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On Hybrid Diffusions

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***Abstract*— For the pressing needs in control and optimization using hybrid systems, this work focuses on weak stochastic stability and ergodicity of regime-switching diffusions. Using Liapunov functions, we derive necessary and sufficient conditions for weak stability. Then, ergodicity of weakly stable regime-switching diffusions is obtained by constructing cycles using the associated discrete-time Markov chains.**

***Index Terms*— Switching diffusion, Liapunov function, weak stochastic stability, positive recurrence, ergodicity.**

I. Introduction

Owing to the increasing demands for modeling large-scale and complex systems, designing optimal control, and conducting optimization tasks, resurgent interest has been directed to hybrid systems. A feature of these systems is the coexistence of continuous dynamics and discrete events. Hybrid diffusion systems (also known as regime-switching diffusions) belongs to such a class. Aiming at capturing random evolutions, recent research efforts stem from emerging applications in financial engineering, wireless communications, manufacturing systems, and other related fields.

The systems in applications are often in operation for a relatively long time, thus it is important to understand the systems' asymptotic properties. For instance, treating average cost per unit time problems, we often wish to "replace" the time-dependent instantaneous measure by a steady state measure. Do the systems possess ergodic property? Under what conditions, do the systems have the desired ergodicity? In accordance with [22], a deterministic system $\dot{x} = g(t, x)$ satisfying appropriate conditions is Lagrange stable, if the solutions are ultimately uniformly bounded. When stochastic systems are treated, almost sure boundedness excludes many systems. Thus, in lieu of such boundedness, one seeks stability in certain weak sense [31]. One question of fundamental importance is: Under what conditions, will the systems return to a prescribed compact region in finite time? In this paper, we focus on asymptotic behaviors and address these issues. More specifically, we deal with such properties as recurrence, positive recurrence, and ergodicity. One of the main features of our approach is the use of appropriate Liapunov functions. We develop Liapunov-function-based general criteria for weak stability, followed by a further study on ergodicity by constructing cycles using discrete-time Markov chains. In what follows, the conditions needed are provided, and the results are stated. However, for detailed proofs of the results, we refer the reader to [35] for verbatim argument. Note that in the proofs, some results from partial differential equations and stochastic systems are used; see [1], [5], [6], [7], [8], [10], [27] and references therein.

A. Formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual condition (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets). Let $x \in \mathbb{R}^r$, $\mathcal{M} = \{1, \dots, m_0\}$, and $Q(x) = (q_{ij}(x))$ an $m_0 \times m_0$ matrix depending on x and satisfying that for any $x \in \mathbb{R}^r$, $q_{ij}(x) \ge 0$ for $i \ne j$ and $\sum_{j=1}^{m_0} q_{ij}(x) = 0$. For each $i \in \mathcal{M}$, and for any twice continuously differentiable function $g(\cdot, i)$, define $\mathcal{L}$ by

$$\begin{aligned}\mathcal{L}g(x,i) =& \frac{1}{2}\mathrm{tr}(a(x,i)\nabla^2 g(x,i)) + \langle b(x,i), \nabla g(x,i)\rangle \\ &+ Q(x)g(x,\cdot)(i),\end{aligned} \tag{1}$$

where $\nabla g(\cdot, i)$ and $\nabla^2 g(\cdot, i)$ denote the gradient and Hessian of $g(\cdot, i)$, respectively, and for $i \in \mathcal{M}$,

$$Q(x)g(x,\cdot)(i) = \sum_{j\ne i, j\in\mathcal{M}} q_{ij}(x)(g(x,j) - g(x,i)). \tag{2}$$

Consider a Markov process $Y(t) = (X(t), \gamma(t))$, whose associated operator is given by $\mathcal{L}$; see [28] for further references. Note that $Y(t)$ has two components, an r-dimensional diffusion component $X(t)$ and a jump component $\gamma(t)$ taking value in $\mathcal{M} = \{1, \dots, m_0\}$.

