ON HYPONORMAL TOEPLITZ OPERATORS WITH POLYNOMIAL AND CIRCULANT-TYPE SYMBOLS

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This paper characterises those hyponormal Toeplitz operators on the Hardy space of the unit circle among all Toeplitz operators that have polynomial symbols with circulant-type sets of coefficients.

A bounded linear operator A on a Hilbert space $\mathfrak{H}$ with inner product $(\cdot,\cdot)$ is said to be hyponormal if its selfcommutator $[A^*,A]=A^*A-AA^*$ induces a positive semidefinite quadratic form on $\mathfrak{H}$ via $\xi\mapsto([A^*,A]\xi,\xi)$, for $\xi\in\mathfrak{H}$. The purpose of this note is to study hyponormality for Toeplitz operators acting on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}=\partial\mathbb{D}$ in the complex plane. In particular, our interest is with Toeplitz operators with polynomial symbols, such that the coefficients of these trigonometric polynomials satisfy certain combinatorial constraints suggested by a recent description [2,6,7] of finite normal Toeplitz matrices. In [2,6,7], circulant matrices arise and lead to a natural definition for "circulant polynomial" and such polynomials are studied in connection with hyponormality. In Theorem 2 of this paper we determine precisely those circulant polynomials (and, more generally, those polynomials of circulant type) that induce hyponormal Toeplitz operators. Although it is not feasible to characterise by properties of coefficients all trigonometric polynomials that induce hyponormal Toeplitz operators, in Theorem 1 of this article some new necessary conditions will be presented. Although quite basic in form, these necessary conditions appear to have gone unobserved in previous literature.

Recall that given $\varphi\in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol φ is the operator T_φ on $H^2(\mathbb{T})$ defined by $T_\varphi f=P(\varphi\cdot f)$, where $f\in H^2(\mathbb{T})$ and P denotes the projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. The problem of determining which symbols induce hyponormal Toeplitz operators was solved by Cowen in [1], however here we shall employ an equivalent variant of Cowen's theorem that was first proposed by Nakazi and Takahashi in [10]. Suppose that $\varphi\in L^\infty(\mathbb{T})$ is arbitrary and consider the following subset of the closed unit ball of $H^\infty(\mathbb{T})$:

$$\mathcal{E}(\varphi)=\left\{k\in H^\infty(\mathbb{T}):||k||_\infty\le 1 \text{ and } \varphi-k\overline{\varphi}\in H^\infty(\mathbb{T})\right\}.$$

The criterion is that T_φ is hyponormal if and only if the set $\mathcal{E}(\varphi)$ is nonempty [1,10]. Cowen's method, then, is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution with specified properties to a certain functional equation involving the operator's symbol φ. This approach has been put to use in the works [3,10,11] to

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study Toeplitz operators on the Hardy space of the unit circle. An abstract version of Cowen's method has been developed in [5].

If φ is a trigonometric polynomial, say $\varphi(e^{i\theta}) = \sum_{-m}^{N} a_n e^{in\theta}$, where a_{-m} and a_N are nonzero, then the nonnegative integers N and m denote the analytic and co-analytic degrees of φ. For arbitrary trigonometric polynomials, Zhu [11] has applied Cowen's criterion and adopted a method based on the classical interpolation theorems of Schur to obtain an abstract characterisation of those trigonometric polynomial symbols that correspond to hyponormal Toeplitz operators. Furthermore, he was able to use this characterisation to give explicit necessary and sufficient conditions for hyponormality in terms of the coefficients of the polynomial φ whenever $N \leq 3$. However, with polynomials of higher analytic degree, the analogous explicit necessary and sufficient conditions (via properties of coefficients) are not known and in fact would be too complicated to be of much value. Nevertheless, certain general features of hyponormal Toeplitz operators with polynomial symbols are known, some of which are listed below in Theorem 1. The lower bound on the rank of the selfcommutator in (ii) and statement (2) appear to be new.

