# On hyperbolic groups

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**Abstract**

We prove that a $\delta$-hyperbolic group for $\delta < \frac{1}{2}$ is a free product $F * G_1 * \ldots * G_n$ where $F$ is a free group of finite rank and each $G_i$ is a finite group.

## 1 Introduction and main result

Let $(X, d)$ be a metric space. The *Gromov product* $(x \cdot y)_v$ of two points $x, y \in X$ with respect to a point $v \in X$ is defined by

$$(x \cdot y)_v = \frac{1}{2}(d(x, v) + d(y, v) - d(x, y)).$$

$X$ is termed $\delta$***-hyperbolic***, where $\delta$ is positif reel number, if for any $x, y, z, v \in X$,

$$(x \cdot y)_v \geq \min \{(x \cdot z)_v, (y \cdot z)_v\} - \delta.$$

If $H$ is a group generated by a set $S$, the Cayley graph $Cay(H, S)$ is a metric space with the path metric and with edges of length one.

A group $H$ is $\delta$***-hyperbolic***, where $\delta \geq 0$, if $H$ is generated by a finite set $S$ such that the Cayley graph $Cay(H, S)$ is $\delta$-hyperbolic as a metric space. $H$ is termed ***hyperbolic*** if $H$ is $\delta$-hyperbolic for some $\delta \geq 0$. It is well-known [1, 2] that being hyperbolic does not depend on a particular generating set $S$, but $\delta$ depends on $S$.

We define $\delta_0(H)$ to be the infimum of $\delta$ for which $H$ is $\delta$-hyperbolic. It is a natural question to ask when we can have $\delta_0(H) = 0$ ? We show that in general we must have $\delta_0(H) \geq \frac{1}{2}$. The main result of this paper is the following.

**Theorem 1.1** *Let $H$ be a $\delta$-hyperbolic group for some $\delta < \frac{1}{2}$. Then $H$ is a free product $F * G_1 * \ldots * G_n$ where $F$ is a free group of finite rank and each $G_i$ is a finite group.*

Let $H$ be a group generated by a finite set $S$. For an element $h \in H$ we denote by $\ell_S(g)$ the shortest word $w$ in the alphabet $S^{\pm 1}$ such that $h = w$ in $H$. For $u, v \in H$ we let $c(u,v) = \frac{1}{2}(\ell_S(u) + \ell_S(v) - \ell_S(uv^{-1}))$. We say that $\ell_S$ is ***δ-hyperbolic***, if

$$c(u,v) \geq \min \{c(u,z), c(z,v)\} - \delta, \quad \text{for all } u,v,z \in H.$$

For $u, v \in H$ we let $d_S(u,v) = \ell_S(uv^{-1})$. It is well-known [1, 2] that the metric space $(H, d_S)$ is isometrically embedded in the Cayley graph $Cay(H,S)$. Therefore if $Cay(H,S)$ is $\delta$-hyperbolic then $\ell_S$ is $\delta$-hyperbolic.

If $Cay(H,S)$ is $\delta$-hyperbolic for some $\delta < \frac{1}{2}$ then, actually, $\ell_S$ is $0$-*hyperbolic*. Indeed. Since

$$c(u,v) - \min \{c(u,z), c(z,v)\} \geq -\delta > -\frac{1}{2}, \quad \text{for all } u,v,z \in H,$$

and since the number $c(u,v) - \min \{c(u,z), c(z,v)\}$ is an integer or a half of integer we get

$$c(u,v) - \min \{c(u,z), c(z,v)\} \geq 0, \quad \text{for all } u,v,z \in H,$$

and thus $\ell_S$ is 0-hyperbolic.

Therefore Theorem 1.1 is a consequence of the following one.

**Theorem 1.2** *Let $H$ be a group generated by a finite set $S$ such that the word length $\ell_S$ is $0$-hyperbolic. Then $H$ is a free product $F * G_1 * \ldots * G_n$ where $F$ is a free group of finite rank and each $G_i$ is a finite group.*

Note that when $H = F * G_1 * \ldots * G_n$ where $F$ is free group with basis $X$ and each $G_i$ is a finite group, if we let $S = X \cup G_1 \ldots \cup G_n$ then the word length $\ell_S$ is 0-hyperbolic.

To prove Theorem 1.2 we follow the general line of the argument used by R.C. Lyndon in [3]. The paper is organized as follows. We end this section by a general lemma about hyperbolic groups. In the next section we prove preparatory lemmas in the particular case when $\ell_S$ is 0-hyperbolic. In the last section we prove Theorem 1.2.

We end this section by the following.

