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ON HOMOTOPY INVARIANCE OF THE TANGENT BUNDLE I

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1. Introduction.

A well-known result of M. F. Atiyah [2] states that if M and M' are compact q-dimensional oriented differentiable manifolds and $f\colon M \to M'$ is an orientation preserving homotopy equivalence, then $f^*\tau'$ is stably fibre homotopy equivalent to τ. Here τ and τ' denote the tangent sphere bundles of M and M' respectively. The problem to be studied in this note is, whether the word "stably" can be cancelled in the above statement.

This was partly done by W. A. Sutherland [16] and we follow the line of his paper. Only we use a method of "least indeterminacy" introduced by W. Browder [5] to define an invariant $b(\xi)$ for certain $(q-1)$-dimensional sphere bundles ξ over M^q, q odd and different from $1,3,7$. This invariant is a substitute for the Euler class in the even case. Unfortunately I am *not* able to show that $b(\xi)$ only depends on ξ except in the case, where $\tau(M^q)$ is stably homotopy trivial, in which case this is a consequence of the solution of the Hopf invariant one problem. Therefore, this paper only gives new information for $q=2^i-1$; but nevertheless I still hope to solve the general case by the same method.

At last we remark that $b(\tau)=\chi^*(M^q)$, the semi-characteristic introduced by M. Kervaire [8].

2. Stably equivalent bundles over a manifold.

$\mathrm{SH}(n)$ is the space of maps $S^{n-1} \to S^{n-1}$ of degree $+1$, and $B_n = \mathrm{BSH}(n)$ is the classifying space, defined by J. Stasheff [14], for oriented $(n-1)$-dimensional sphere bundles.

Consider a q-dimensional manifold M and an embedded disk $D^q \subset M$ with boundary S^{q-1}. According to J. Milnor [10, § 8] the triad

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After this was written it was pointed out to me that the problem was solved in general by different methods by René Benlian and John Wagoner in C. R. Acad. Sci. Paris Série A–B 265 (1967), A 207–A 209.

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$(M\setminus D^\circ, S^{q-1}, \emptyset)$ can be given a "self-indexing" Morse function with no points of index q. So $N=M\setminus D^\circ$ has the homotopy type of a $(q-1)$-dimensional CW-complex L, and M has the homotopy type of $K=L\cup_\theta e^q$, where $\theta\colon S^{q-1}\to N$ is the injection, and e^q denotes a q-cell.

Pinching $S^{q-1}\subset M$ defines a map $c\colon M\to M\vee S^q$, which defines an action of $\pi_q(X)$ on $[M,X]$ for any space X (see P. Hilton [6, chapter XV]): If $v\in[M,X]$ and $\mu\in\pi_q(X)$, we denote the composite

$$M\xrightarrow{c} M\vee S^q\xrightarrow{v\vee\mu} X\vee X\xrightarrow{\nabla} X$$

by $v^\mu\in[M,X]$. This action has the property that if $v_1,v_2\in[M,X]$ such that v_1 and v_2 restricted to N are homotopic, then there exists $\mu\in\pi_q(X)$ such that $v_2=v_1{}^\mu$ in $[M,X]$.

Furthermore let $u\colon N\to X$; W. D. Barcus and M. G. Barratt [3, § 2] have defined a map $\alpha_u\colon\pi_1(X^N,u)\to\pi_q(X)$ such that if $v\colon M\to X$ is an extension of u, then $v^\mu=v$ iff $\mu\in\operatorname{Im}\alpha_u$.

Now consider two stably equivalent $(q-1)$-dimensional oriented sphere bundles ξ_1,ξ_2 over M^q.

Proposition 2.1. *Let v_1 and v_2 be the classifying maps from M into B_q. Then $v_2=v_1{}^{\mu_0}$ in $[M,B_q]$, where $\mu_0\in\pi_q(B_q)$ is in the kernel of*

$$j_*\colon\ \pi_q(B_q)\to\pi_q(B_{q+1})\,.$$

Here $j\colon B_q\to B_{q+1}$ is the natural inclusion.

Proof. Restricting to N, v_1 and v_2 become homotopic, because N has the homotopy type of a $(q-1)$-dimensional complex. Therefore, $v_2=v_1{}^\mu$, where $\mu\in\pi_q(B_q)$. Without loss of generality we can assume v_1 and v_2 restricted to N to equal a map $u\colon N\to B_q$.