Theorem 1. *Suppose that φ is a trigonometric polynomial of co-analytic and analytic degrees m and N.*

1. *If T_φ is a hyponormal operator, then*
   (i) $m \leq N$ and $|a_{-m}| \leq |a_N|$,
   (ii) $N - m \leq$ rank $[T_\varphi^*, T_\varphi] \leq N$, and
   (iii) T_φ is subnormal and nonnormal if and only if $m = 0$.
2. *The hyponormality of T_φ is independent of the particular values of the Fourier coefficients $a_0, a_1, \ldots, a_{N-m}$ of φ.*

Proof. For statement (1), assume that φ is as above and that T_φ is hyponormal. The proof of (i) is given in [10] and [11], while the proof of (iii) and the upperbound N in (ii) is known from [8]. To prove the lower bound in (ii), we use Theorem 10 of [10], which states that there exists a finite Blaschke product $b \in \mathcal{E}(\varphi)$ such that the degree of b – meaning the number of zeros of b (in the open unit disc $\mathbb{D}$) [4, page 6] – equals the rank of $[T_\varphi^*, T_\varphi]$. The function b is of the form

$$b(z) = \alpha z^l \prod_{i=1}^{n} \left(\frac{z - \alpha_i}{1 - \overline{\alpha}_i z} \right) ,$$

where $|\alpha| = 1$, $0 < |\alpha_i| < 1$ for $i = 1, \ldots, n$, and $\deg(b) = l + n$. ¿From $b \in \mathcal{E}(\varphi)$ we have that $g = \varphi - b\overline{\varphi} \in H^\infty$. Hence

$$z^m b\overline{\varphi} = z^m \varphi - z^m g \in H^2 ,$$

because $z^m \varphi \in H^2$. Cross multiplication by the denominator of b leads to the polynomial

$$\alpha z^l \prod_{i=1}^{n} (z - \alpha_i) z^m \overline{\varphi} \in H^2 .$$

But the only way this polynomial can be analytic is if $l + n + m - N \geq 0$. Hence $\deg(b) = l + n \geq N - m$.

For statement (2), suppose that $\varphi(e^{i\theta}) = \sum_{n=-m}^{N} a_n e^{in\theta}$, where $m \leq N$ and $a_N \neq 0$. By Cowen's theorem there is a function k in the closed unit ball of $H^\infty(\mathbb{T})$ such that $\varphi - k\overline{\varphi} \in H^\infty$. Because k satisfies $\varphi - k\overline{\varphi} \in H^\infty$, then k necessarily has the property that

$$k \sum_{n=1}^{N} \overline{a}_n e^{-in\theta} - \sum_{n=1}^{m} a_{-n} e^{-in\theta} \in H^\infty . \tag{1.1}$$

¿From (1.1) one computes the Fourier coefficients $\hat{k}(0), \ldots, \hat{k}(N-1)$ of k to be $\hat{k}(n) = c_n$, for $n = 0, 1, \ldots, N-1$, where $c_0, c_1, \ldots, c_{N-1}$ are determined uniquely from the coefficients of φ by the recurrence relation

$$\begin{aligned} &c_0 = c_1 = \cdots = c_{N-m-1} = 0 \\ &c_{N-m} = \frac{a_{-m}}{\overline{a_N}} \\ (1.2) \qquad &c_n = (\overline{a_N})^{-1}\Big(a_{-N+n} - \sum_{j=N-m}^{n-1} c_j \overline{a_{N-n+j}}\Big)\ , \text{ for } n = N-m+1, \cdots, N-1\ . \end{aligned}$$

Therefore if $k_1, k_2 \in \mathcal{E}(\varphi)$, then $c_n = \hat{k}_1(n) = \hat{k}_2(n)$ for all $n = 0, 1, \ldots, N-1$, and $k_p(z) = \sum_{j=0}^{N-1} c_j z^j$ is the unique (analytic) polynomial of degree less than N satisfying $\varphi - k\overline{\varphi} \in H^\infty$. But since the coefficients in (1.2) are independent of the values of the coefficients $a_0, \cdots, a_{N-m}$ of φ, it follows that if $k \in \mathcal{E}(\varphi)$, then k is independent of $a_0, \cdots, a_{N-m}$. This completes the proof of (2). $\square$

To motivate our interest in the circulant-type symbols that follow, let us recall that the characterisation of finite normal Toeplitz matrices in [2,6,7] indicates that every finite normal Toeplitz matrix whose eigenvalues are not collinear must be a generalised circulant, which is a normal matrix of the form

$$\begin{pmatrix} a_0 & e^{i\omega}a_N & \ldots & \ldots & e^{i\omega}a_1 \\ a_1 & a_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_0 & e^{i\omega}a_N \\ a_N & \ldots & \ldots & a_1 & a_0 \end{pmatrix}.$$

