On hypersurfaces in spaces of constant curvature satisfying a certain condition on the curvature tensor

By

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1. Introduction

If a Riemannian manifold M is locally symmetric, then its curvature tensor R satifies

(*) $\qquad R(X, Y)\cdot R=0 \qquad$ for all tangent vectors X and Y,

where the endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M. Conversely, does this algebraic condition on the curvature tensor field R imply that M is locally symmetric?

We conjecture that the answer is affirmative in the case where M is irreducible and complete and dim $M\geqq 3$.

Recently, K. Nomizu [4], has given an affirmative answer in the case where M is a complete hypersurface in a Euclidean space.

In this paper, let $\tilde{M}$ be a $(n+1)$-dimensional connected pseudo-Riemannian manifold with constant curvature c, and the main purpose is to consider the hypersurfaces of $\tilde{M}$ satisfying the condition (*).

Now, we give a short summary of those parts of the theory of hypersurfaces which are necessary for what follows.

Let M be a real hypersurface immersed in $\tilde{M}$ and g be the induced pseudo-Riemannian metric from the pseudo-Riemannian metric $\tilde{g}$ of $\tilde{M}$. And let H be the second fundamental form with respect to this immersion and A be a field of endomorphism which corresponds to H, that is, $H(X, Y)=g(AX, Y)$, where X and Y are tangent vectors to M.

By definition, the directions of the lines of curvature of M are given by the vectors $\rho_m{}^i$ which satisfy

(1.1) $\qquad (H_{ij}-\lambda_m g_{ij})\rho_m{}^i=0, \qquad$ for $m,\ i,\ j=1, 2, \ldots, n.$

where H_{ij} and g_{ij} are the components of H and g, respectively, and λ_m is a principal normal curvature and is a root of the determinant equation

(1.2) $$|H_{ij}-\lambda g_{ij}|=0.$$

If the principal normal curvature are real and none of the lines of curvature are tangent to null vectors, we call M a proper hypersurface of $\tilde{M}$ and term the immersion a proper immersion respectively. In this paper, we assume that an immersion means always a proper immersion.

The equation of Gauss expresses the curvature tensor R of M by means of A

(1.3) $$R(X, Y)=\varepsilon\cdot AX\wedge AY+cX\wedge Y,$$

where, in general, $X\wedge Y$ denotes the endomorphism which maps Z upon $g(Z, Y)X-g(Z, X)Y$, and $\varepsilon=+1$, or -1.
The type number $k(x)$ at x is, by definition, the rank of A at x

2. Reduction of condition (*)

At a point $x\in M$, let $\{e_1,\ldots, e_n\}$ be an orthonormal basis of the tangent space $T_x(M)$ such that $Ae_i=\lambda_i e_i$, $1\leqq i\leqq n$. Then the equation of Gauss implies

$$R(e_i, e_j)=(\varepsilon\lambda_i\lambda_j+c)e_i\wedge e_j.$$

By computing

$$(R(e_i, e_j)\cdot R)(e_k, e_l)=[R(e_i, e_j), R(e_k, e_l)]$$
$$-R(R(e_i, e_j)e_k, e_l)-R(e_k, R(e_i, e_j)e_l),$$

we find that it is zero except possibly in the case where $k=i$ and $l\neq i$, $j(i\neq j)$. For this case we have

(2.1) $$(R(e_i, e_j)\cdot R)(e_i, e_l)=\delta(\varepsilon\lambda_i\lambda_j+c)\lambda_l(\lambda_j-\lambda_i)e_j\wedge e_l,$$

where $\delta=+1$, or -1.
Thus we see that condition (*) is equivalent to

(2.2) $$(\varepsilon\lambda_i\lambda_j+c)\lambda_l(\lambda_j-\lambda_i)=0 \qquad \text{for } l\neq i,\ j, \text{ where } i\neq j.$$

Suppose that the type number $k(x)$ is $\geqq 2$ at a point $x\in M$ and assume that $\lambda_1,\ldots, \lambda_k$ are non-zero principal normal curvatures, that is, non-zero eigenvalues of A at x and $\lambda_{k+1}=\ldots=\lambda_n=0$.
When $c\neq 0$, from (2.2), we can see that non-zero eigenvalues of A are classified into at most two classes, that is,

$$\lambda_{i_1}=\ldots\ldots\ldots=\lambda_{i_s}(=\lambda),$$
$$\lambda_{j_1}=\ldots\ldots\ldots=\lambda_{j_t}(=\mu), \qquad \text{where } s+t=n, \text{ and } \varepsilon\lambda\mu+c=0.$$