The process $Y(t) = (X(t), \gamma(t))$ can be described by the following equations:

$$\begin{aligned}&dX(t) = b(X(t),\gamma(t))dt + \sigma(X(t),\gamma(t))dw(t),\\ &X(0) = x,\ \gamma(0) = \gamma,\end{aligned} \tag{3}$$

and

$$\begin{aligned}&\mathbf{P}\{\gamma(t+\Delta t) = j | \gamma(t) = i, X(s), \gamma(s), s \le t\}\\ &\quad = q_{ij}(X(t))\Delta t + o(\Delta t), \quad i \ne j,\end{aligned} \tag{4}$$

where $w(t)$ is a d-dimensional standard Brownian motion, $b(\cdot,\cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^r$, and $\sigma(\cdot,\cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^{r\times d}$ satisfying $\sigma(x,i)\sigma'(x,i) = a(x,i)$ (where z' denotes the transpose of z for $z \in \mathbb{R}^{\iota_1\times\iota_2}$ with $\iota_1,\ \iota_2 \ge 1$). [We refer the reader to [28] for related stochastic differential equations involving Poisson measures describing the evolution of the jump process. In this paper, our study will mainly be concerned with the use of the operator $\mathcal{L}$ given in (1).] Throughout the paper, we assume that both $b(\cdot, i)$ and $\sigma(\cdot, i)$ satisfy the usual local Lipschitz condition and linear growth condition for each $i \in \mathcal{M}$ and that $Q(\cdot)$ is bounded and

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continuous. It is well known that under these conditions, the system (3)–(4) has a unique strong solution; see [12] or [28] for details. In what follows, denote the solution of (3)–(4) by $(X^{x,\gamma}(t), \gamma(t))$ if the emphasis on the initial data is needed. To study recurrence and ergodicity of the process $Y(t) = (X(t), \gamma(t))$, we further assume that the following condition (A) holds throughout the paper. For convenience, we also collect the boundedness and continuity of $Q(\cdot)$ in (A).

(A) The operator $\mathcal{L}$ satisfies the following conditions:

- For each $i \in \mathcal{M}$, $a(x,i) = (a_{jk}(x,i))$ is symmetric and satisfies
$$\kappa_1|\xi|^2 \le \langle a(x,i)\xi, \xi\rangle \le \kappa_1^{-1}|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^r, \tag{5}$$
with some constant $\kappa_1 \in (0,1]$ for all $x \in \mathbb{R}^r$.
- For $i \ne j$, $q_{ij}(x) > 0$. The matrix-valued function $Q(\cdot)$ is bounded and continuous.

As mentioned earlier, the motivation of our study stems from recent interests in regime-switching diffusion processes that include a random process with a finite state space in addition to the usual diffusion component. The finite-state process depicts random environment that has right-continuous sample paths and that cannot be described by a diffusion. Consequently, both continuous dynamics (diffusions) and discrete events (jumps) coexist yielding hybrid dynamic systems, which provide more realistic formulation for many applications.

Regime-switching diffusions have received much attention lately. For instance, optimal controls of switching diffusions were studied in [4] using a martingale problem formulation; jump-linear systems were considered in [13]; stability of semi-linear stochastic differential equation with Markovian switching was considered in [2]; ergodic control problems of switching diffusions were studied in [9]; stability of stochastic differential equations with Markovian switching was treated in [23], [25], [33]; asymptotic expansions for solutions of integro-differential equations for transition densities of singularly perturbed switching-diffusion processes were developed in [11]; switching diffusions were used for stock liquidation models in [34]. For some recent applications of hybrid systems in communication networks, air traffic management, and control problems, etc., we refer the reader to [14], [15], [24], [26], [29] and references therein.

In [2], [23], [33], [34], $Q(x) = Q$, a constant matrix. In such cases, $\gamma(\cdot)$ is a continuous-time Markov chain. Moreover, it is assumed that the Markov chain $\gamma(\cdot)$ is independent of the Brownian motion. In our formulation, x-dependent $Q(x)$ is considered, and as a result, the transition rates of the discrete event $\gamma(\cdot)$ depend on the continuous dynamic $X(\cdot)$, as depicted in (4). Although the pair $(X(\cdot), \gamma(\cdot))$ is a Markov process, for x-dependent $Q(x)$, only for each fixed x, the discrete-event process $\gamma(\cdot)$ is a Markov chain. Such formulation enables us to describe complex systems and their inherent uncertainty and randomness in the environment. However, it adds much difficulty in analysis. Our formulation is motivated by the fact that in many applications, the discrete event and continuous dynamic are intertwined, and the independence assumption of the discrete-event process and the Brownian motion appears to be restrictive.