Of course the matrix above is simply a matrix representation of the compression of the Toeplitz operator T_φ to the subspace of $H^2(\mathbb{T})$ spanned by the functions $1, z, z^2, \ldots, z^{N-1}$, where φ is the polynomial $\varphi(e^{i\theta}) = \sum_{-N}^{N} b_n e^{in\theta}$ with

$$\begin{pmatrix} b_0 & b_{-1} & \ldots & \ldots & b_{-N} \\ b_1 & b_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_0 & b_{-1} \\ b_N & \ldots & \ldots & b_1 & b_0 \end{pmatrix} = \begin{pmatrix} a_0 & e^{i\omega}a_N & \ldots & \ldots & e^{i\omega}a_1 \\ a_1 & a_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_0 & e^{i\omega}a_N \\ a_N & \ldots & \ldots & a_1 & a_0 \end{pmatrix}.$$

We say, therefore, that φ is a circulant polynomial.

Definition. A trigonometric polynomial $\varphi(e^{i\theta}) = \sum_{-m}^{N} a_n e^{in\theta}$ of analytic degree N and co-analytic degree m is said to be a *circulant polynomial with argument* ω if (i) $m \le N$ and (ii) there exists $\omega \in [0, 2\pi)$ such that $a_{-k} = e^{i\omega} a_{N-k+1}$ for every $1 \le k \le m$. In other words, φ is a circulant polynomial if and only if $m \le N$ and the compression of the Toeplitz operator T_φ to the subspace of $H^2(\mathbb{T})$ spanned by the functions $1, z, z^2, \ldots, z^{N-1}$ is, with respect to the standard basis, a generalised circulant matrix.

The polynomial $\varphi(e^{i\theta}) = e^{-i2\theta} - 3e^{-i\theta} + e^{i7\theta} - 3e^{i8\theta}$ is a circulant polynomial, however we know from Theorem 1 that T_φ is hyponormal if and only if T_ψ is hyponormal, where $\psi(e^{i\theta}) = e^{-i2\theta} - 3e^{-i\theta} + 2e^{i4\theta} - e^{i6\theta} + e^{i7\theta} - 3e^{i8\theta}$. So although ψ is not a circulant polynomial *per se*, it is convenient to view ψ as of “circulant type.” More general symbols of circulant type are described in the hypothesis of Theorem 2, which is the main result of this paper.

Theorem 2. *Let $a_0, a_1, \ldots, a_n$ be fixed complex numbers and let $\alpha \in \mathbb{C}$ be such that either $\alpha = 0$ or $|\alpha| = 1$. Suppose that $\varphi(e^{i\theta}) = \sum_{-m}^{N} b_n e^{in\theta}$ is a trigonometric polynomial, with $m \leq N$, whose coefficients b_n satisfy the following combinatorial constraints:*

$$b_n = \begin{cases} a_n & (0 \leq n \leq N-m) \\ a_n + \alpha a_{n+1} & (N-m+1 \leq n \leq N-1) \\ a_N & (n = N) \end{cases} \tag{2.1}$$

and

$$b_{-n} = \begin{cases} e^{i\omega}(a_{N-n+1} + \bar{\alpha} a_{N-n}) & (1 \leq n \leq m-1) \\ e^{i\omega} a_{N-m+1} & (n = m)\,. \end{cases} \tag{2.2}$$

If f denotes the analytic polynomial $f(z) = a_{N-m+1} + a_{N-m+2}z + \cdots + a_N z^{m-1}$, then the following statements are equivalent.

1. *T_φ is a hyponormal operator.*
2. *For every root ζ of f such that $|\zeta| > 1$, the number $1/\overline{\zeta}$ is a root of f in $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ.*

In the cases where T_φ is a hyponormal operator, we have that

$$\mathcal{E}(\varphi) = \left\{ \frac{e^{i\omega} a_N}{\overline{a_N}} z^{N-m} \prod_{j=1}^{m-1} \left(\frac{z - \zeta_j}{1 - \overline{\zeta_j} z} \right) \right\},$$

where $\zeta_1, \cdots, \zeta_{m-1}$ denote the roots (repeated according to multiplicity) of the analytic polynomial f. Moreover the rank of the selfcommutator of T_φ is computed from the formula

$$\operatorname{rank}\,[T_\varphi^*, T_\varphi] = N - m + Z_{\mathbb{D}} - Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}\,,$$

where $Z_{\mathbb{D}}$ and $Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ are the number of zeros of f in $\mathbb{D}$ and in $\mathbb{C}\setminus\overline{\mathbb{D}}$ counting multiplicity.