Then, by the observations we made above, we can define three linear subspaces, say, $T_0(M)$, $T_1(M)$, $T_2(M)$ of the tangent space $T_x(M)$ at $x \in M$ as follows:

$$T_0(x) = \{X \in T_x(M)\,;\ AX = 0\}$$

$$T_1(x) = \{X \in T_x(M)\,;\ AX = \lambda X\}$$

$$T_2(x) = \{X \in T_x(M)\,;\ AX = \mu X\}.$$

We have $T_x(M) = T_0(M) + T_1(M) + T_2(M)$ (direct sum). For any $Z \in T_x(M)$, Z_0, Z_1, and Z_2 will denote the components of Z in $T_0(x)$, $T_1(x)$ and $T_2(x)$, respectively. We shall only consider the following two cases in this paper.

I. $c \neq 0$, and $k(x) \geqq 3$ at a point $x \in M$, $s \geqq 1$, $t \geqq 1$.

II. $c = 0$, and $k(x) \geqq 3$ at a point $x \in M$.

Then we have the following for the case I.

LEMMA 2.1. *If $c \neq 0$ and the rank of A is $\geqq 3$ at $x_0 \in M$, then there is a neighborhood U of x_0 on which the dimension of $T_1(x)$ and the dimension of $T_2(x)$ are constant at each point $x \in U$, and the non-zero eigenvalues $\lambda(x)$ and $\mu(x)$ of A are differentiable functions. Where $\varepsilon\lambda(x)\mu(x) + c = 0$ and $\lambda(x) > \mu(x)$, at $x \in U$.*

And we have also the following for the case II.

LEMMA 2.2. *If $c = 0$ and $k(x_0) \geqq 3$ at $x_0 \in M$, then then there is a neighborhood U of x_0 on which $k(x) \geqq 3$ at each point $x \in U$, and the non-zero eigenvalue $k(x)$ is a differentiable function.*

3. Lemmas

In this paper, we shall assume that M is oriented (so that a unit normal field is defined on the whole M) and the type number $k(x)$ is $\geqq 3$ everywhere on M. By the observations we made in §2, for the case I, the functions $s(x)$ and $t(x)$ are locally constant and hence are constant functions, say, s and t, respectively, since M is connected, and moreover, for the case II, the function $k(x)$ is locally constant and hence is a constant function, say, k.

LEMMA 3.1. *For the case I, T_1 and T_2 are differentiable.*

PROOF. For any point $x_0 \in M$, let $\{X_1, \ldots, X_s\}$ be a basis of $T_1(x_0)$ and $\{X_{s+1}, \ldots X_n\}$ be a basis of $T_2(x)$. We extend X_i's to vector fields on M and define vector fields

$$Y_i = (A - \mu I)X_i \qquad \text{for } 1 \leqq i \leqq s$$

and

$$Y_j = (A - \lambda I)X_j \qquad \text{for } s+1 \leqq j \leqq n,$$

where I denotes the identity transformation. At x_0, we have $Y_i=(\lambda-\mu)X_i$ for $1\leqq i\leqq s$ and $Y_j=-(\lambda-\mu)X_j$ for $s+1\leqq j\leqq n$.

Thus $Y_1, \dots\dots, Y_n$ are linearly independent at x_0 and hence in a neighborhood U of x_0. At each point of U, we have

$$(A-\lambda I)Y_i=(A-\lambda I)(A-\mu I)X_i=0 \qquad \text{for } 1\leqq i\leqq s,$$

$$(A-\mu I)Y_j=(A-\mu I)(A-\lambda I)X_j=0 \qquad \text{for } s+1\leqq j\leqq n.$$

Hence $Y_1, \dots\dots, Y_s$ from a basis of T_1 and $Y_{s+1}, \dots\dots, Y_n$ from a basis of T_2.