**Lemma 1.3** *Let $H$ be a group generated by a finite set $S$ such that the word length $\ell_S$ (denoted simply $\ell$) is $\delta$-hyperbolic for some $\delta \geq 0$. Let $h_1, h_2, h_3$ be elements of $H$ satisfying*

$$c(h_1, h_2^{-1}) + c(h_2, h_3^{-1}) < \ell(h_2) - \delta.$$

*Then*

$$|\ell(h_1 h_2 h_3) - (\sum_{i=1}^{3} \ell(h_i) - 2\sum_{i=1}^{2} c(h_i, h_{i+1}^{-1}))| \leq 2\delta.$$

*In particuliar if $\delta < \frac{1}{2}$ then*

$$\ell(h_1h_2h_3) = \sum_{i=1}^{3} \ell(h_i) - 2\sum_{i=1}^{2} c(h_i, h_{i+1}^{-1}).$$

**Proof**

We have

$$c(h_2, h_3^{-1}) = \frac{1}{2}(\ell(h_2) + \ell(h_3) - \ell(h_2h_3))$$

$$= \frac{1}{2}(\ell(h_2) + \ell(h_2) - \ell(h_2) + \ell(h_3) - \ell(h_2h_3))$$

$$= \ell(h_2) - \frac{1}{2}(\ell(h_2) + \ell(h_2h_3) - \ell(h_3))$$

$$= \ell(h_2) - c(h_2^{-1}, (h_2h_3)^{-1}).$$

From the hypothesis $c(h_1, h_2^{-1}) + c(h_2, h_3^{-1}) < \ell(h_2) - \delta$ we get

$$c(h_1, h_2^{-1}) < c(h_2^{-1}, (h_2h_3)^{-1}) - \delta. \tag{1}$$

By the $\delta$-hyperbolicity of $\ell$ we have

$$c(h_1, h_2^{-1}) \geq \min\,\{c(h_1, (h_2h_3)^{-1}), c(h_2^{-1}, (h_2h_3)^{-1})\} - \delta,$$

and by (1) we get

$$c(h_1, h_2^{-1}) \geq c(h_1, (h_2h_3)^{-1}) - \delta,$$

therefore

$$\ell(h_1) + \ell(h_2) - \ell(h_1h_2) \geq \ell(h_1) + \ell(h_2h_3) - \ell(h_1h_2h_3) - 2\delta,$$

and thus

$$\ell(h_1h_2h_3) \geq \ell(h_2h_3) - \ell(h_2) + \ell(h_1h_2) - 2\delta.$$

But we have

$$\sum_{i=1}^{3} \ell(h_i) - 2\sum_{i=1}^{2} c(h_i, h_{i+1}^{-1})$$

$$= \sum_{i=1}^{3} \ell(h_i) - \Big((\ell(h_1) + \ell(h_2) - \ell(h_1h_2)) + (\ell(h_2) + \ell(h_3) - \ell(h_2h_3))\Big)$$

$$= \ell(h_2h_3) - \ell(h_2) + \ell(h_1h_2). \tag{2}$$

Therefore

$$\ell(h_1h_2h_3) \geq \sum_{i=1}^{3} \ell(h_i) - 2\sum_{i=1}^{2} c(h_i, h_{i+1}^{-1}) - 2\delta. \tag{3}$$

An other side, by the $\delta$-hyperbolicity

$$c(h_1,(h_2h_3)^{-1}) \geq \min\{c(h_1,h_2^{-1}), c(h_2^{-1},(h_2h_3)^{-1})\} - \delta,$$

and by (1) we find

$$c(h_1,(h_2h_3)^{-1}) \geq c(h_1,h_2^{-1}) - \delta.$$

Therefore

$$\ell(h_1)+\ell(h_2h_3)-\ell(h_1h_2h_3) \geq \ell(h_1)+\ell(h_2)-\ell(h_1h_2)-2\delta,$$

and thus

$$\ell(h_2h_3)-\ell(h_1h_2h_3) \geq \ell(h_2)-\ell(h_1h_2)-2\delta,$$

thus by (2) we get

$$\ell(h_1h_2h_3) \leq \sum_{i=1}^{3}\ell(h_i) - 2\sum_{i=1}^{2}c(h_i,h_{i+1}^{-1}) + 2\delta. \tag{4}$$

By (3) and (4) we conclude

$$|\ell(h_1h_2h_3) - (\sum_{i=1}^{3}\ell(h_i) - 2\sum_{i=1}^{2}c(h_i,h_{i+1}^{-1}))| \leq 2\delta.$$

Now the last assertion of the lemma follows from the fact that $\delta < \frac{1}{2}$ and that $\ell(h_1h_2h_3) - (\sum_{i=1}^{3}\ell(h_i) - 2\sum_{i=1}^{2}c(h_i,h_{i+1}^{-1}))$ is an integer. □

**Assumption** In the rest of this paper we let $H$ to be a group generated by a finite set $S$ such that the word length $\ell_S$ is 0-**hyperbolic**. To simplify notations we denote by $\ell$ the word length $\ell_S$. For convenience we suppose also that $1 \notin S$ and $S = S^{-1}$.