One of the important problems concerning switching models is their longtime behavior. Despite the growing interests in treating regime-switching systems (see the works mentioned in the previous paragraphs and references therein), the results regarding such issues as recurrence and positive recurrence (or weak stochastic stability as termed in [31]) are still scarce. Furthermore, these are not simple extensions of their diffusion counterpart. Due to the coupling and interactions, elliptic systems instead of a single elliptic equation must be treated. Moreover, even the classical approaches such as Liapunov function methods and the Dykin's formula are still applicable for switching diffusions, the analysis is much more delicate than the diffusion counterparts. It requires careful handling of discrete-event component $\gamma(\cdot)$; see, for example, the proofs of Lemma 3.7, Lemma 3.8, and Theorem 3.12 in [35].

In addition to recurrence, many applications in control and optimization require minimizing an expected cost of certain objective function. The computation is difficult and complicated. Significant effort has been devoted to approximating such expected values by replacing the measure with stationary measures when the time horizon is long enough. To justify such a replacement, ergodicity is needed. For diffusion processes, much effort has been devoted to ergodicity; see for example [3], [20], among others. For regime-switching diffusions, asymptotic stability for the density of the so-called two-state diffusion precess $(X(t), \gamma(t))$ was established in [25]; asymptotic stability in distribution for the process $(X(t), \gamma(t))$ was obtained in [33], where the jump component $\gamma(\cdot)$ is generated by some constant matrix Q and is independent of the Brownian motion. In this work, we will address ergodicity for $(X(t), \gamma(t))$ under different conditions than those in [25], [33]. Moreover, our work is applicable to more general settings. The discrete component $\gamma(\cdot)$ has x-dependent generator $Q(x)$ and takes value in a finite state space $\mathcal{M} = \{1, 2, \dots, m_0\}$. Another highlight of the paper is that we obtain the explicit representation of the invariant measure of the process $(X(t), \gamma(t))$ by considering certain cylinder sets and by defining cycles appropriately. As a byproduct, we demonstrate a strong law of large number type theorem for positive recurrent regime-switching diffusions.

Compared with the existing work in the literature, the novelty and contribution of this paper are as follows. (a) By considering x-dependent generator $Q(x)$, our model provides more realistic formulation allowing the switching component depending on the continuous states. This, in turn, allows the coupling and correlation between $X(t)$ and $\gamma(t)$. (b) By appropriately defining cycles, we establish the ergodicity of the underlying process. (c) Moreover, explicit representation of the invariant measure for positive-recurrent regime-switching diffusions is given.

The rest of the paper is arranged as follows. In Section II, in addition to introducing certain notations, we also provide definitions of regularity, recurrence, positive recurrence, and null recurrence. Section III focuses on positive recurrence. We present results of necessary and sufficient conditions for recurrence using Liapunov functions, along with a couple of examples as applications of the general results. Section IV develops ergodicity of switching diffusion processes. Discussions and further remarks are made in Section V.

II. Regularity, Recurrence, and Positive Recurrence

This section is devoted to the definitions of regularity, recurrence, positive recurrence, and null recurrence. For simplicity, we introduce some notations as follows. For any $U = D \times J \subset \mathbb{R}^r \times \mathcal{M}$, where $D \subset \mathbb{R}^r$ and $J \subset \mathcal{M}$, denote

$$\begin{aligned} \tau_U &:= \inf\{t \geq 0 : (X(t), \gamma(t)) \notin U\}, \\ \sigma_U &:= \inf\{t \geq 0 : (X(t), \gamma(t)) \in U\}. \end{aligned} \tag{6}$$

In particular, if $U = D \times \mathcal{M}$ is a "cylinder", we set

$$\begin{aligned} \tau_D &:= \inf\{t \geq 0 : X(t) \notin D\}, \\ \sigma_D &:= \inf\{t \geq 0 : X(t) \in D\}. \end{aligned} \tag{7}$$

Definition 2.1: Regularity. A Markov process $(X^{x,\gamma}(t), \gamma(t))$ is said to be *regular*, if for any $0 < T < \infty$,

$$\mathbf{P}\{\sup_{0\leq t\leq T} |X^{x,\gamma}(t)| = \infty\} = 0. \tag{8}$$

Remark 2.2: Let β_n be the first exit time of the process $(X^{x,\gamma}(t), \gamma(t))$ from the bounded set $\{\widetilde{x} : |\widetilde{x}| < n\} \times \mathcal{M}$, that is,

$$\beta_n = \inf\{t : |X^{x,\gamma}(t)| = n\}. \tag{9}$$

Then, the sequence $\{\beta_n\}$ is monotonically increasing and hence has a (finite or infinite) limit. It is not difficult to see that the process $(X^{x,\gamma}(t), \gamma(t))$ is regular if and only if

$$\beta_n \to \infty \text{ almost surely as } n \to \infty. \tag{10}$$

In what follows, we assume that the process $(X^{x,\gamma}(t), \gamma(t))$ is regular. Subsequently we will use equation (10) often.

