s_2 \leq r_2 s_1 \leq r_2 s'_1 \leq c_2 s'_2 \\ c_3 s_3 \leq r_3 s_2 \leq r_3 s'_2 \leq c_3 s'_3 \\ \ldots \\ c_n s_n \leq r_n s_{n-1} \leq r_n s'_{n-1} \leq c_n s'_n \\ s = s s_n = s s'_n, \end{gathered} \tag{4}$$

(iii) for every element $s \in S$ there exist $r, r_1, \ldots, r_n, s_1, \ldots, s_n, s'_1, \ldots, s'_n \in S$ and left po-cancellable elements $c_1, \ldots, c_n \in S$ such that

$$\begin{gathered} c_1 s_1 \leq r_1 s \leq c_1 s'_1 \\ c_2 s_2 \leq r_2 s_1 \leq c_2 s'_2 \\ c_3 s_3 \leq r_3 s_2 \leq c_3 s'_3 \\ \ldots \\ c_n s_n \leq r_n s_{n-1} \leq c_n s'_n \\ s = s s_n = s s'_n. \end{gathered} \tag{5}$$

Proof. $(i) \Rightarrow (ii)$. Assume that all regularly divisible right S-posets are regularly principally weakly injective. For an element $s \in S$, let $sS^{(2,C)}$ be the regularly divisible extension of sS obtained as in Construction 4.1. In our case

$$\begin{aligned} \Gamma_i &= \Gamma_i^1 = \{(c,b) \in C \times (sS)_i \mid \text{ for all } u,v \in S \;\; cu \le cv \text{ implies } bu \le_i bv\}, \\ H_i &= \{((c,b)c,b) \in F_i \times A_i \mid (c,b) \in \Gamma_i\}. \end{aligned}$$

Note that every element $b = [d]_{\theta_{i-1}} \in (sS)_i = (F_{i-1} \amalg (sS)_{i-1})/\theta_{i-1}$, $d \in F_{i-1} \amalg (sS)_{i-1}$ can be presented in the form

$$b = [(c,b')s]_{\theta_{i-1}} \quad \text{where} \quad (c,b') \in \Gamma_{i-1} \quad \text{and } s \in S. \tag{6}$$

If $d \in F_{i-1}$, this is clear. If $d \in (sS)_{i-1}$ then $(1,d) \in \Gamma_{i-1}$, $((1,d)1,d) \in H_{i-1}$, hence $(1,d)1\theta_{i-1}d$ and $b = [d]_{\theta_{i-1}} = [(1,d)1]_{\theta_{i-1}}$.

By assumption, $sS^{(2,C)}$ is regularly principally weakly injective. Thus there exists an S-poset morphism $g : S \longrightarrow sS^{(2,C)}$ such that the diagram

$$\begin{tikzcd} sS \arrow[rr, "\iota"] \arrow[dr, "f"'] & & S \arrow[dl, dashed, "g"] \\ & sS^{(2,C)} & \end{tikzcd}$$

commutes, where ι and f are the inclusion mappings. Then

$$s = f(s) = g\iota(s) = g(s) = g(1)s$$

where $g(1) \in sS^{(2,C)}$. Let $n \in \mathbb{N}_0$ be such that $g(1) \in (sS)_n$. If $n = 0$ then $g(1) \in sS$, hence $s \in sSs$, i.e. s is regular and therefore there exist $c_1 = 1, r_1 = x, s_1 = s_1' = xs$, where $s = sxs, x \in S$ such that the inequalities and equalities in 4 are fulfilled.

Suppose that $n > 0$. Then, by (6), $g(1) = [(c_1,b_1)r_1]_{\theta_{n-1}} \in (sS)_n = (F_{n-1} \amalg (sS)_{n-1})/\theta_{n-1}$, where $r_1 \in S$ and $(c_1,b_1) \in \Gamma_{n-1}$; in particular $c_1 \in C$ and $b_1 \in (sS)_{n-1}$. Then $s\theta_{n-1}(c_1,b_1)r_1s$, that is $s \underset{\rho_{n-1}}{\le} (c_1,b_1)r_1s \underset{\rho_{n-1}}{\le} s$. By Lemma 4.8,