## 2 Preparatory lemmas

**Definition 2.1** We define a relation $\sim$ on $S$ by

$$s \sim t \text{ if and only if } c(s,t) \geq \frac{1}{2}.$$

**Lemma 2.2** *The relation $\sim$ is an equivalence relation on $S$.*

**Proof**

Clearly $\sim$ is reflexive and symmetric. Let $s,t,u \in S$ such that $s \sim t$ and $t \sim u$. Then

$$c(s,t) \geq \frac{1}{2}, \quad c(t,u) \geq \frac{1}{2}.$$

Since $\ell$ is 0-hyperbolic we get $c(s,u) \geq \frac{1}{2}$ and thus $s \sim u$. □

**Notation.** For $s \in S$ we denote by $q(s)$ the equivalence class of $s$ under $\sim$. We let

$$S_1 = \{s \in S \mid s \sim s^{-1}\}, \quad S_2 = S \setminus S_1.$$

**Lemma 2.3** *For each $s \in S_1$ the set $H(s) = (q(s) \cap S_1) \cup \{1\}$ is a finite subgroup of $H$.*

**Proof**

Let $u, v \in H(s)$ and prove that $uv^{-1} \in H(s)$. Clearly if $u = 1$ or $v = 1$ or $u = v$ then $uv^{-1} \in H(s)$. Thus we may assume that $u \neq 1$, $v \neq 1$ and $u \neq v$; thus $u, v \in q(s) \cap S_1$ and $uv^{-1} \neq 1$.

Since $u \sim v$ we have $c(u, v) \geq \frac{1}{2}$ and a simplification shows $\ell(uv^{-1}) \leq 1$. As $uv^{-1} \neq 1$ we get $\ell(uv^{-1}) = 1$ and thus $uv^{-1} \in S$.

We first prove that $uv^{-1} \in q(s)$. We have

$$c(s, v^{-1}) \geq \frac{1}{2}, \quad \text{as } s \sim v \text{ and } v \sim v^{-1},$$

$$c(v^{-1}, uv^{-1}) = \frac{1}{2}(\ell(v) + \ell(uv^{-1}) - \ell(u^{-1})) = \frac{1}{2}\ell(uv^{-1}) \geq \frac{1}{2}.$$

Therefore $s \sim v^{-1}$ and $v^{-1} \sim uv^{-1}$. Thus $s \sim uv^{-1}$, so $uv^{-1} \in q(s)$.

We prove now that $uv^{-1} \in S_1$. From $u \sim u^{-1}$ and $u \sim uv^{-1}$ we conclude $u^{-1} \sim uv^{-1}$. An another side

$$c(u^{-1}, (uv^{-1})^{-1}) = \frac{1}{2}(\ell(u) + \ell(uv^{-1}) - \ell(v^{-1})) \geq \frac{1}{2}.$$

Therefore $u^{-1} \sim uv^{-1}$ and $u^{-1} \sim (uv^{-1})^{-1}$. Thus $uv^{-1} \sim (uv^{-1})^{-1}$ and so $uv^{-1} \in S_1$.

Now since $H(s) \subseteq S \cup \{1\}$ and $S$ is finite, $H(s)$ is a finite subgroup. □

**Lemma 2.4** *Let $\leq$ be a well ordering of the set of equivalence classes of $S$ under $\sim$. Then there is a well ordering $\preceq$ of $S$ satisfying the following conditions:*

*(i) if $q(s) < q(t)$ then $s \prec t$,*

*(ii) if $q(s) = q(t)$ and $q(s^{-1}) < q(t^{-1})$ then $s \prec t$.*

**Proof**

For an equivalence class $q(s)$ we let $\preceq_{q(s)}$ be a well ordering of $q(s)$. We define $\preceq$ on $S$ by

$$s \prec t \text{ if and only if } \begin{cases} q(s) < q(t), & \text{or;} \\ q(s) = q(t),\ q(s^{-1}) < q(t^{-1}), & \text{or;} \\ q(s) = q(t),\ q(s^{-1}) = q(t^{-1}), s \prec_{q(s)} t & . \end{cases}$$

The verification that $\preceq$ is a well ordering is left to the reader. □

**Assemption.** We assume henceforth that the set of equivalence classes of $S$ is well ordered by a fixed ordering $\leq$ and $S$ is well ordered by a fixed ordering $\preceq$ satisfying the conditions $(i)$-$(ii)$ of the above lemma.