Definition 2.3: Recurrence and positive and null recurrence are defined as follows.

- *Recurrence.* For $U := D \times J$, where $J \subset \mathcal{M}$ and $D \subset \mathbb{R}^r$ is an open set with compact closure, let $\sigma_U^{x,\gamma} = \inf\{t : (X^{x,\gamma}(t), \gamma(t)) \in U\}$. A regular process $(X^{x,\gamma}(\cdot), \gamma(\cdot))$ is *recurrent* with respect to U if $\mathbf{P}\{\sigma_U^{x,\gamma} < \infty\} = 1$ for any $(x, \gamma) \in D^c \times \mathcal{M}$, where D^c denotes the complement of D.
- *Positive Recurrence and Null Recurrence.* A recurrent process with finite mean recurrence time for some set $U = D \times J$, where $J \subset \mathcal{M}$ and $D \subset \mathbb{R}^r$ is a bounded open set with compact closure, is said to be *positive recurrent* with respect to U, otherwise, the process is *null recurrent* with respect to U.

III. Positive Recurrence

This section takes up the positive recurrence issue. It entails the use of appropriate Liapunov functions. We begin this section with certain preparatory results, which indicate that the process $Y(t) = (X(t), \gamma(t))$ is recurrent (resp. positive recurrent) with respect to some "cylinder" $D \times \mathcal{M}$ if and only if it is recurrent (resp. positive recurrent) with respect to $D \times \{\ell\}$, where $D \subset \mathbb{R}^r$ is a nonempty open set with compact closure and $\ell \in \mathcal{M}$. We will also prove that the properties of recurrence or positive recurrence do not depend on the choice of the open set $D \subset \mathbb{R}^r$ or $\ell \in \mathcal{M}$. After the preparatory results, two subsections follow. The first presents Liapunov-function-based criteria on positive recurrence. As applications of the general results, a subsection of two examples is provided. Note that Example 3.3 is quite interesting, which shows that the combination of a transient diffusion and a positive recurrent diffusion is a positive recurrent switching diffusion.

It can be shown that recurrence (positive recurrence) is independent of the set. Thus, we have the following remarks.

- A regular process $Y(t) = (X(t), \gamma(t))$ with the associated generator $\mathcal{L}$ satisfying (A) is said to be *recurrent*, if it is recurrent with respect to some $U = D \times \{\ell\}$, where $D \subset \mathbb{R}^r$ is a nonempty bounded open set and $\ell \in \mathcal{M}$; otherwise it is said to be *transient*.
- Henceforth, we call a recurrent process $Y(t) = (X(t), \gamma(t))$ *positive recurrent* if it is positive recurrent with respect to some bounded domain $U = D \times \{\ell\} \subset \mathbb{R}^r \times \mathcal{M}$; otherwise, we have a *null recurrent* process.

A. General Criteria

Theorem 3.1: A necessary and sufficient condition for positive recurrence with respect to a domain $U = D \times \{\ell\} \subset \mathbb{R}^r \times \mathcal{M}$ is: For each $i \in \mathcal{M}$, there exists a nonnegative function $V(\cdot, i) : D^c \mapsto \mathbb{R}$ such that $V(\cdot, i)$ is twice continuously differentiable and that

$$\mathcal{L}V(x, i) = -1, \quad (x, i) \in D^c \times \mathcal{M}. \tag{11}$$

Let $u(x, i) = \mathbf{E}_{x,i}\sigma_D$. Then $u(x, i)$ is the smallest positive solution of

$$\begin{cases} \mathcal{L}u(x, i) = -1, & (x, i) \in D^c \times \mathcal{M}, \\ u(x, i) = 0, & (x, i) \in \partial D \times \mathcal{M}, \end{cases} \tag{12}$$

where ∂D denotes the boundary of D.