Then $j_*v_2=(j_*v_1)^{j_*\mu}$, and according to the q-dimensionality of M, $j_*v_2=j_*v_1$, because ξ_2 and ξ_1 are stably equivalent. So $j_*\mu\in\operatorname{Im}\alpha_{ju}$. It follows easily (e.g. by using Theorem 22 in Spanier [11, chapter VII, § 6]) that

$$j_\sharp\colon\ \pi_1(B_q{}^N,u)\to\pi_1(B_{q+1}^N,ju)\quad\text{is onto}\,.$$

Using this,

$$j_*\mu=\alpha_{ju}(z)\,,$$

where $z=j_\sharp(x)$ for some $x\in\pi_1(B_q{}^N,u)$. Let $\mu'=\alpha_u(x)$ and $\mu_0=\mu-\mu'\in\pi_q(B_q)$. Then

$$v_2=v_1{}^\mu=(v_1{}^\mu)^{-\mu'}=v_1{}^{\mu-\mu'}=v_1{}^{\mu_0}\,.$$

Finally

$$j_*\mu'=j_*\alpha_u(x)=\alpha_{ju}j_\sharp(x)=j_*\mu\,,$$

so $j_*\mu_0=0$.

The following proposition is a well-known consequence of the solution of the Hopf invariant one problem:

PROPOSITION 2.2. *The kernel of* $j_* : \pi_q(B_q) \to \pi_q(B_{q+1})$ *is cyclic generated by* τ_q, *the map classifying the tangent bundle of* S^q.

$$\operatorname{Ker} j_* = \begin{cases} \mathsf{Z} & q \ \ even \\ 0 & q = 1,3,7 \\ \mathsf{Z}_2 & q \ \ odd \quad q \neq 1,3,7 \,. \end{cases}$$

Returning to the problem stated in § 1, let $\xi_1 = \tau$, $\xi_2 = j^* \tau'$. If v_1 and v_2 are the classifying maps, $v_2 = v_1^{\mu_0}$, $\mu_0 \in \ker j_*$. So in case $q = 1,3,7$ it is trivial that ξ_1 and ξ_2 are equivalent. Now consider the even case. Letting $E(v_i)$ denote the Euler class of ξ_i evaluated on the orientation class, it is easily shown that $E(v_2) = E(v_1) + E(\mu_0)$ for $v_2 = v_1^{\mu_0}$. Hence in the present case $E(\mu_0) = 0$ and consequently $\mu_0 = 0$, so also in this case τ and $f^*(\tau')$ are fibre homotopy equivalent.

3. Definition of $b(\xi)$ in the stably trivial case.

In this section M is a q-dimensional manifold, q odd and different from 1,3,7, and we assume $\tau(M)$ to be stably fibre homotopy trivial. Let ξ denote a stably fibre homotopy trivial bundle over M. This means that for k large there exists a Thom map

$$\eta : \ S^{2q+k} \to T(\nu_M) \cong \Sigma^{k+q}(M_+) \cong T(\xi + k) \,,$$

that is, a map inducing isomorphism by taking $H_{2q+k}(\cdot, \mathsf{Z})$. Let

$$U_\xi : \ T(\xi) \to K(\mathsf{Z}_2, q)$$

denote the map representing the Thom class. Put

$$\delta = \Sigma^k U_\xi \circ \eta : \ S^{2q+k} \to \Sigma^k K(\mathsf{Z}_2, q) \,.$$

DEFINITION 3.1. $b(\xi) = Sq_\delta^{q+1}(\Sigma^k \iota) \in H^{2q+k}(S^{2q+k}, \mathsf{Z}_2) = \mathsf{Z}_2$.

Here $\iota \in H^q(K(\mathsf{Z}_2, q))$ and $b(\xi)$ is defined as the functional Steenrod square. (See W. Browder [5, § 1].)

PROPOSITION 3.2. $b(\xi)$ *is independent of the choice of* k *and*

$$\eta : \ S^{2q+k} \to T(\xi + k) \,.$$

PROOF. Suspending η we evidently get the same $b(\xi)$, and thus it is sufficient to prove the last part of the proposition.

Let $\eta' : S^{2q+k} \to T(\xi + k)$ be another map of degree one. Then $\eta = \eta' + \gamma$ in $\pi_{2q+k}(\Sigma^k T(\xi))$, where $\gamma : S^{2q+k} \to \Sigma^k T(\xi)$. Consider the cofibration

$$T(\xi_{|N}) \xrightarrow{i} T(\xi) \xrightarrow{p} S^{2q} .$$

Using the covariant Puppe sequence for stable homotopy, we see that $\gamma = (\Sigma^k i) \circ \gamma'$ for k large, where $\gamma' : S^{2q+k} \to T(\xi_{|N})$. In fact $(\Sigma^k p) \circ \gamma$ has degree 0.