Remark. If $\alpha = 0$ in (2.1) and (2.2) and if $a_1 = \cdots = a_{N-m} = 0$ when $m < N$, then φ is a circulant polynomial with argument ω.

Proof. In view of Theorem 1, we will assume without loss of generality that

$$a_0 = \begin{cases} \bar{\alpha} e^{i\omega} a_N & (m < N) \\ \alpha a_1 + \bar{\alpha} e^{i\omega} a_N & (m = N), \end{cases} \qquad a_1 = \cdots = a_{N-m} = 0.$$

The connection between the given trigonometric polynomial φ and the analytic polynomial f is explained by a factorisation of $z^m \varphi$:

$$z^m \varphi(z) = f(z) \Big(z^N (z + \alpha) + e^{i\omega}(1 + \bar{\alpha} z) \Big)\,.$$

So, φ vanishes on at least N points on $\mathbb{T}$, namely on each of the $(N+1)$-roots of $-e^{i\omega}$ if $\alpha = 0$ or on each of the N-roots of $-\bar{\alpha} e^{i\omega}$ if $|\alpha| = 1$. Consider the function $|\varphi|$ as a continuous function $\theta \mapsto |\varphi(e^{i\theta})|$ on the real interval $[-\pi, \pi]$. Then

$$\log|\varphi(e^{i\theta})| = \log|f(e^{i\theta})| + \log|e^{iN\theta}(e^{i\theta} + \alpha) + e^{i\omega}(1 + \bar{\alpha} e^{i\theta})|$$

fails to be Lebesgue integrable on $[-\pi, \pi]$ (because $\log |e^{iN\theta}(e^{i\theta}+\alpha)+e^{i\omega}(1+\bar{\alpha}e^{i\theta})|$ grows exponentially near the $(N+1)$-roots of $-e^{i\omega}$ if $\alpha = 0$ or near the N-roots of $-\bar{\alpha}e^{i\omega}$ if $|\alpha| = 1$). We conclude, then, that $\log |\varphi|$ is not integrable for every polynomial φ with the coefficients obeying (2.1); therefore, a Toeplitz operator T_φ is hyponormal if and only if $\varphi = q\overline{\varphi}$ for some inner function $q \in H^\infty(\mathbb{T})$ ([10, Theorem 4(1)]). Because φ is a trigonometric polynomial, the inner function q that arises must be rational. Therefore T_φ is hyponormal if and only if $q = \frac{\varphi}{\overline{\varphi}}$ is analytic on the open unit disc $\mathbb{D}$. Observe that

$$\begin{aligned} q = \frac{\varphi}{\overline{\varphi}} &= \frac{\frac{1}{z^m} f(z)\left(z^{N+1}+\alpha z^N + e^{i\omega} + e^{i\omega}\bar{\alpha}z\right)}{z^m \overline{f(z)}\left(\frac{1}{z^{N+1}}+\frac{\bar{\alpha}}{z^N}+e^{-i\omega}+\frac{e^{-i\omega}\alpha}{z}\right)} \\ &= \frac{f(z)\left(z^{N+1}+\alpha z^N + e^{i\omega} + e^{i\omega}\bar{\alpha}z\right)}{\overline{f(z)}\left(z^{2m-N-1}+\bar{\alpha}z^{2m-N}+e^{-i\omega}z^{2m}+e^{-i\omega}\alpha z^{2m-1}\right)} \\ &= \frac{f(z)\left(z^{N+1}+\alpha z^N + e^{i\omega} + e^{i\omega}\bar{\alpha}z\right)}{e^{-i\omega}z^{2m-N-1}\overline{f(z)}\left(e^{i\omega}+e^{i\omega}\bar{\alpha}z+z^{N+1}+\alpha z^N\right)} \\ &= e^{i\omega}z^{N-m}\frac{f(z)}{z^{m-1}\overline{f(z)}}. \end{aligned}$$