One of the nice things about the above result is: It converts the problem to a boundary value problem. Thus we can use analytic tools to resolve the problem. We also note that sometimes, the equation above may still be difficult to use. It would be nice if we have an upper bound instead. This brings us to the following alternative result.

Theorem 3.2: A necessary and sufficient condition for positive recurrence with respect to a domain $U = D \times \{\ell\} \subset \mathbb{R}^r \times \mathcal{M}$ is: For each $i \in \mathcal{M}$, there exists a nonnegative function $V(\cdot, i) : D^c \mapsto \mathbb{R}$ such that $V(\cdot, i)$ is twice continuously differentiable and that for some $\alpha > 0$,

$$\mathcal{L}V(x, i) \leq -\alpha, \quad (x, i) \in D^c \times \mathcal{M}. \tag{13}$$

B. Examples

In this subsection, we provide a couple of examples to illustrate Theorem 3.1 and Theorem 3.2.

Example 3.3: To illustrate the utility of Theorem 3.2, consider a real-valued process

$$dX(t) = b(X(t), \gamma(t))dt + \sigma(X(t), \gamma(t))dw(t), \tag{14}$$

where $\gamma(t)$ is a 2-state random jump process, with x-dependent generator

$$Q(x) = \begin{pmatrix} -\frac{1}{3} - \frac{1}{4}\cos x & \frac{1}{3} + \frac{1}{4}\cos x \\ \frac{7}{3} + \frac{1}{2}\sin x & -\frac{7}{3} - \frac{1}{2}\sin x \end{pmatrix},$$

and

$$b(x,1) = -x, \quad \sigma(x,1) = 1, \quad b(x,2) = x, \quad \sigma(x,2) = 1.$$

Thus (14) can be regarded as the result of the following two diffusions:

$$dX(t) = -X(t)dt + dw(t), \quad \text{and} \tag{15}$$

$$dX(t) = X(t)dt + dw(t), \tag{16}$$

switching back and forth from one to the other according to the movement of $\gamma(t)$.

Note that (15) is positive recurrent while (16) is a transient diffusion process. But, the switching diffusion (14) is positive recurrent. We verify these as follows. Consider the Liapunov function $V(x,1) = |x|$. Let $\mathcal{L}_1$ be the operator associated with (15). Then we have for all $|x| \geq 1$, $\mathcal{L}_1 V(x,1) = -x \cdot \text{sign}x = -|x| \leq -1 < 0$. Thus it follows from [16, Theorem 3.7.3] that (15) is positive recurrent. Recall that the real-valued diffusion process $dX(t) = b(X(t))dt + \sigma(X(t))dw(t)$ with $\sigma(x) \neq 0$ for all $x \in \mathbb{R}$, is recurrent if and only if $\int_0^x \exp\{-2\int_0^u \frac{b(z)}{\sigma^2(z)}dz\}du \to \pm\infty$ as $x \to \pm\infty$; see [16, p. 105]. Direct computation shows that (16) fails to satisfy this condition and hence is transient.

Next, we use Theorem 3.2 to demonstrate that the switching diffusion (14) is positive recurrent for appropriate Q. Consider Liapunov functions

$$V(x,1) = |x|, \quad V(x,2) = \frac{7}{3}|x|.$$

Then we can show

$$\mathcal{L}V(x,1) \leq -\frac{2}{9}, \quad \mathcal{L}V(x,2) \leq -\frac{1}{9},$$

for all $|x| \geq 1$. Then the switching diffusion (14) is positive recurrent by Theorem 3.2.

IV. Ergodicity

In this section, we study the ergodic properties of the process $Y(t) = (X(t), \gamma(t))$ under the assumption that the process is positive recurrent with respect to some bounded domain $U = E \times \{\ell\}$, where $E \subset \mathbb{R}^r$ and $\ell \in \mathcal{M}$ are fixed throughout this section. We also assume that the boundary ∂E of E is sufficiently smooth. Let the operator $\mathcal{L}$ satisfy (A). Then it follows that the process is positive recurrent with respect to any nonempty open set.