$$s \le_{n-1} b_1s_1, \quad c_1s_1 \le r_1s, \quad \text{and} \quad r_1s \le c_1s_1', \quad b_1s_1' \le_{n-1} s,$$

for some $s_1, s_1' \in S$. Again by (6), $b_1 = [(c_2,b_2)r_2]_{\theta_{n-2}}$, where $r_2 \in S$ and $(c_2,b_2) \in \Gamma_{n-2}$, in particular, $c_2 \in C$ and $b_2 \in (sS)_{n-2}$. Now $s \le_{n-1} b_1s_1$ and $b_1s_1' \le_{n-1} s$ mean that $s \underset{\rho_{n-2}}{\le} (c_2,b_2)r_2s_1$ and $(c_2,b_2)r_2s_1' \underset{\rho_{n-2}}{\le} s$. Lemma 4.8 implies that

$$s \le_{n-2} b_2s_2, \quad c_2s_2 \le r_2s_1, \quad \text{and} \quad r_2s_1' \le c_2s_2', \quad b_2s_2' \le_{n-2} s$$

for some $s_2, s_2' \in S$. Continuing in a similar manner, we finally obtain $b_n = sr \in sS = (sS)_0$, $c_n \in C$, $r_n, s_n, s_n' \in S$ such that

$$s \le b_ns_n = srs_n, \quad c_ns_n \le r_ns_{n-1}, \quad \text{and} \quad r_ns_{n-1}' \le c_ns_n', \quad srs_n' = b_ns_n' \le s.$$

Now $c_1s_1 \leq r_1s \leq c_1s_1'$ implies $s_1 \leq s_1'$, $c_2s_2 \leq r_2s_1 \leq r_2s_1' \leq c_2s_2'$ implies $s_2 \leq s_2'$, and so on. Finally we obtain $s_n \leq s_n'$ and hence $s \leq srs_n \leq srs_n' \leq s$, which yields $s = srs_n = srs_n'$. The inequality $s_n \leq s_n'$ also implies $rs_n \leq rs_n'$, and thus we have obtained

$$c_1s_1 \leq r_1s \leq c_1s_1'$$

$$c_2s_2 \leq r_2s_1 \leq r_2s_1' \leq c_2s_2'$$

$$\dots$$

$$c_ns_n \leq r_ns_{n-1} \leq r_ns_{n-1}' \leq c_ns_n'$$

$$1(rs_n) \leq rs_n \leq rs_n' \leq 1(rs_n')$$

$$s = s(rs_n) = s(rs_n').$$

$(ii) \Rightarrow (iii)$. This is clear.

$(iii) \Rightarrow (i)$. Assume (iii) holds. Let A_S be a regularly divisible right S-poset, $s \in S$, and $f : sS \to A$ an S-poset morphism. Then for s we have inequalities and equalities as in (5). Hence $f(s) = f(s)s_n = f(s)s_n'$. Using regular divisibility of A, there exists $a_1 \in A$ such that $f(s) = a_1c_n$. Consequently,

$$f(s) = a_1c_ns_n \leq a_1r_ns_{n-1} \leq a_1c_ns_n' = f(s),$$

and so $f(s) = a_1r_ns_{n-1}$. Again, by the regular divisibility of A, $a_1r_n = a_2c_{n-1}$ for some $a_2 \in A$. Thus

$$f(s) = a_2c_{n-1}s_{n-1} \leq a_2r_{n-1}s_{n-2} \leq a_2c_{n-1}s_{n-1}' = f(s)$$

and $f(s) = a_2r_{n-1}s_{n-2}$. In this way we finally arrive at $f(s) = a_nr_1s$ for some $a_n \in A$, i.e. $f = \lambda_{a_nr_1}$. So A is regularly principally weakly injective by Proposition 3.3.

Definition 5.3. We say that an element s of a pomonoid S is *regularly right almost regular* if there exist elements such that equalities and inequalities in (4) hold. We call a pomonoid *regularly right almost regular*, if all its elements are regularly right almost regular.

If $s \in S$ is a regular element then $s = sxs$ for some $x \in S$ and hence we have

$$1s \leq (sx)s \leq 1s$$

$$1(xs) \leq xs \leq xs \leq 1(xs)$$

$$s = s(xs) = s(xs).$$

So every regular element of a pomonoid is regularly right almost regular. It is also easy to see that every left po-cancellable element of a pomonoid is regularly right almost regular.