And we have also the following

LEMMA 3.2. *For the case II, T_0 and T_1 are differentiable.*

LEMMA 3.3. *For the case I, T_1 and T_2 are involutive.*

PROOF. We recall the Codazzi equation: $(\nabla_X A)Y=(\nabla_Y A)X$, where ∇ denotes the Levi-Civita connection with respect to the pseudo-Riemannian metric g of M.

Suppose that X and Y are vector fields belonging to T_1. Then

$$(\nabla_X A)Y=\nabla_X(AY)-A(\nabla_X Y)=(X\lambda)Y+\lambda(\nabla_X Y)-A(\nabla_X Y),$$

and

$$(\nabla_Y A)X=(Y\lambda)X+\lambda(\nabla_Y X)-A(\nabla_Y X).$$

Thus we get

$$(X\lambda)Y-(Y\lambda)X+(\lambda I-A)[X, Y]=0.$$

Since $(X\lambda)Y-(Y\lambda)X\in T_1$ and $(\lambda I-A)[X, Y]=(\lambda-\mu)[X, Y]_2$, we get

$$(X\lambda)Y-(Y\lambda)X=0 \text{ and } [X, Y]_2=0.$$

The second identity shows that $[X, Y]\in T_1$, proving that T_1 is involutive. Similarly, T_2 is involutive.

And we have also the following

LEMMA 3.4. *For the case II, T_0 and T_1 are involutive.*

4. Some results

For the case I, let $M_1(x)$ and $M_2(x)$ be the maximal integral manifolds of T_1 and T_2, respectively.

Then we have the following

LEMMA 4.1. *For the case I, if $s\geqq 2$ and $t\geqq 2$, then $\lambda(x)$ and $\mu(x)$ are constant functions on M.*

PROOF. Since $s\geqq 2$, we may choose $X, Y\in T_1(x)$ such that X and Y are linearly independent. Extending X and Y to vector fields belonging to $T_1(x)$, we have $(X\lambda)Y-(X\lambda)X=0$ at x. Thus $X\lambda=Y\lambda=0$ at x. Therefore, the function $\lambda(x)$ is constant on each $M_1(x)$. Similarly, since $t\geqq 2$, the function $\mu(x)$ is also constant on each $M_2(x)$. However, $\varepsilon\lambda\mu+c=0$, thus it follows that $\lambda(x)$ is a constant function on M, say, λ. And moreover, $\mu(x)$ is also a constant function on M, say, μ.

From Lemma 4.1., we can easily prove the following

LEMMA 4.2. *If $s\geqq 2$ and $t\geqq 2$, then $M_1(x)$ and $M_2(x)$ are both totally geodesic and $T_1(x)$ and $T_2(x)$ are paprallel.*

Thus, by virtue of above two lemmas, we can see that $M_1(x)$ and $M_2(x)$ are spaces of constant curvatures c_1, c_2, respectively, where

$$c_1=\varepsilon\lambda^2+c$$

$$c_2=\varepsilon\mu^2+c.$$

Then, we see that c_1 and c_2 satisfy the following equation

$$1/c_1+1/c_2=1/c. \tag{4.1}$$

And moreover, we can show that for a point $x\in M$, there is a coordinate system $\{x^1,\ldots\ldots, x^s, x^{s+1},\ldots\ldots, x^n\}$ with origin x in a neighborhood U of x such that $\{\partial/\partial x^1, \ldots\ldots, \partial/\partial x^s\}$ and $\{\partial/\partial x^{s+1},\ldots\ldots, \partial/\partial x^n\}$ are local bases for T_1 and T_2, and with respect to this coordinate system the first fundamental form of M is given as follows

$$ds^2=\frac{\varepsilon_1(dx^1)^2+\ldots\ldots+\varepsilon_s(dx^s)^2}{\left[1+\frac{c_1}{4}(\varepsilon_1(x^1)^2+\ldots+\varepsilon_s(x^s)^2)\right]^2} + \frac{\varepsilon_{s+1}(dx^{s+1})^2+\ldots\ldots+\varepsilon_n(dx^n)^2}{\left[1+\frac{c_2}{4}(\varepsilon_{s+1}(x^{s+1})^2+\ldots+\varepsilon_n(x^n)^2)\right]^2}, \tag{4.2}$$