Let $D \subset \mathbb{R}^r$ be a bounded ball with sufficiently smooth boundary ∂D such that $E \cup \partial E \subset D$. Let $\varsigma_0 = 0$ and define the stopping times $\varsigma_1, \varsigma_2, \ldots$ inductively as: ς_{2n+1} is the first time after ς_{2n} at which the process $Y(t) = (X(t), \gamma(t))$ reaches the set $\partial E \times \{\ell\}$ and ς_{2n+2} is the first time after ς_{2n+1} at which the path reaches the set $\partial D \times \{\ell\}$. Now we can divide an arbitrary sample path of the process $Y(t) = (X(t), \gamma(t))$ into cycles:

$$[\varsigma_0, \varsigma_2), [\varsigma_2, \varsigma_4), \ldots, [\varsigma_{2n}, \varsigma_{2n+2}), \ldots \tag{17}$$

Figure 1 presents a demonstration of such cycles when the discrete component $\gamma(\cdot)$ has three states.

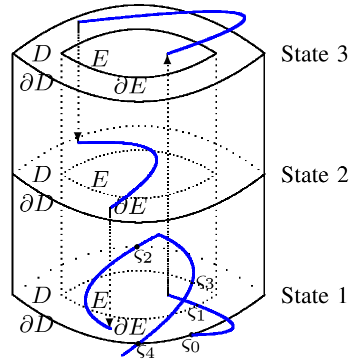

Fig. 1. A "Sample Path" of the Process $Y(t) = (X(t), \gamma(t))$ when $m_0 = 3$

The process $Y(t) = (X(t), \gamma(t))$ is positive recurrent with respect to $E \times \{\ell\}$ and hence positive recurrent with respect to $D \times \{\ell\}$. It follows that all the stopping times $\varsigma_0 < \varsigma_1 < \varsigma_2 < \varsigma_3 < \varsigma_4 < \cdots$ are finite a.s. Since the process $Y(t) = (X(t), \gamma(t))$ is positive recurrent, we may assume without loss of generality that $Y(0) = (X(0), \gamma(0)) = (x, \ell) \in \partial D \times \{\ell\}$. It follows from the strong Markov property of the process $Y(t) = (X(t), \gamma(t))$ that the sequence $\{Y_n\}$ is a Markov chain on $\partial D \times \{\ell\}$, where $Y_n = Y(\varsigma_{2n}) = (X_n, \ell)$, $n = 0, 1, 2, \ldots$ Let $\widetilde{P}(x, A)$ denote the one-step transition probabilities of this Markov chain, that is,

$$\widetilde{P}(x, A) = \mathbf{P}\left(Y_1 \in (A \times \{\ell\}) \mid Y_0 = (x, \ell)\right)$$

for any $x \in \partial D$ and $A \in \mathcal{B}(\partial D)$, where $\mathcal{B}(\partial D)$ denotes the collection of Borel measurable sets on ∂D. Note that the process $Y(t) = (X(t), \gamma(t))$, starting from (x, ℓ), may jump many times before it reaches the set (A, ℓ); see [28] for more details. Denote by $\widetilde{P}^{(n)}(x, A)$ the n-step transition probability of the Markov chain for any $n \geq 1$. For any Borel measurable function $f : \mathbb{R}^r \mapsto \mathbb{R}$, set

$$\mathbf{E}_x f(X_1) := \mathbf{E}_{x,\ell} f(X_1) = \int_{\partial D} f(y)\widetilde{P}(x, dy). \tag{18}$$

Throughout this section, we write $\mathbf{E}_x$ for $\mathbf{E}_{x,\ell}$ for simplicity. We will show that the process $Y(t) = (X(t), \gamma(t))$ possesses a unique stationary distribution. To this end, we need the following lemma.

Lemma 4.1: The Markov chain $Y_i = (X_i, \ell)$ has a unique stationary distribution $m(\cdot)$ such that

$$\left|\widetilde{P}^{(n)}(x, A) - m(A)\right| < \lambda^n, \quad \text{for any} \quad A \in \mathcal{B}(\partial D), \tag{19}$$

for some constant $0 < \lambda < 1$.

Theorem 4.2: The positive recurrent process $Y(t) = (X(t), \gamma(t))$ has a unique stationary distribution $\widehat{\nu}(\cdot, \cdot) = (\widehat{\nu}(\cdot, i) : i \in \mathcal{M})$.