**Definition 2.5** For each $s \in S$ we define $K(s)$ to be the subgroup generated by the set $\{t \in S \mid t \prec s\}$. We let $U = \{s \in S \mid s \notin K(s)\}$.

**Lemma 2.6** *$U$ generates $H$.*

**Proof**

Let $L$ be the subgroup generated by $U$. Suppose that $L \neq H$ and let $s$ be the least element of $S$ which is not in $L$. Then $\{t \in S \mid t \prec s\} \subseteq L$ and thus $K(s) \subseteq L$. Hence $s \notin K(s)$. Therefore $s \in U$, a contradiction. □

**Notation.** For $s \in S$ we denote by $\bar{s}$ the earlier in the order $\preceq$ of $S$ of $s$ and $s^{-1}$.

We note that $u \in U^{\pm 1}$ if and only if $\bar{u} \in U$. Indeed, if $u \in U$ then $u \preceq u^{-1}$ (as if $u^{-1} \prec u$ then $u \in K(u)$ contradicting $u \in U$) and thus $\bar{u} = u \in U$; if $u \in U^{-1}$ then $u^{-1} \in U$, as before, $u^{-1} \preceq u$ and thus $\bar{u} = u^{-1} \in U$. Clearly if $\bar{u} \in U$ then $u \in U^{\pm 1}$.

We let

$$U_1 = \bigcup\{H(s) \mid s \in U \cap S_1\}, \quad U_2 = (U \cap S_2)^{\pm 1}.$$

**Lemma 2.7** *If $u, v$ are non-trivial elements of $U_1 \cup U_2$ and $\ell(uv) = 1$ then $q(v^{-1}) \leq q(v)$.*

**Proof**

We begin with the following claim.

**Claim.** *We may assume $v \in U_2$ and $\bar{u} \in U$.*

*Proof.* If $v \in U_1$ then $v \in H(s)$ for some $s \in U \cap S_1$. Since $H(s) = q(s) \cap S_1 \cup \{1\}$ and $v$ is non-trivial we get $v \in q(s) \cap S_1$. Thus $v \sim v^{-1}$, hence $q(v) = q(v^{-1})$. So we may assume $v \in U_2$.

We prove now that we may assume $\bar{u} \in U$. If $\bar{u} \notin U$ then $u \notin U_2$ and thus $u \in H(s) = q(s) \cap S_1 \cup \{1\}$ for some $s \in U \cap S_1$. We prove that $\ell(sv) = 1$.

Since $u \in H(s)$ and $u$ is non-trivial, $u \in q(s) \cap S_1$. Therefore we have

$$c(u, s) \geq \frac{1}{2}. \tag{1}$$

By $\ell(uv) = 1$, we get

$$c(u, v^{-1}) = \frac{1}{2}(\ell(u) + \ell(v) - \ell(uv)) \geq \frac{1}{2}. \tag{2}$$

By (1)-(2) we get $c(s, v^{-1}) \geq \frac{1}{2}$. After simplifications $\ell(sv) \leq 1$. Since $v \notin U_1$ we have $sv \neq 1$ and thus $\ell(sv) = 1$. Thus replacing $u$ by $s$ if necessary we assume that $\bar{u} \in U$.

This ends the proof of our claim. $\square$

Thus we suppose $v \in U_2$ and $\bar{u} \in U$. Suppose now towards a contradiction $q(v) < q(v^{-1})$. By $\ell(uv) = 1$ we have

$$c(u, v^{-1}) = \frac{1}{2},$$

$$c(uv, v) = \frac{1}{2}(\ell(uv) + \ell(v) - \ell(u)) = \frac{1}{2},$$

$$c((uv)^{-1}, u^{-1}) = \frac{1}{2}(\ell(uv) + \ell(u) - \ell(v)) = \frac{1}{2},$$

so $u \sim v^{-1}$, $uv \sim v$, $(uv)^{-1} \sim u^{-1}$.