On the other hand, since

$$z^{m-1}\overline{f(z)} = \overline{a_N} + \overline{a_{N-1}}z + \cdots + \overline{a_{N-m+1}}z^{m-1},$$

we have that

$$\zeta \text{ is a zero of } f \iff (\overline{\zeta})^{-1} \text{ is a zero of } z^{m-1}\overline{f}.$$

Thus we can write

$$q(z) = \frac{e^{i\omega}a_N}{\overline{a_N}} z^{N-m} \prod_{j=1}^{m-1}\left(\frac{z-\zeta_j}{1-\overline{\zeta_j}z}\right), \tag{2.3}$$

where $\zeta_1, \cdots, \zeta_{m-1}$ denote the roots (repeated according to multiplicity) of the analytic polynomial f. Therefore T_φ is hyponormal if and only if q is analytic on $\mathbb{D}$ if and only if the zeros of the analytic polynomial f have the property that for every root ζ of f with $|\zeta| > 1$, the complex number $1/\overline{\zeta}$ is a root of f in $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ. This is precisely the criterion that was sought.

Suppose now that T_φ is a hyponormal operator, where φ is described by the hypothesis of the theorem, but where we no longer assume (as we did in this first part of the proof) that a_0 has a special value and that $a_1 = \cdots = a_{N-m} = 0$ in the case that $m < N$. Then, as an argument of Nakazi and Takahashi ([10, Theorem 10]) shows, for every $k \in \mathcal{E}(\varphi)$,

$$\operatorname{rank}\,[T_\varphi^*, T_\varphi] \;\le\; \text{degree of } k. \tag{2.4}$$

The selfcommutator of T_φ is independent of the constant term, and so without loss of generality we assume once again that

$$a_0 = \begin{cases} \bar{\alpha}e^{i\omega}a_N & (m < N) \\ \alpha a_1 + \bar{\alpha}e^{i\omega}a_N & (m = N)\,. \end{cases}$$

Next we shall perturb φ to a trigonometric polynomial φ_0 such that in the case where $m < N$ the Fourier coefficients of φ_0 vanish at $1, 2, \ldots, N-m$. To achieve this, let

$$\varphi_0 = \varphi - g\,, \qquad \text{where} \quad g(e^{i\theta}) = \sum_{n=0}^{N-m} a_n e^{in\theta}\,.$$

We are now in a position to apply the results obtained in the first part of the proof to the polynomial φ_0. By that which has preceded above, there is a finite Blaschke product q that satisfies $\varphi_0 - q\overline{\varphi}_0 = 0$ and so

$$\varphi - q\overline{\varphi} = \varphi_0 - q\overline{\varphi}_0 + g - q\overline{g} = 0 + q - q\overline{g}\,.$$

Because $z^{N-m}\overline{g(z)}$ is analytic, so is $g(z) - q(z)\overline{g(z)}$. Thus, $\varphi - q\overline{\varphi} = g - q\overline{g} \in H^\infty$ and hence $q \in \mathcal{E}(\varphi)$.

Formula (2.3) shows that the degree of q is $N - m + Z_{\mathbb{D}} - Z_{\mathbb{C}\backslash\overline{\mathbb{D}}}$, where $Z_{\mathbb{D}}$ and $Z_{\mathbb{C}\backslash\overline{\mathbb{D}}}$ are the number of zeros of f in $\mathbb{D}$ and in $\mathbb{C}\backslash\overline{\mathbb{D}}$ counting multiplicity. This together with (2.4) gives that

$$\text{rank}\,[T_\varphi^*, T_\varphi] \;\le\; N - m + Z_{\mathbb{D}} - Z_{\mathbb{C}\backslash\overline{\mathbb{D}}}. \tag{2.5}$$

Now we will show that $\mathcal{E}(\varphi)$ has exactly one element k, namely q. To the contrary, assume that $\mathcal{E}(\varphi)$ contains at least two elements. Then by another argument of Nakazi and Takahashi ([10, Proposition 11]), $\ker[T_\varphi^*, T_\varphi] = \ker H_{\overline{\varphi}}$, where $H_{\overline{\varphi}}$ is a Hankel operator with symbol $\overline{\varphi}$. Thus

$$\text{rank}\,[T_\varphi^*, T_\varphi] = \dim\,(H^2 \ominus \ker H_{\overline{\varphi}}).$$