The form of $\widehat{\nu}$ is given as follows. For any $A \in \mathcal{B}(\mathbb{R}^r)$ and $i \in \mathcal{M}$, denote by $\tau^{A \times \{i\}}$ the time spent by the path of $Y(t) = (X(t), \gamma(t))$ in the set $(A \times \{i\})$ during the first cycle. Set $\nu(A, i) := \int_{\partial D} m(dx)\mathbf{E}_x \tau^{A\times\{i\}}$, where $m(\cdot)$ is the stationary distribution of $Y_i = (X_i, \ell)$. We proved in [35] that

$$\widehat{\nu}(A, i) = \frac{\nu(A, i)}{\sum_{j=1}^{m_0} \nu(\mathbb{R}^r, j)},\ i \in \mathcal{M}, \tag{20}$$

defines the desired stationary distribution.

Theorem 4.3: Denote by $\mu(\cdot,\cdot)$ the stationary density associated with the stationary distribution $\widehat{\nu}(\cdot,\cdot)$ constructed in Theorem 4.2 and let $f(\cdot,\cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}$ be a Borel measurable function such that

$$\sum_{i=1}^{m_0} \int_{\mathbb{R}^r} |f(x, i)|\mu(x, i)dx < \infty. \tag{21}$$

Then

$$\mathbf{P}_{x,i}\left(\frac{1}{T}\int_0^T f(X(t), \gamma(t))dt \to \overline{f}\right) = 1, \tag{22}$$

for any $(x, i) \in \mathbb{R}^r \times \mathcal{M}$, where

$$\overline{f} = \sum_{i=1}^{m_0} \int_{\mathbb{R}^r} f(x, i)\mu(x, i)dx. \tag{23}$$

As a consequence of Theorem 4.3, we obtain the following corollary.

Corollary 4.4: Let the assumptions of Theorem 4.3 be satisfied and let $u(t, x, i)$ be the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t, x, i)}{\partial t} &= \mathcal{L}u(x, i),\ i \in \mathcal{M}, \\ u(0, x, i) &= f(x, i). \end{cases} \tag{24}$$

Then as $T \to \infty$,

$$\frac{1}{T}\int_0^T u(t, x, i)dt \to \sum_{i=1}^{m_0} \int_{\mathbb{R}^r} f(x, i)\mu(x, i)dx. \tag{25}$$

V. Discussions and Remarks

A. Discussions

The recurrence and ergodicity obtained enable us to undertake further study on asymptotic properties of hybrid diffusion systems, and to carry out control and optimization tasks. We outline several directions in what follows.

Easily Verifiable Conditions. In many applications, it is often more convenient to be able to analyze weak stability through conditions on the coefficients of the corresponding stochastic differential equations. Assume that $X(\cdot)$ is a real-valued process for simplicity; assume also condition (A) holds. Motivated by Example 3.3, next we present easily verifiable conditions for positive recurrence when the coefficients of the switching diffusions (3)–(4) are linearizable in an x-neighborhood of ∞. Suppose that for each $i \in \mathcal{M}$, there exists $b_i \in \mathbb{R}$ such that

$$\frac{b(x, i)}{x} = b_i + o(1), \quad \text{and} \quad Q(x) \to \widetilde{Q}, \quad \text{as} \quad |x| \to \infty,$$

where $\widetilde{Q} = (\widetilde{q}_{ij})$ is the generator of a continuous-time ergodic Markov chain $\widetilde{\gamma}(t)$ whose stationary distribution is $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^{1\times m}$. Then using Theorem 3.2, we can prove that the process is positive recurrent if $\sum_{i=1}^m \mu_i b_i < 0$. The result can be strengthened if in addition,

$$\frac{\sigma(x, i)}{x} = \sigma_i + o(1) \quad \text{as} \quad |x| \to \infty,$$

where $\sigma_i^2 > 0$. Then in this case, the process is positive recurrent if

$$\sum_{i=1}^m \mu_i\Big(b_i - \frac{\sigma_i^2}{2}\Big) < 0.$$

The details are omitted for brevity.