Since $q(uv) = q(v) < q(v^{-1}) = q(u)$, by the conditions $(i)$-$(ii)$ of Lemma 2.4, we have

$$uv \prec u, \quad v \prec v^{-1}. \tag{3}$$

Since $v \prec v^{-1}$ and $v \in U_2$ we get

$$v \in U. \tag{4}$$

Since $q((uv)^{-1}) = q(u^{-1})$ and $q(uv) < q(u)$, by the conditions $(i)$-$(ii)$ of Lemma 2.4, we have

$$(uv)^{-1} \prec u^{-1}. \tag{5}$$

Thus, by (3)-(4)-(5), we conclude

$$uv \prec u, \quad v \prec v^{-1}, \quad v \in U, \quad (uv)^{-1} \prec u^{-1}.$$

Since $uv \prec u$ and $(uv)^{-1} \prec u^{-1}$ we get $\overline{uv} \prec \bar{u}$.

Now we treat the following cases:

- If $v = \bar{u}$ then either $u = v$ or $u = v^{-1}$. Therefore either $\ell(uv) = \ell(v^2) = 1$ and thus $v \in U_1$, or $\ell(uv) = \ell(v^{-1}v) = 0$. In both cases we have a contradiction as $v \in U_2$ and $\ell(uv) = 1$.
- If $v \prec \bar{u}$ then, since $\overline{uv} \prec \bar{u}$, we get $\bar{u} \in K(\bar{u})$. Contradiction with $\bar{u} \in U$.
- If $\bar{u} \prec v$ then, since $\overline{uv} \prec \bar{u} \prec v$, we get $v \in K(v)$. Contradiction with $v \in U$.

So our supposition is false. Thus $q(v^{-1}) \leq q(v)$. $\square$

**Lemma 2.8** *Let $s, t, u$ be non-trivial elements of $U_1 \cup U_2$ such that $\ell(st) = 1$, $\ell(tu) = 1$. Then $t \sim t^{-1}$ and thus $t \in U_1$.*

**Proof**

Since $\ell(st) = 1$, by Lemma 2.7 we have $q(t^{-1}) \leq q(t)$. Since $\ell(u^{-1}t^{-1}) = 1$, by the same lemma we have $q(t) \leq q(t^{-1})$. Therefore $q(t) = q(t^{-1})$. $\square$

**Lemma 2.9** *Let $s, u \in U_2$, $t \in U_1 \cup U_2 \setminus \{1\}$ such that $\ell(st) = 1$, $\ell(tu) = 1$. Then $\ell(stu) = 1$.*

**Proof**
**Claim 1.** $\ell(stu) \leq 1$.

*Proof.* We have

$$c(s, t^{-1}) = \frac{1}{2}(\ell(s) + \ell(t) - \ell(st)) = \frac{1}{2}, \tag{1}$$

$$c(t^{-1}, (tu)^{-1}) = \frac{1}{2}(\ell(t) + \ell(tu) - \ell(u)) = \frac{1}{2},$$

so

$$c(s, (tu)^{-1}) \geq \frac{1}{2}.$$

Thus $s \sim (tu)^{-1}$. A simple count shows $\ell(stu) \leq 1$ as claimed. □

**Claim 2.** $s \in U$, $u^{-1} \in U$, $t \prec s$.

*Proof.* We first prove

$$q(s) < q(s^{-1}), \quad q(u^{-1}) < q(u).$$

Since $\ell(t^{-1}s^{-1}) = 1$, by Lemma 2.7, we get $q(s) \leq q(s^{-1})$. If $q(s) = q(s^{-1})$ then we get $s \sim s^{-1}$ and thus $\ell(s^2) \leq 1$. Therefore $s \in S_1$ contradicting the fact that $s \in U_2 = (U \cap S_2)^{\pm 1} \subseteq S_2^{\pm 1}$. Therefore $q(s) < q(s^{-1})$.

Since $\ell(tu) = 1$, by Lemma 2.7, we have $q(u^{-1}) \leq q(u)$. If $q(u) = q(u^{-1})$ then we get $u \sim u^{-1}$ and thus $\ell(u^2) \leq 1$. Therefore $u \in S_1$ contradicting the fact that $u \in U_2 = (U \cap S_2)^{\pm 1} \subseteq S_2^{\pm 1}$. Therefore $q(u^{-1}) < q(u)$.

Since $q(s) < q(s^{-1})$, $q(u^{-1}) < q(u)$, by conditions $(i)$-$(ii)$ of Lemma 2.4 we get $s \prec s^{-1}$ and $u^{-1} \prec u$. Therefore $s \in U$ and $u^{-1} \in U$.

By Lemma 2.8 we have $q(t) = q(t^{-1})$. By (1) we have $q(s) = q(t^{-1})$. Now since

$$q(s) = q(t) = q(t^{-1}), \; q(t^{-1}) = q(s) < q(s^{-1}),$$

we get, by conditions $(i)$-$(ii)$ of Lemma 2.4, $t \prec s$.