Corollary 5.4. *For a pomonoid S, the following conditions are equivalent:*
(i) all right S-posets are regularly principally weakly injective,
(ii) all right ideals of S are regularly principally weakly injective,
(iii) all finitely generated right ideals of S are regularly principally weakly injective,
(iv) all principal right ideals of S are regularly principally weakly injective,
(v) S is a regular pomonoid.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$. These are clear.

$(iv) \Rightarrow (v)$. For any $s \in S$, by (iv), since sS_S is regularly principally weakly injective, there exists an S-poset morphism $g : S_S \to sS_S$ such that $g\iota = 1_{sS}$, where ι is the inclusion mapping from sS to S and 1_{sS} is the identity mapping of sS. Consequently, one has that $s = g(s) = g(1)s$. Since $g(1) \in sS$, it follows that s is regular.

$(v) \Rightarrow (i)$. If S is regular then all right S-posets are regularly principally weakly injective by Proposition 5.1 and Theorem 5.2.

It is known that every right almost regular monoid is a right PP monoid (see [8]). We can prove an analogue of this result for commutative pomonoids. Recall that a pomonoid S is a right PP monoid if and only if for every $s \in S$ there exists an idempotent $e \in S$ such that $s = se$ and $su \leq sv$ implies $eu \leq ev$ for all $u, v \in S$ (see Proposition 3.2 of [9]).

Lemma 5.5. *If S is a regularly right almost regular pomonoid then for every element $s \in S$ there exist $p, q \in S$ such that $s = sp = sq$ and $su \leq sv$ implies $pu \leq qv$ for all $u, v \in S$.*

Proof. For every element $s \in S$ there exist elements as in (4). Suppose $su \leq sv$, $u, v \in S$. Then

$$c_1 s_1 u \leq r_1 su \leq r_1 sv \leq c_1 s_1' v$$

implies $s_1 u \leq s_1' v$. Next,

$$c_2 s_2 u \leq r_2 s_1 u \leq r_2 s_1' v \leq c_2 s_2' v$$

implies $s_2 u \leq s_2' v$. Continuing in this manner we arrive at $s_n u \leq s_n' v$.

Corollary 5.6. *Every commutative regularly (right) almost regular pomonoid is a (right) PP pomonoid.*

Proof. For an element $s \in S$ let $p, q \in S$ such that $s = sp = sq$ and $su \leq sv$ implies $pu \leq qv$ for all $u, v \in S$. Denote $e = pq$. Then $sq = s = s(p^2 q)$ and $s(pq^2) = s = sp$ imply $pq \leq qp^2 q$ and $p^2 q^2 \leq qp$. Hence $e = e^2$ by commutativity and $s = se$. If now $su \leq sv$ then $s(qu) \leq s(pv)$ and hence $eu = pqu \leq qpv = ev$.

5.3 When regularly principally weakly injective S-posets are regularly fg-weakly injective

Lemma 5.7. *Let A_S be an S-poset and let $A^{(2)}$ be constructed as in Construction 4.1. If $A \subseteq bS$ for some $b \in A_n$, $n \in \mathbb{N}$, then $A \subseteq dS$ for some $d \in A_{n-1}$.*

Proof. We may assume that $b \in A_n \backslash A_{n-1}$. Then $b = [y]_{n-1}$ for some $y = \delta v \in F_{n-1}$ where $v \in S$ and $\delta = (z, d) \in \Gamma_{n-1}$. For every $a \in A$, there exists $t \in S$ such that $a = [\delta v]_{n-1} t$. So $a \theta_{n-1} \delta v t$, i.e. $a \underset{\rho_{n-1}}{\leq} \delta v t \underset{\rho_{n-1}}{\leq} a$. By Lemma 4.8, there exist $s_1, s_2, z_1, z_2 \in S$, $b_1, b_2 \in A_{n-1}$ such that $a \leq b_1 s_1$, $z_1 s_1 \leq vt$, $vt \leq z_2 s_2$, $b_2 s_2 \leq a$ and $\delta = (z_1, b_1) = (z_2, b_2)$. Hence $z = z_1 = z_2$, $d = b_1 = b_2$, and $z s_1 = z_1 s_1 \leq z_2 s_2 = z s_2$ implies $d s_1 \leq d s_2$ because $\delta \in \Gamma^1_{n-1}$. Consequently, $a \leq b_1 s_1 = d s_1 \leq d s_2 = b_2 s_2 \leq a$, i.e. $a = d s_1 \in dS$.