where each ε is $+1$, or -1. (See for example A. Fialkow [1].)
Then M may be immersed in $\tilde{M}$ by means of the algebraic equations

$$\begin{aligned} &e_1(z^1)^2+\ldots\ldots+e_{s+2}(z^{s+1})^2=1/c_1, \\ &e_{s+2}(z^{s+2})^2+\ldots\ldots+e_{n+2}(z^{n+2})^2=1/c_2, \end{aligned} \tag{4.3}$$

where the e's are $+1$, or -1 and the $\tilde{M}$ is defined by

$$e_1(z^1)^2+\ldots\ldots+e_{n+2}(z^{n+2})^2=1/c. \tag{4.4}$$

The z's are the point coordinates of Weierstrass. Since $n\geqq 4$ and $\lambda\mu\neq 0$, the M is indeformable in $\tilde{M}$.
The equations (4.3) show that M may also be considered as the intersection of two hypersylinders in a flat space of $n+2$ dimensions.

Thus we have the following

THEOREM 4.3. *Let $\tilde{M}$ be complete, connected and of constant curvature $c \neq 0$ and defined by* (4.4). *If M is connected complete proper hypersurface of $\tilde{M}$ of the case I and $s \geqq 2$, $t \geqq 2$, then M may be considered as the intersection of two hypersylinders in a flat space of $n+2$ dimensions.*

As a result, M is, of course, symmetric.

Next, for the case II, let $M_0(x)$ and $M_1(x)$ be the maximal integral manifolds of T_0 and T_1 respectively.

LEMMA 4.4. If X belongs to $T_1(x)$, then $X\lambda = 0$.

From this lemma, we see that the function λ is constant on each maximal integral manifold of T_1, that is, on each $M_1(x)$.

We now let $X \in T_1$, $Y \in T_0$ and compute the both sides of the Codazzi equation:

$$(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y) = -A(\nabla_X Y) = -\lambda(\nabla_X Y)_1,$$

$$\begin{aligned}(\nabla_Y A)X &= \nabla_Y(AX) - A(\nabla_Y X) = \nabla_Y(\lambda X) - A(\nabla_Y X)\\ &= Y\lambda \cdot X + \lambda(\nabla_Y X) - A(\nabla_Y X)\\ &= Y\lambda \cdot X + \lambda(\nabla_Y X)_0.\end{aligned}$$

Therefore we have

$$(\nabla_Y X)_0 = 0, \qquad \text{that is,} \quad \nabla_Y X \in T_1$$

and

$$(Y\lambda)X = -\lambda(\nabla_X Y)_1 = -A(\nabla_X Y).$$

We have hence

LEMMA 4.5. *If $X \in T_1$, $Y \in T_0$, then $A(\nabla_X Y) = -(Y\lambda)X$.*

LEMMA 4.6.

(i) *If $Y \in T_0$, then $\nabla_Y(T_1) \subset T_1$.*

(ii) *If $Y \in T_0$, then $\nabla_Y(T_0) \subset T_0$.*

(iii) *If $Y \in T_0$, $X \in T_1$ and $[X, Y] = 0$, then $\nabla_X Y \in T_1$.*

By virtue of Lemma 4.5. and Lemma 4.6., we have

$$\nabla_Y X = -\frac{Y\lambda}{\lambda} \cdot X. \qquad \text{for } X \in T_1,\ Y \in T_0,\ [X, Y] = 0. \tag{4.5}$$

By virtue of Lemma 3.4., for a point $x \in M$, there is a coordinate system $\{x^1, \ldots, x^n\}$ in a neighborhood U of x such that $\{\partial/\partial x^1, \ldots\ldots, \alpha/\partial x^k\}$ and $\{\partial/\partial x^{k+1}, \ldots\ldots, \partial/\partial x^n\}$ are local bases for T_1 and T_0, respectively.