Path Excursions. Applications of the positive recurrence criteria enable us to establish path excursions of the underlying processes. Suppose that $Y(t) = (X(t), \gamma(t))$ is positive recurrent. Suppose that the Liapunov functions $V(x, i)$ (with $i \in \mathcal{M}$) are given in Theorem 3.2, so is the set D. Let D_0 be a bounded open set with compact closure satisfying $D \subset D_0$, and τ be a random time such that $(X(\tau), \gamma(\tau)) \in D_0^c \times \mathcal{M}$, and $\tau_1 = \min\{t > \tau : (X(t), \gamma(t)) \in D_0 \times \mathcal{M}\}$. We can obtain for $\kappa > 0$,

$$\begin{aligned} &\mathbf{P}(\sup_{\tau\le t\le\tau_1} V(X(t), \gamma(t)) \ge \kappa) \le \frac{\mathbf{E}V(X(\tau), \gamma(\tau))}{\kappa}, \\ &\mathbf{E}(\tau_1 - \tau) \le \frac{\mathbf{E}V(X(\tau), \gamma(\tau))}{\alpha}, \end{aligned}$$

where α is as given in Theorem 3.2.

Tightness. Under positive recurrence, we may obtain tightness (or boundedness in the sense of in probability) of the underlying process. Suppose that $(X(t), \gamma(t))$ is positive recurrent. It is then possible to prove that for any compact set $\overline{D}$, the set $\cup_{x\in\overline{D}}\{(X(t), \gamma(t)) : t \ge 0,\ X(0) = x, \gamma(0) = \gamma\}$ is tight (or bounded in probability). For a study on the diffusion counter part, we refer the reader to [19, p. 146].

Occupation Measures. To illustrate the utility of Theorem 4.3, take $f(x, i) = \chi_{[B\times J]}(x, i)$, the indicator function of the set $B \times J$, where $B \subset \mathbb{R}^r$ and $J \subset \mathcal{M}$. Then Theorem 4.3 becomes a result regarding occupation measure. In fact, we have

$$\frac{1}{T}\int_0^T \chi_{[B\times J]}(X(t), \gamma(t))dt \to \sum_{i\in J}\int_B \mu(x, i)dx \text{ a.s.}$$

as $T \to \infty$.

Stochastic Approximation. Consider a parameter optimization problem. We wish to find θ_*, a vector-valued parameter so that the cost function

$$J(\theta) = \lim_{T\to\infty} E\frac{1}{T}\int_0^T \widehat{J}(\theta, Y(t))dt$$

is minimized, where $Y(t)$ is a positive recurrent switching diffusion as considered in this paper and for each θ, $\widehat{J}(\theta, \cdot, \cdot)$ satisfies the conditions of Theorem 4.3. For simplicity, we assume that the gradient of $\widehat{J}(\cdot, x, i)$ with respect to θ is available for each x and each $i \in \mathcal{M}$. Then we consider a constant stepsize recursive algorithm

$$\theta_{n+1} = \theta_n - \varepsilon \frac{1}{T} \int_{nT}^{nT+T} \nabla \widehat{J}(\theta_n, Y(t))dt,$$

or a decreasing stepsize algorithm

$$\theta_{n+1} = \theta_n - \varepsilon_n \frac{1}{T} \int_{nT}^{nT+T} \nabla \widehat{J}(\theta_n, Y(t))dt,$$

where $\varepsilon > 0$, and $\varepsilon_n \to 0$ as $n \to \infty$ and $\sum_n \varepsilon_n = \infty$. Modifications and variants are possible. For example, we may include additional measurement noise, and the gradient of $\widehat{J}(\cdot)$ may be changed to its gradient estimates. The motivation for such algorithms stems from optimization of average cost per unit time problems arising from parameter estimations in switching systems of SDEs, manufacturing systems, and queueing networks; see related work in [21, Chapter 9] and [32]. The ergodicity of the switching diffusion is crucial in the study of the asymptotic behavior of the algorithms.

Further Remarks. This work developed asymptotic properties of positive recurrent switching diffusions. Under general conditions, necessary and sufficient conditions for positive recurrence were developed. Then ergodicity was established for positive recurrent Markov processes with switching. Also provided were explicit representations of the invariant measures. For new results on related problems of stability for regime-switching diffusions, we refer the reader to our recent work [18]. Related work on randomly switching ODEs can be found in [36], in which different phenomena than the well-known Hartman-Grobman theorem have been observed. A number of problems remain open. Obtaining large deviations type of bounds is a worthwhile under taking, which will have important impact on studying the associated control and optimization problems. Next, concerning null-recurrent switching diffusions (see [16], [17]), can we obtain necessary and sufficient conditions? It appears that the desired criteria will be more difficult to obtain compared to a single diffusion process since one needs to solve systems of boundary value problems.

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