Theorem 5.8. *Let S be a pomonoid and $\alpha > 1$ a cardinal. Then all regularly principally weakly injective S-posets are regularly α-injective if and only if all right α-ideals are principal.*

Proof. Necessity. Consider a right α-ideal $I = \bigcup_{j \in J} s_j S$, where $|J| < \alpha$. By assumption, its regularly principally weakly injective extension $I^{(2)}$ is regularly α-injective. Hence there exists an S-poset morphism $g : S \to I^{(2)}$ such that the diagram

$$\begin{tikzcd} I \arrow[rr, "\iota"] \arrow[dr, "f"'] & & S \arrow[dl, dashed, "g"] \\ & I^{(2)} & \end{tikzcd}$$

is commutative, where $\iota : I \to S$ and $f : I \to I^{(2)}$ are inclusion mappings. Then, for every $j \in J$,

$$s_j = f(s_j) = g\iota(s_j) = g(s_j) = g(1)s_j,$$

and hence

$$I = \bigcup_{j \in J} s_j S = \bigcup_{j \in J} g(1) s_j S \subseteq g(1)S.$$

Now $g(1) \in I_n$ for some $n \in \mathbb{N}_0$. If $n = 0$ then $g(1) \in I$. Otherwise, by applying Lemma 5.7 n times we obtain $d \in I$ such that $I \subseteq dS$. So in both cases $I \subseteq sS$ for some $s \in I$, which implies $I = sS$.

Sufficiency. This is obvious.

Corollary 5.9. *Let α be any cardinal such that $2 < \alpha \leq \aleph_0$. Then the following conditions are equivalent for a pomonoid S:*

(i) all regularly principally weakly injective S-posets are regularly fg-weakly injective,

(ii) all regularly principally weakly injective S-posets are regularly α-injective,

(iii) all regularly principally weakly injective S-posets are regularly 3-injective,

(iv) all right 3-ideals are principal,
(v) all finitely generated right ideals of S are principal.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$, $(iv) \Rightarrow (v)$. These are evident.
$(iii) \Rightarrow (iv)$, $(v) \Rightarrow (i)$. These follow from Theorem 5.8.

Corollary 5.10. *All regularly principally weakly injective S-posets are regularly weakly injective if and only if S is a principal right ideal pomonoid.*

From Corollary 5.9 and Corollary 5.4 we obtain the following result.

Corollary 5.11. *All S-posets are regularly fg-weakly injective if and only if S is a regular pomonoid all of whose finitely generated right ideals are principal.*

From Corollary 5.10 and Corollary 5.4 we obtain the following result.

Corollary 5.12. *All S-posets are regularly weakly injective if and only if S is a regular principal right ideal pomonoid.*

5.4 When regularly fg-weakly injective S-posets are regularly weakly injective

Lemma 5.13. *Let A be an S-poset and let $A^{(\aleph_0)}$ be constructed as in Construction 4.1. If A is contained in a finitely generated S-subposet of A_n for some $n \in \mathbb{N}$ then A is contained in a finitely generated S-subposet of A_{n-1}.*

Proof. Let $n \in \mathbb{N}$ and $b_1, \ldots, b_m \in A_n$ be such that $A \subseteq \bigcup_{i=1}^m b_i S$. If $b_1, \ldots, b_m \in A_{n-1}$ then there is nothing to prove. Assume that $r \in \{1, \ldots, m\}$ is such that $b_1, \ldots, b_r \in A_n \setminus A_{n-1}$ and $b_{r+1}, \ldots, b_m \in A_{n-1}$. Then $b_i = [\delta_i v_i]_{n-1}$ for some $\delta_i \in \Gamma_{n-1}$ and $v_i \in S$, for every $i \in \{1, \ldots, r\}$. By the definition of Γ_{n-1}, for every $i \in \{1, \ldots, r\}$ there exists $p_i \in \mathbb{N}$ such that

$$\delta_i = ((s_{i1}, a_{i1}), \ldots, (s_{ip_i}, a_{ip_i})) \in \Gamma_{n-1}^{p_i}.$$