This ends the proof of our claim. □

By Claim 1 we have $\ell(stu) = 0$ or $\ell(stu) = 1$. Suppose towards a contradiction that $\ell(stu) = 0$; thus $stu = 1$.

Since $t \neq 1$ we have $s \neq u^{-1}$.

• If $s \prec u^{-1}$, then, since $t \prec s$ and $u^{-1} = st$, we get $u^{-1} \in K(u^{-1})$. Contradiction with $u^{-1} \in U$.

• If $u^{-1} \prec s$, then, since $t \prec s$ and $s = u^{-1}t^{-1}$, we get $s \in K(s)$. Contradiction with $s \in U$.

Therefore our supposition is false and thus $\ell(stu) = 1$. □

**Lemma 2.10** *Let $u_1, u_2, u_3$ be non-trivial elements of $U_1 \cup U_2$ such that $\ell(u_1u_2) = 1$, $\ell(u_2u_3) = 1$, $u_1 \not\sim u_2$ and $u_2 \not\sim u_3$. Then $u_1, u_3 \in U_2$, $u_2 \in U_1$ and $\ell(u_1u_2u_3) = 1$.*

**Proof**

By Lemma 2.8, $u_2 \in U_1$.

Prove $u_1 \in U_2$ and $u_3 \in U_2$. Suppose towards a contradiction $u_1 \in U_1$. Then, since $\ell(u_1u_2) = 1$, a simple count shows $c(u_1, u_2^{-1}) \geq \frac{1}{2}$. Thus $u_1 \sim u_2^{-1}$. Since $u_2 \in U_1$ we have $u_2 \sim u_2^{-1}$. Therefore $u_1 \sim u_2$. Contradiction as $u_1 \not\sim u_2$. Thus $u_1 \in U_2$. By the same argument we get $u_3 \in U_2$.

By Lemma 2.9 we get $\ell(u_1u_2u_3) = 1$. □

**Definition 2.11** *A sequence* $(u_1, \ldots, u_n)$ *of* $U_1 \cup U_2$ *is said to be* ***pseudo-reduced*** *if it satisfies the following conditions:*

*(i)* $u_i \neq 1$, $u_iu_{i+1} \neq 1$,
*(ii)* $u_i, u_{i+1} \in U_1 \Rightarrow u_i \not\sim u_{i+1}$.

**Lemma 2.12** *If* $(u_1, \ldots, u_n)$, $n \geq 2$, *is a pseudo-reduced sequence of* $U_1 \cup U_2$ *then*

$$\ell(u_1 \cdots u_n) = \sum_{i=1}^{n} \ell(u_i) - 2\sum_{i=1}^{n-1} c(u_i, u_{i+1}^{-1}).$$

**Proof**

The proof is by induction on $n$. The lemma is trivial for $n = 2$.

**For** $n = 3$. We consider the following two cases.

**Case 1.** $\ell(u_1u_2) = 1$ and $\ell(u_2u_3) = 1$.

By Lemma 2.8 $u_2 \in U_1$. Since the sequence $(u_1, u_2, u_3)$ is pseudo-reduced, $u_1 \not\sim u_2$ and $u_2 \not\sim u_3$. Thus, by Lemma 2.9, we have $\ell(u_1u_2u_3) = 1$. Therefore

$$\ell(u_1u_2u_3) = 1 = \ell(u_1) + \ell(u_2) + \ell(u_3) - 2(c(u_1, u_2^{-1}) + c(u_2, u_3^{-1})),$$

and we find the desired conclusion.

**Case 2.** $\ell(u_1u_2) = 2$ or $\ell(u_2u_3) = 2$.

Then $c(u_1, u_2^{-1}) = 0$ or $c(u_2, u_3^{-1}) = 0$. Therefore

$$c(u_1, u_2^{-1}) + c(u_2, u_3^{-1}) \leq \frac{1}{2}.$$

Since $\frac{1}{2} < \ell(u_2) - \delta = 1$ (as $\delta = 0$), by Lemma 1.3, we get the desired conclusion.