$\xi_{|N}$ is trivial and the trivialisation defines a map $g : T(\xi_{|N}) \to S^q$. Let $l : S^q \to K(Z_2, q)$ be such that $l^* \iota = \sigma_q$, the generator of $H^q(S^q)$. Then

$$U_\xi \circ i = l \circ g : T(\xi_{|N}) \to K(Z_2, q) .$$

Putting

$$\delta = \Sigma^k U_\xi \circ \eta, \qquad \delta' = \Sigma^k U_\xi \circ \eta', \qquad \beta = (\Sigma^k g) \circ \gamma' ,$$

we have

$$\delta = \delta' + (\Sigma^k l) \circ \beta .$$

Hence by Lemma 1.6 in Browder [5]

$$Sq_\delta^{q+1} \iota = Sq_{\delta'}^{q+1} \iota + Sq^{q+1}_{\Sigma^k l \circ \beta} \iota .$$

According to Adams [1]

$$Sq^{q+1}_{\Sigma^k l \circ \beta} \iota = Sq_\beta^{q+1}(\Sigma^k \sigma_q) = 0$$

when q is odd $\neq 1, 3, 7$.

PROPOSITION 3.3. *Let ξ_1, ξ_2 be stably fibre homotopy trivial bundles over M, and ζ a third such over S^q. Assume the classifying maps v_1, v_2 and μ_0 satisfy $v_2 = v_1^{\mu_0}$.*

Then $b(\xi_2) = b(\xi_1) + b(\zeta)$.

PROOF. If $c : M \to M \vee S^q$ is the pinching map and $\xi_1 \vee \zeta$ denotes the bundle over $M \vee S^q$, which is ξ_1 over M and ζ over S^q, then $c^*(\xi_1 \vee \zeta) = \xi_2$ and so there is a natural map

$$h : T(\xi_2 + k) \to T((\xi_1 + k) \vee (\zeta + k)) .$$

This is equivalent to a cofibration with cofibre S^{2q+k}, which we for convenience write

$$T(\xi_2 + k) \xrightarrow{h} T((\xi_1 + k) \vee (\zeta + k)) \xrightarrow{p} S^{2q+k} .$$

By the Thom isomorphism theorem,

$$H_{2q+k}(T(\xi_2 + k), Z) = Z(d_2) ,$$
$$H_{2q+k}((\xi_1 + k) \vee (\zeta + k)) = Z(d_1) \oplus Z(d_0) ,$$

and

$$h_* d_2 = d_1 + d_0 .$$

Consider

$$\begin{array}{ccccc} S^{2q+k} & \xrightarrow{\Delta} & S^{2q+k} \vee S^{2q+k} & \xrightarrow{\eta_1 \vee \eta_0} & T(\xi_1+k) \vee T(\zeta+k) \\ \Big| \gamma & & & & \Big\downarrow g \\ \Big\lfloor & \longrightarrow & \longrightarrow & \longrightarrow & T((\xi_1+k) \vee (\zeta+k)) . \end{array}$$

Here Δ is the usual pinching map.

$$\eta_1 : S^{2q+k} \to T(\xi_1+k), \qquad \eta_0 : S^{2q+k} \to T(\zeta+k)$$

are maps of degree one, and g is induced by the map

$$M \cup S^q \to M \vee S^q .$$

Under this map g the bottom cells of the two Thom complexes are identified. Further

$$\gamma_* \sigma_{2q+k} = d_1 + d_0 ,$$

so $p \circ \gamma$ has degree 0. Again by the covariant Puppe sequence there exists a map

$$\eta_2 : S^{2q+k} \to T(\xi_2+k)$$

such that $h \circ \eta_2 = \gamma$. Clearly η_2 has degree one. The proposition follows from an easy computation using the commutative diagram:

$$\begin{array}{ccccc} S^{2q+k} \vee S^{2q+k} & \xrightarrow{\eta_1 \vee \eta_0} & \Sigma^k T(\xi_1) \vee \Sigma^k T(\zeta) & \xrightarrow{\Sigma^k U_{\xi_1} \vee \Sigma^k U_\zeta} & \Sigma^k K(Z_2,q) \vee \Sigma^k K(Z_2,q) \\ \uparrow \Delta & & \downarrow g & & \downarrow \\ & \xrightarrow{h} & \Sigma^k T(\xi_1 \vee \zeta) & \xrightarrow[\Sigma^k U_{\xi_1 \vee \zeta}]{} & \Sigma^k K(Z_2,q) \\ & & & & \uparrow \\ S^{2q+k} & \xrightarrow[\eta_2]{} & T(\xi_2+k) & \xrightarrow[\Sigma^k U_{\xi_2}]{} & \end{array}$$