We claim that

$$A \subseteq \left(\bigcup_{\substack{1 \leq i \leq r \\ 1 \leq l \leq p_i}} a_{il} S \right) \cup \left(\bigcup_{r < i \leq m} b_i S \right) \subseteq A_{n-1}.$$

Consider an element $a \in A$. If $a \in b_i S$ for some $i \in \{1, \ldots, r\}$ then there exists $t \in S$ such that $a \equiv [a]_{n-1} = [\delta_i v_i t]_{n-1}$. By Lemma 4.8, $a \underset{\rho_{n-1}}{\leq} \delta_i v_i t$ and $\delta_i v_i t \underset{\rho_{n-1}}{\leq} a$ imply that

$$a \leq_{n-1} bs, zs \leq v_i t \quad \text{and} \quad v_i t \leq z's', b's' \leq_{n-1} a$$

for some $s, s', z, z' \in S$, $b, b' \in A_{n-1}$, where $(\delta_i)_l = (z, b)$ and $(\delta_i)_k = (z', b')$ for some

$l, k \in \{1, \ldots, p_i\}$. Hence

$$s_{il}s = zs \leq v_i t \leq z's' = s_{ik}s',$$

which implies $bs = a_{il}s \leq_{n-1} a_{ik}s' = b's'$. It follows that $a \leq_{n-1} bs \leq_{n-1} b's' \leq_{n-1} a$, and thus $a = bs = a_{il}s \in a_{il}S \subseteq A_{n-1}$.

Theorem 5.14. *Let S be a pomonoid and let $\alpha \geq \aleph_0$ be a cardinal. Then all regularly fg-weakly injective S-posets are regularly α-injective if and only if all right α-ideals of S are finitely generated.*

Proof. Necessity. Let I be a right α-ideal of S. Then $I^{(\aleph_0)}$ is an α-injective S-poset by assumption. Thus there exists an S-poset morphism $g : S \to I^{(\aleph_0)}$ such that the diagram

$$\begin{tikzcd} I \arrow[rr, "\iota"] \arrow[dr, "f"'] & & S \arrow[dl, dashed, "g"] \\ & I^{(\aleph_0)} & \end{tikzcd}$$

commutes, where ι and f are the inclusion mappings. If $r \in I$ then

$$r = f(r) = g\iota(r) = g(r) = g(1)r.$$

Hence $I \subseteq g(1)S$. If $g(1) \in I$ then $I \subseteq g(1)S \subseteq IS \subseteq I$ and so $I = g(1)S$ is a principal right ideal. Otherwise $g(1) \in I_n \setminus I_{n-1}$ for some $n \in \mathbb{N}$. Then $g(1)S \subseteq I_n$ and $g(1)S$ is a finitely generated S-subposet of I_n. Applying Lemma 5.13 n times we conclude that I is contained in a finitely generated S-subposet of I, but then I must also be finitely generated.

Sufficiency. It is clear.

A pomonoid S is called *right noetherian* (see [7], Def. 4.3.5) if it satisfies the ascending chain condition on right ideals. This is equivalent to all right ideals of S being finitely generated.

From Theorem 5.14 we obtain the following result.

Corollary 5.15. *All regularly fg-weakly injective S-posets are regularly weakly injective if and only if S is right noetherian.*

5.5 Summary

The homological classification results of this section can be summarized in the following table (compare it with Table IV.2 of [7]).

$\Rightarrow$	reg. w. inj.	reg. fg-w. inj.	reg. princ. w. inj.	reg. divisible
reg. fg-w. inj.	right noetherian Cor. 5.15			
reg. princ. w. inj.	right ideals are principal Cor. 5.10	f.g. right ideals are principal Cor. 5.9		
reg. divisible			regularly right almost regular Thm. 5.2	
All	Cor. 5.12	Cor. 5.11	regular Cor. 5.4	left po-canc. $\Rightarrow$left inv. Prop. 5.1

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