**We go from** $n$ **to** $n + 1$. We treat the two following cases.

**Case 1.** $\ell(u_1u_2) = 1$ and $\ell(u_2u_3) = 1$. Put

$$a = u_1u_2, \quad b = u_3, \quad d = u_4 \cdots u_{n+1}.$$

We claim $c(b, d^{-1}) = c(u_3, u_4^{-1})$. We have

$$c(b, d^{-1}) = \frac{1}{2}(\ell(u_3) + \ell(u_4 \cdots u_{n+1}) - \ell(u_3 \cdots u_{n+1})). \tag{1}$$

By induction we have

$$\ell(u_3 \cdots u_{n+1}) = \sum_{i=3}^{n+1} \ell(u_i) - 2\sum_{i=3}^{n} c(u_i, u_{i+1}^{-1}),$$

$$\ell(u_4 \cdots u_{n+1}) = \sum_{i=4}^{n+1} \ell(u_i) - 2\sum_{i=4}^{n} c(u_i, u_{i+1}^{-1}).$$

By replacing in (1) we get

$$c(b, d^{-1}) = \frac{1}{2}(\ell(u_3) - \ell(u_3) + 2c(u_3, u_4^{-1})) = c(u_3, u_4^{-1}),$$

as claimed.

We claim $\ell(u_3u_4) = 2$ and $\ell(u_1u_2u_3) = 1$. Since the sequence $(u_1, u_2, u_3)$ is pseudo-reduced and $\ell(u_1u_2) = \ell(u_2u_3) = 1$, by Lemma 2.10, $\ell(u_1u_2u_3) = 1$ and $u_3 \in U_2$.

Suppose that $\ell(u_3u_4) = 1$. Since $\ell(u_2u_3) = 1$ we get, by Lemma 2.8, $u_3 \in U_1$. Contradiction with $u_3 \in U_2$. This ends the proof of our claim.

Since $\ell(u_3u_4) = 2$ we have $c(u_3, u_4^{-1}) = 0$ and thus $c(b, d^{-1}) = 0$. Therefore

$$c(a, b^{-1}) + c(b, d^{-1}) = \frac{1}{2}(\ell(u_1u_2) + \ell(u_3) - \ell(u_1u_2u_3)) = \frac{1}{2}.$$

Since $\frac{1}{2} < \ell(b) - \delta = 1$ (as $\delta = 0$), by Lemma 1.3, we get

$$\ell(abc) = \ell(a) + \ell(b) + \ell(c) - 2(c(a, b^{-1}) + c(b, d^{-1})). \tag{2}$$

By induction we have

$$\ell(c) = \sum_{i=4}^{n+1} \ell(u_i) - 2\sum_{i=4}^{n} c(u_i, u_{i+1}^{-1}).$$

By replacing in (2), and since $c(b, d^{-1}) = c(u_3, u_4^{-1})$, $\ell(u_1 \cdots u_{n+1})$ is equal to

$$\ell(u_1u_2) + \ell(u_3) + \sum_{i=4}^{n+1} \ell(u_i) - 2\sum_{i=4}^{n} c(u_i, u_{i+1}^{-1}) - 2(c(u_1u_2, u_3^{-1}) + c(u_3, u_4^{-1})).$$

But, since $\ell(u_1u_2u_3) = 1$, a simple count shows

$$\ell(u_1u_2) + \ell(u_3) - 2c(u_1u_2, u_3^{-1}) = \ell(u_1u_2u_3) = \sum_{i=1}^{3} \ell(u_i) - 2\sum_{i=1}^{2} c(u_i, u_{i+1}^{-1}).$$

Therefore we find

$$\ell(u_1 \cdots u_{n+1}) = \sum_{i=1}^{3} \ell(u_i) - 2\sum_{i=1}^{2} c(u_i, u_{i+1}^{-1}) + \sum_{i=4}^{n+1} \ell(u_i) - 2\sum_{i=4}^{n} c(u_i, u_{i+1}^{-1}) - 2c(u_3, u_4^{-1})$$

$$= \sum_{i=1}^{n+1} \ell(u_i) - 2\sum_{i=1}^{n} c(u_i, u_{i+1}^{-1}).$$

**Case 2.** $\ell(u_1u_2) = 2$ or $\ell(u_2u_3) = 2$.

Put

$$a = u_1, \quad b = u_2, \quad d = u_3 \cdots u_{n+1}.$$

We claim $c(b, d^{-1}) = c(u_2, u_3^{-1})$. We have

$$c(b, d^{-1}) = \frac{1}{2}(\ell(u_2) + \ell(u_3 \cdots u_{n+1}) - \ell(u_2 \cdots u_{n+1})).$$

By induction

$$\ell(u_3 \cdots u_{n+1}) = \sum_{i=3}^{n+1} \ell(u_i) - 2\sum_{i=3}^{n} c(u_i, u_{i+1}^{-1}),$$

$$\ell(u_2 \cdots u_{n+1}) = \sum_{i=2}^{n+1} \ell(u_i) - 2\sum_{i=2}^{n} c(u_i, u_{i+1}^{-1}),$$

after calculating, as in the precedent case, we get $c(b, d^{-1}) = c(u_2, u_3^{-1})$.