PROPOSITION 3.4. *Let M be a q-dimensional compact differentiable manifold, q odd $\neq 1,3,7$. If the tangent sphere bundle τ is stably fibre homotopy trivial, then*

$$b(\tau) = \chi^*(M) \equiv \sum_{i=0}^{(q-1)/2} \dim H^i(M, Z_2) \pmod 2 .$$

PROOF. Choosing a tubular neighborhood of the diagonal in $M \times M$ we get a map

$$j : M \times M \to T(\tau)$$

by pinching everything outside. Let $\sigma_M \in H^q(M, Z_2)$ be the generator.

Following E. Thomas [17, § 4] we choose a basis $\{\alpha_1,\ldots,\alpha_d,\beta_d,\ldots,\beta_1\}$ for $H^*(M,Z_2)$ such that if

$$\deg \alpha_i + \deg \beta_j = q ,$$

then

$$\alpha_i \cup \beta_j = \delta_{ij}\sigma_M ,$$

where

$$d \equiv \chi^*(M) \pmod 2 .$$

Let $t\colon M\times M \to M\times M$ be the transposition map, and put

$$A = \sum_{i=1}^{d} \alpha_i \otimes \beta_i \in H^q(M\times M) .$$

Then

$$A \cup t^*A = d\sigma_{M\times M} \quad \text{and} \quad j^*U_\tau = A + t^*A .$$

Now let $g\colon S^{q+k} \to \Sigma^k M_+$ be a map of degree one. The map

$$g \wedge g : S^{2q+2k} \to \Sigma^{2k}(M\times M_+)$$

defines an $\boldsymbol{S}$-orientation of $M\times M$, in the sense of Browder [5, § 1] ($\boldsymbol{S}$ is the sphere cospectrum). This defines an operation

$$\psi : H^q(M\times M) \to Z_2 .$$

The map

$$\eta : S^{2q+2k} \xrightarrow{g\wedge g} \Sigma^{2k}(M\times M_+) \xrightarrow{\Sigma^{2k}j} \Sigma^{2k}(T(\tau))$$

has clearly degree one. By Theorem 1.4 in Browder [5], it follows that

$$\begin{aligned} b(\tau) &= \psi(j^*U_\tau) = \psi(A+t^*A) \\ &= \psi(A) + \psi(t^*A) + (g\wedge g)^*(A\cup t^*A) . \end{aligned}$$

From the commutativity of the diagram

$$\begin{array}{ccc} S^{2q+2k} & \xrightarrow{g\wedge g} & \Sigma^{2k}(M\times M_+) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \Sigma^{2k}t} \\ S^{2q+2k} & \xrightarrow[g\wedge g]{} & \Sigma^{2k}(M\times M_+) \end{array}$$

it follows that $\psi(A)=\psi(t^*A)$. Hence

$$b(\tau) = (g\wedge g)^*(A\cup t^*A) = d\sigma_{2q+2k} .$$

COROLLARY 3.5. *Let ζ be a stably fibre homotopy trivial $(q-1)$-dimensional sphere bundle over S^q. $\tau_q=\tau(S^q)$. Then*

$$\zeta=\tau_q \Leftrightarrow b(\zeta)=1, \qquad q \text{ odd} \neq 1,3,7 .$$

PROOF. $b(\tau_q)=1$ according to Proposition 3.4. It follows from Proposition 2.2 that either ζ is trivial or $\zeta=\tau_q$. If ζ is trivial, $b(\zeta)=0$ according to Proposition 3.3.

Using b in stead of E in the concluding remarks of § 2, we get

THEOREM 3.6. *Consider M, M' q-dimensional oriented compact differentiable manifolds, q odd $\neq 1,3,7$. Assume the stable fibre homotopy classes of the tangent sphere bundles are trivial. Let $f\colon M \to M'$ be an orientation preserving homotopy equivalence.*

Then $f^*\tau'$ and τ are fibre homotopy equivalent.

In fact, τ is fibre homotopy trivial iff $\chi^*(M)=0$.

Finally we remark that, if we use BSO(n) instead of BSH(n) in Section 2, then we can prove in the same way

THEOREM 3.7. *Consider M, M' and $f\colon M \to M'$ satisfying the conditions of Theorem* 3.6. *Let τ and τ' denote the tangent q-plane bundles of M and M' respectively, and assume further τ and $f^*\tau'$ to be stably isomorphic. Then τ and $f^*\tau'$ are isomorphic.*

Especially we get the following corollary implicitly proved in G. Bredon and A. Kosinski [4].