But since $H_{\overline{\varphi}}$ has the following matrix representation

$$H_{\overline{\varphi}} = \begin{pmatrix} \overline{b_1} & \overline{b_2} & \overline{b_3} & \dots & \overline{b_N} & 0 & \dots \\ \overline{b_2} & \overline{b_3} & & \cdot & 0 & & \\ \overline{b_3} & & \cdot & 0 & & & \\ \vdots & \cdot & 0 & & & & \\ \overline{b_N} & 0 & & & & & \\ 0 & & & & & & \\ \vdots & & & & & & \end{pmatrix} \;:\; l^2 \;\longrightarrow\; l^2$$

and $\overline{b_N} = \overline{a_N} \neq 0$, it follows that

$$\ker H_{\overline{\varphi}} = \{(\xi_1, \xi_2, \xi_3, \cdots) \in l^2 \;:\; \xi_1 = \cdots = \xi_N = 0\}.$$

Therefore we have that $\text{rank}\,[T_\varphi^*, T_\varphi] = N$, which leads a contradiction because from (2.5) we must have that $\text{rank}\,[T_\varphi^*, T_\varphi] \le N - 1$. Therefore we can conclude that

$$\mathcal{E}(\varphi) = \left\{ \frac{e^{i\omega} a_N}{\overline{a_N}} z^{N-m} \prod_{j=1}^{m-1} \left(\frac{z - \zeta_j}{1 - \overline{\zeta_j} z} \right) \right\}.$$

Furthermore, by [10, Theorem 10], we have that $\text{rank}\,[T_\varphi^*, T_\varphi] = N - m + Z_{\mathbb{D}} - Z_{\mathbb{C}\backslash\overline{\mathbb{D}}}$. This completes the proof. $\square$

Corollary 3. *If $a_0, \dots, a_N$ are fixed and if $\varphi(e^{i\theta}) = \sum_{-N}^{N} b_n e^{in\theta}$ is a trigonometric polynomial for which the Toeplitz matrix $B = (b_{i-j})_{0 \le i,j \le N}$ is equal to*

$$\begin{pmatrix} a_0 & a_{N-1} + e^{i\omega} a_N & a_{N-2} + e^{i\omega} a_{N-1} & \dots & a_1 + e^{i\omega} a_2 & e^{i\omega} a_1 \\ a_1 + e^{i\omega} a_2 & a_0 & \ddots & & & \vdots \\ a_2 + e^{i\omega} a_3 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ a_{N-1} + e^{i\omega} a_N & & & & a_0 & a_{N-1} + e^{i\omega} a_N \\ a_N & \dots & \dots & \dots & a_1 + e^{i\omega} a_2 & a_0 \end{pmatrix},$$

and if f denotes the analytic polynomial $f(z) = a_1 + a_2 z + \cdots + a_N z^{N-1}$, then T_φ is a hyponormal operator if and only if for every root ζ of f such that $|\zeta| > 1$, the number $1/\overline{\zeta}$ is a root of f in $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ.

Proof. Take $m = N$ and $\alpha = e^{i\omega}$ in Theorem 2. $\square$

Corollary 4. *If $a_0, \ldots, a_N$ are fixed and if $\varphi(e^{i\theta}) = \sum_{-N}^{N} b_n e^{in\theta}$ is a trigonometric polynomial for which the Toeplitz matrix $B = (b_{i-j})_{0 \le i,j \le N}$ satisfies*

$$B = \begin{pmatrix} a_0 & a_{N-1} & \ldots & a_1 & -(a_1 + \cdots + a_N) \\ a_1 & a_0 & \ddots & & a_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_0 & a_{N-1} \\ a_N & \ldots & \ldots & a_1 & a_0 \end{pmatrix},$$

and if f denotes the analytic polynomial

$$f(z) = (a_1 + \cdots + a_N) + (a_2 + \cdots + a_N)z + \cdots + (a_{N-1} + a_N)z^{N-2} + a_N z^{N-1},$$

then T_φ is a hyponormal operator if and only if for every root ζ of f such that $|\zeta| > 1$, the number $1/\overline{\zeta}$ is a root of f in $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ.