Since $\ell(u_1u_2) = 2$ or $\ell(u_2u_3) = 2$, then $c(u_1, u_2^{-1}) = 0$ or $c(u_2, u_3^{-1}) = 0$. Thus

$$c(u_1, u_2^{-1}) + c(u_2, u_3^{-1}) \leq \frac{1}{2}.$$

We have

$$c(a, b^{-1}) + c(b, d^{-1}) = c(u_1, u_2^{-1}) + c(u_2, u_3^{-1}) \leq \frac{1}{2}.$$

Therefore, as in the above case, by Lemma 1.3 we get

$$\ell(abc) = \ell(a) + \ell(b) + \ell(c) - 2(c(a, b^{-1}) + c(b, d^{-1})),$$

thus

$$\ell(u_1 \cdots u_{n+1}) = \ell(u_1) + \ell(u_2) + \sum_{i=3}^{n+1} \ell(u_i) - 2\sum_{i=3}^{n} c(u_i, u_{i+1}^{-1}) - 2(c(u_1, u_2^{-1}) + c(u_2, u_3^{-1})).$$

Thus we have the desired conclusion. □

# 3 Proof of Theorem 1.2

We have

$$U_1 = \bigcup\{H(s) \mid s \in U \cap S_1\}, \quad U_2 = (U \cap S_2)^{\pm 1},$$
$$H(s) = q(s) \cap S_1 \cup \{1\}.$$

By Lemma 2.3, $H(s)$ is a finite subgroup.

The set $U_1$ can be written in the following manner

$$U_1 = H(s_1) \cup \ldots \cup H(s_n),$$

where $s_1, \ldots, s_n$ are in $U \cap S_1$ and $s_i \not\sim s_j$ for $i \neq j$.

Let $F$ be the subgroup generated by $U_2$. We are going to prove that $H$ is the free product $H(s_1) * \cdots * H(s_n) * F$ and that $F$ is free with basis $U \cap S_2$.

As $U = (U \cap S_1) \cup (U \cap S_2) \subseteq U_1 \cup U_2$, $H$ is generated by $H(s_1) \cup \ldots \cup H(s_n) \cup U_2$.

Now to prove that $H = H(s_1) * \cdots * H(s_n) * F$ and that $U \cap S_2$ is a basis of $F$ we must show that if $(u_1, \ldots, u_n)$ is a sequence of $H(s_1) \cup \ldots \cup H(s_n) \cup U_2$ satisfying:

*(i)* $u_i \neq 1$, $u_i u_{i+1} \neq 1$,
*(ii)* if $u_i \in H(s_j)$ then $u_{i+1} \notin H(s_j)$,

then $u_1 \cdots u_n \neq 1$.

To this end we must prove that if $(u_1, \ldots, u_n)$ is a sequence of $U_1 \cup U_2$ satisfying the conditions:

*(i)* $u_i \neq 1$, $u_i u_{i+1} \neq 1$,
*(ii)* $u_i, u_{i+1} \in U_1 \Rightarrow u_i \not\sim u_{i+1}$,

then $u_1 \cdots u_n \neq 1$.

Thus we must prove that if $(u_1, \ldots, u_n)$ is a pseudo-reduced sequence of $U_1 \cup U_2$ then $u_1 \cdots u_n \neq 1$.

Now if $(u_1, \ldots, u_{n+1})$ is a pseudo-reduced sequence of $U_1 \cup U_2$ then, by Lemma 2.12, we have

$$\ell(u_1 \cdots u_{n+1}) = \sum_{i=1}^{n+1} \ell(u_i) - 2 \sum_{i=1}^{n} c(u_i, u_{i+1}^{-1})$$

$$= \sum_{i=1}^{n} \ell(u_i) - 2 \sum_{i=1}^{n-1} c(u_i, u_{i+1}^{-1}) + \ell(u_{n+1}) - 2c(u_n, u_{n+1})$$

$$= \ell(u_1 \cdots u_n) + \ell(u_n u_{n+1}) - \ell(u_n).$$

Since $\ell(u_n u_{n+1}) - \ell(u_n) \geq 0$ we get

$$\ell(u_1 \cdots u_{n+1}) \geq \ell(u_1 \cdots u_n).$$

Therefore by induction on $n$ we get $\ell(u_1 \cdots u_n) \geq \ell(u_1)$. Since $\ell(u_1) \neq 0$ then $\ell(u_1 \cdots u_n) \neq 0$ and thus $u_1 \cdots u_n \neq 1$. □

**Acknowledgements.** Many thanks to Tuna Altinel to have read an earlier version of this paper and to Gilbert Levitt for his remarks and improvements.

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