COROLLARY 3.8. *Let M and M' be π-manifolds and $f\colon M \to M'$ a homotopy equivalence. Then $f^*\tau'$ and τ are isomorphic.*

In fact, for q odd $\neq 1,3,7$ τ is trivial iff $\chi^*(M)=0$.

4. Remarks concerning the general case.

In this section we will explain the difficulty in the general case. First we recall some notation of W. Browder [5]: We assume q odd.

$$v_{q+1}:\ B_n \to K(\mathbf{Z}_2, q+1)$$

represents the Wu class v_{q+1}. Consider the fibration

$$B_n\langle v_{q+1}\rangle \xrightarrow{\pi} B_n$$

induced by v_{q+1} from the path fibration over $K(\mathbf{Z}_2, q+1)$ with fibre

$$\Omega K(\mathbf{Z}_2, q+1) = K(\mathbf{Z}_2, q)$$

and let $\bar{\gamma}_n = \pi^*(\gamma_n)$ denote the pull back of the universal sphere bundle over B_n. Further $Y_n = T(\bar{\gamma}_n)$ defines a Wu spectrum, and $\{X_n\}$ is the dual Wu cospectrum.

Now consider M^q a compact differentiable oriented manifold with normal bundle ν and let ξ be an oriented $(q-1)$-dimensional sphere bundle over M^q. Choose a bundle ξ' such that $\xi+\xi'$ is trivial (such one exists according to M. Spivak [13] or C. T. C. Wall [18]), and assume that the classifying map $\varphi: M \to B_n$ (n large) for $\nu+\xi'$ is given a specific lifting φ' through π. Then $\varphi=\pi\varphi'$ and $\nu+\xi'=\varphi'^*(\bar{\gamma}_n)$. This defines maps $T(\nu+\xi') \to Y_n$ and thus dual maps

$$X_{-2q-k} \xrightarrow{g_k} \Sigma^k T(\xi)$$

for k large such that

$$g_{k*}: H_{2q+k}(X_{-2q-k}, \mathbf{Z}) \to H_{2q+k}(\Sigma^k T(\xi), \mathbf{Z})$$

is an isomorphism. Such a system of maps we call an $\boldsymbol{X}$-orientation for ξ.

Let $U_\xi \in H^q(T(\xi), \mathbf{Z}_2)$. Assume $g_k^*(\Sigma^k U_\xi)=0$. (Using S-duality, see E. Spanier [12], this is seen to be equivalent to the following condition: If $i_1+\ldots+i_s=q$, then $w_{i_1}(\nu+\xi')\cup\ldots\cup w_{i_s}(\nu+\xi')=0$. This is clearly fullfilled if ξ is stably equivalent to τ.) Consider the map

$$\delta = \Sigma^k U_\xi \circ g_k : X_{-2q-k} \to \Sigma^k K(\mathbf{Z}_2, q)$$

and define

$$b_g(\xi) = Sq_\delta^{q+1}(\Sigma^k \iota) \in H^{2q+k}(X_{-2q-k}) = \mathbf{Z}_2 .$$

As in Browder [5] the indeterminacy is 0. A priori $b_g(\xi)$ might depend on the orientation g_k. In fact it does for $q=1,3,7$.

In turn the orientation depends on the following choices:

I a) ν and the trivialization of $\nu+\tau$.
 b) ξ' and the trivialization of $\xi+\xi'$.

II The lifting φ' of φ.

It turns out that II is not very serious, and the problem concerning I can be reduced to the following:

Let $\psi: M\times S^{k-1} \to M\times S^{k-1}$ be a fibre-homotopy equivalence. This induces a map

$$\alpha = T(id\oplus\psi) : T(\xi+k) \to T(\xi+k) ,$$

and $g_k'=\alpha\circ g_k$ defines a new orientation for ξ.

If $b_{g'}(\xi)=b_g(\xi)$, then Section 3 goes through with minor changes, and proves the conclusions of Theorems 3.6 and 3.7 without the assumption on the stable fibre-homotopy class to be trivial. However, as pointed out by the referee there are examples where $\xi^{\tau_q}=\xi$ contradicting Proposition 3.3 but not 3.6 and 3.7. Is this the only case? Or stated in another way: If there is a map

$$X_{-2q-k} \to T(\xi + k_{|N})$$

with a non-zero functional Sq^{q+1}, is it true then that $\xi^{\tau q} = \xi$? We will discuss this in a later paper.

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