By setting $g_{\alpha\beta}(x^1, \ldots\ldots, x^n) = g(\partial/\partial x^\alpha, \partial/\partial x^\beta)$ for $k+1 \leqq \alpha$, $\beta \leqq n$, we have

$$\partial g_{\alpha\beta}/\partial x^r = g(\nabla_{\partial/\partial x^r}(\partial/\partial x^\alpha),\ \partial/\partial x^\beta) + g(\partial/\partial x^\alpha,\ \nabla_{\partial/\partial x^r}(\partial/\partial x^\beta)).$$

But since Lemma 4.6 (iii), implies $\nabla_{\partial/\partial x^r}(\partial/\partial x^\alpha)\in T_1$ for $1\leqq r\leqq k$, we have

$$g(\nabla_{\partial/\partial x^r}(\partial/\partial x^\alpha),\ \partial/\partial x^\beta)=0$$

and, similarly, $g(\partial/\partial x^\alpha,\ \nabla_{\partial/\partial x^r}(\partial/\partial x))=0$. We have thus $\partial g_{\alpha\beta}/\partial x^r=0$. Moreover, by setting $g_{pq}(x^1, \ldots\ldots, x^n)=g(\partial/\partial x^p,\ \partial/\partial x^q)$ for $1\leqq p,\ q\leqq k$,
we have

$$\partial g_{pq}/\partial x^\alpha = g(\nabla_{\partial/\partial x^\alpha}(\partial/\partial x^p),\ \partial/\partial x^q) + g(\partial/\partial x^p,\ \nabla_{\partial/\partial x^\alpha}(\partial/\partial x_p)).$$

But (4.5) implies $\nabla_{\partial/\partial x^\alpha}(\partial/\partial x^p) = \nabla_{\partial/\partial x^p}(\partial/\partial x^\alpha) = -\dfrac{\partial\lambda}{\partial x^\alpha}\partial/\partial x^p$ for $k+1\leqq\alpha\leqq n$.
Hence

$$\partial g_{pq}/\partial x^\alpha = -(2\partial\lambda/\partial x^\alpha)g_{pq}. \tag{4.6}$$

Thus we have the following

THEOREM 4.7. *For the case II, there is a coordinate system in a neighborhood U of x such that with respect to this coordinate system the metric tensor of M is given as follows*

$$g_{ij}(x)=\begin{pmatrix} g_{pq}(x^1, \ldots\ldots, x^n) & 0 \\ 0 & g_{\alpha\beta}(x^{k+1}, \ldots\ldots, x^n) \end{pmatrix} \tag{4.7}$$

where $g_{pq}(x^1, \ldots\ldots, x^n)$ must satisfy (4.6), *and $g_{\alpha\beta}$ are constant on U.*

Lastly, let us give an example of hypersurface, of the case I, in the hyperbolic space of dimension $n+1$, say, H^{n+1}. Then H^{n+1} is isometric to the Riemannian half space

$$ds^2=(1/z^1)^2((dz^1)^1+\ldots\ldots+(dz^{n+1})^2),\ z\in R^{n+1},\ z^1>0.$$

Let M be a hypersurface of H^{n+1}, defined by the equations:

$$z^1=e^u, \qquad z^i=e^u w^i(x^1, \ldots\ldots, x^{n-1}), \qquad i=2, \ldots\ldots, n+1, \qquad u\in R, \qquad n\geqq 3$$

where e^u denotes the exponential function of u.
Then the Riemannian metric of M is expressed as follows

$$g_{pq}=\sum_{i=2}^{n+1}(\partial w^i/\partial x^p)(\partial w^i/\partial x^q), \qquad g_{**}=1+\sum_{i=2}^{n+1}(w^i)^2, \qquad p,\ q=1, \ldots\ldots, n-1,$$

where the first fundamental from of M is written in the form, $ds^2=g_{pq}dx^p dx^q + g_{**}dudu$.
Then, if M satisfies the condition (*), then, s or $t=1$ and from the similar arguments as before, we have

$$\sum_{i=2}^{n+1}(w^i)=r^2, \qquad \text{where } r \text{ is a constant.}$$

Therefore, M is homothetic to the direct product Riemannian space of $(n-1)$-dimensional sphere with radius $|r|$ and a real line R.

Remarks, (i) If $c \neq 0$ and $k(x) \geqq 3$ at some point $x \in M$, then we can show that the condition (*) is equivalent to the condition, $R(X, Y) \cdot S=0$, at x. Where S denotes the Ricci tensor of M.

(ii) The assumption, $k(x) \geqq 3$ everywhere on M, can be replaced by the assumption, $k(x) \geqq 3$ at some point $x \in M$.

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