Proof. This is an immediate result from Corollary 3. $\square$

The criteria in Theorem 2, Corollary 3, and Corollary 4 are readily applicable, as illustrated by the following simple examples.

Example 1. Consider the polynomial $\varphi(e^{i\theta}) = \alpha e^{-i2\theta} + \beta e^{-i\theta} + \alpha e^{i\theta} + \beta e^{i2\theta}$. This is a circulant polynomial and Theorem 2 shows that T_φ is hyponormal if and only if $|\alpha| \le |\beta|$. If $\beta \ne 0$, then the solution q to the functional equation $\varphi - q\overline{\varphi} = 0$ is $q(z) = (z + \frac{\alpha}{\beta})(1 + \overline{(\frac{\alpha}{\beta})}z)^{-1}$.

Example 2. Consider the polynomial $\varphi = \alpha e^{-i2\theta} + (\alpha+\beta)e^{-i\theta} + (\alpha+\beta)e^{i\theta} + \beta e^{i2\theta}$. Corollary 3 shows that T_φ is hyponormal if and only if $|\alpha| \le |\beta|$.

Example 3. With the circulant polynomial

$$\varphi(e^{i\theta}) = -2e^{-i4\theta} + 9e^{-i3\theta} - 12e^{-i2\theta} + 4e^{-i\theta} - 2e^{i2\theta} + 9e^{i3\theta} - 12e^{i4\theta} + 4e^{i5\theta},$$

the analytic polynomial f is $f(z) = 4(z - \frac{1}{2})^2(z-2)$, and so by Theorem 2, T_φ is hyponormal. The rank of the selfcommutator is $N - m + Z_{\mathbb{D}} - Z_{\mathbb{C}\setminus\overline{\mathbb{D}}} = 5 - 4 + 2 - 1 = 2$. The solution q to the functional equation $\varphi - q\overline{\varphi} = 0$ is in this case $q(z) = z(z - \frac{1}{2})(1 - \frac{1}{2}z)^{-1}$.

Example 4. Consider the following polynomial of circulant type:

$$\varphi(e^{i\theta}) = 2e^{-i4\theta} - 11e^{-i3\theta} + 21e^{-i2\theta} - 16e^{-i\theta} - 11e^{i\theta} + 21e^{i2\theta} - 16e^{i3\theta} + 4e^{i4\ theta}.$$

Then the analytic polynomial f in Corollary 4 is

$$f(z) = 4z^3 - 12z^2 + 9z - 2 = 4(z - \frac{1}{2})^2(z-2)$$

and hence T_φ is hyponormal with rank-1 selfcommutator.

The normality of Toeplitz matrices involves circularity and symmetry, but it is only circularity that we have considered herein. For a study of relations between the hyponormality of Toeplitz operators with polynomial symbols and various symmetry-type properties of the coefficients, see [3,9].

References

1. C. Cowen, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103** (1988), 809–812.
2. D.R. Farenick, M. Krupnik, N. Krupnik, and W.Y. Lee, *Normal Toeplitz matrices*, SIAM J. Matrix Anal. Appl. **17** (1996), 1037-1043.
3. D.R. Farenick and W.Y. Lee, *Hyponormality and spectra of Toeplitz operators*, Trans. Amer. Math. Soc. **348** (1996), 4153-4174.
4. J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.
5. C. Gu, *A generalization of Cowen's characterization of hyponormal Toeplitz operators*, J. Funct. Anal. **124** (1994), 135–148.
6. Kh.D. Ikramov and V.N. Chugunov, *Normality conditions for a complex Toeplitz matrix*, Zh. Vychisl. Mat. i Mat. Fiz. **36** (1996), 3-10.
7. T. Ito, *Every normal Toeplitz matrix is either of type (I) or type (II)*, SIAM J. Matrix Anal. Appl. **17** (1996), 998-1006.
8. T. Ito and T.K. Wong, *Subnormality and quasinormality of Toeplitz operators*, Proc. Amer. Math. Soc. **34** (1972), 157–164.
9. I.H. Kim and W.Y. Lee, *On hyponormal Toeplitz operators with polynomial and symmetric-type symbols*, preprint (1997).
10. T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338** (1993), 753–769.
11. K. Zhu, *Hyponormal Toeplitz operators with polynomial symbols*, Integral Equations Operator Theory **21** (1995), 376–381.

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