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On $\mathcal{I}$-Baire spaces

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Abstract. In this paper, the concept of $\mathcal{I}$-Baire spaces is introduced, and characterizations and properties of these spaces are given. It is shown that (X, τ) is Baire if and only if $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire for any ideal $\mathcal{I}$ on X.

1. Introduction

It is well known that a Baire space is defined as a space in which every countable intersection of dense open subsets is dense, or equivalently a space with the property that every nonempty open subspace is nonmeager. Baire spaces have various applications in complete metric spaces. To develop applications of Baire spaces, some researchers have studied some spaces such as hyperspaces, Volterra spaces (see [10, 19]). Recently, Chakrabarti and Dasgupta [2] have investigated Baire spaces with minimal structure.

Ideals on topological spaces were studied by Kuratowski [17] and Vaidyanathaswamy [22]. Their applications have been investigated intensively (see [3, 6, 7, 13, 16, 18, 20]).

The aim of this paper is to introduce and study $\mathcal{I}$-Baire spaces. Some characterizations and properties of $\mathcal{I}$-Baire spaces, including their subspaces, are investigated. Finally, some mapping theorems and a topological sum theorem on $\mathcal{I}$-Baire spaces are discussed.

2. Preliminaries

Let X be a nonempty set, let 2^X be a family of all subsets of X and let $\mathcal{I} \subset 2^X$. $\mathcal{I}$ is called an ideal (resp. a σ-ideal) on X, if it satisfies the following conditions:

(1) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$;

(2) If $A, B \in \mathcal{I}$ (resp. $\{A_n : n \in N\} \subset \mathcal{I}$), then $A \cup B \in \mathcal{I}$ (resp. $\bigcup\limits_{n\in N} A_n \in \mathcal{I}$).

If τ is a topology on X and $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space or simply an ideal space.

Let $(X, \tau, \mathcal{I})$ be an ideal space. An operator $(\cdot)^* : 2^X \longrightarrow 2^X$, called a local function [17] of A with respect to τ and $\mathcal{I}$, is defined as follows: for any $A \subset X$,

$$A^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$$

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where $\tau(x) = \{U \in \tau : x \in U\}$.

An operator $cl^*(\cdot) : 2^X \longrightarrow 2^X$ is defined as follows: for any $A \subset X$,

$$cl^*(A)(\mathcal{I}, \tau) = A \cup A^*(\mathcal{I}, \tau).$$

Since $cl^*(\cdot)$ is a Kuratowski closure operator, $cl(\cdot)^*$ generates a topology $\tau^*(\mathcal{I}, \tau)$, called $*$-topology. It is easy to prove that $\tau^*(\mathcal{I}, \tau) \supset \tau$.

When there is no chance for confusion, we will simply write τ^* for $\tau^*(\mathcal{I}, \tau)$, A^* for $A^*(\mathcal{I}, \tau)$, c^*A for $cl^*(A)(\mathcal{I}, \tau)$ and i^*A for $int^*(A)(\mathcal{I}, \tau)$, where

$$int^*(A)(\mathcal{I}, \tau) = X - cl^*(X - A)(\mathcal{I}, \tau).$$

$A \subset X$ is called $*$-closed [16] if $c^*A = A$, and A is called $*$-open (i.e., $A \in \tau^*$) if $X - A$ is $*$-closed. Obviously, A is $*$-open if and only if $i^*A = A$.

Throughout this paper, spaces always mean topological spaces or ideal spaces on which no separation axiom is assumed, and all mappings are onto. Sometimes, (X, τ) and $(X, \tau, \mathcal{I})$ are simply written by X. N denotes the set of all natural integers. Let $\mathcal{U} \subset 2^X$, $A \subset X$ and $x \in X$. $\mathcal{U}_A$ denotes $\{U \bigcap A : U \in \mathcal{U}\}$ and $\mathcal{U}(x)$ denotes $\{U \in \mathcal{U} : x \in U\}$. The closure of A and the interior of A are denoted by cA and iA respectively, and we have

$$iA \subset i^*A \subset A \subset c^*A \subset cA.$$

Let $(X, \tau, \mathcal{I})$ be an ideal space and let $Y \subset X$. Then $(Y, \tau_Y, \mathcal{I}_Y)$ is an ideal space, where $\tau_Y = \{U \cap Y : U \in \tau\}$ and $\mathcal{I}_Y = \{I \cap Y : I \in \mathcal{I}\} = \{I \in \mathcal{I} : I \subset Y\}$. For a space $(X, \tau, \mathcal{I})$ (resp. $(X, \tau^*, \mathcal{I})$) with $A \subset Y \subset X$, the closure of A and the interior of A in the subspace $(Y, \tau_Y, \mathcal{I}_Y)$ (resp. $(Y, \tau^*_Y, \mathcal{I}_Y)$) are denoted by $c_Y A$ and $i_Y A$ (resp. $c^*_Y A$ and $i^*_Y A$), respectively.

Given $A \subset X$ and some operators $\gamma_i : 2^X \longrightarrow 2^X$ $(i = 1, 2, \cdots, n)$. For convenience, we simply denote $\gamma_1(\gamma_2(\cdots(\gamma_n(A))\cdots))$ by $\gamma_1\gamma_2\cdots\gamma_n A$.

Lemma 2.1. ([12]) *Let $(X, \tau, \mathcal{I})$ be an ideal space and let $A \subset X$. If $U \in \tau$, then $U \cap c^*A \subset c^*(U \cap A)$.*

Lemma 2.2. ([12]) *Let $(X, \tau, \mathcal{I})$ be an ideal space and let $A \subset Y \subset X$. Then $c^*_Y(A) = c^*A \cap Y$.*

3. $*$-denseness and nowhere $*$-denseness

Definition 3.1. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is called

(1) *$*$-dense* [11], if $c^*A = X$;

(2) *nowhere $*$-dense* [1], if $i^*cA = \emptyset$.

The family of all nowhere $*$-dense subsets of an ideal space X shall be denoted by $\mathcal{N}^*(X)$ or simply by $\mathcal{N}^*$ when no ambiguity is present.

Remark 3.2. Let $(X, \tau, \mathcal{I})$ be an ideal space and let $B \subset A \subset X$.

(1) If A is $*$-dense in X, then A is dense in X.

(2) If A is nowhere $*$-dense in X, then A is nowhere dense in X.

(3) If $A \in \mathcal{N}^*$, then $B \in \mathcal{N}^*$.

(4) $A \in \mathcal{N}^*$ if and only if $cA \in \mathcal{N}^*$.

Proposition 3.3. *Let $(X, \tau, \mathcal{I})$ and $(X, \tau, \mathcal{J})$ be two ideal spaces with $\mathcal{I} \subset \mathcal{J}$ and $A \subset X$. If A is $*$-dense in $(X, \tau, \mathcal{J})$, then A is $*$-dense in $(X, \tau, \mathcal{I})$.*

Proof. This follows from the fact that $\mathcal{I} \subset \mathcal{J}$ implies $A^*(\mathcal{J}, \tau) \subset A^*(\mathcal{I}, \tau)$. □

Proposition 3.4. *Let $(X, \tau, \mathcal{I})$ and $(X, \sigma, \mathcal{I})$ be two ideal spaces with $\tau \subset \sigma$ and $A \subset X$. If A is $*$-dense in $(X, \sigma, \mathcal{I})$, then A is $*$-dense in $(X, \tau, \mathcal{I})$.*

Proof. This follows from the fact that $\tau \subset \sigma$ implies $A^*(\mathcal{I}, \sigma) \subset A^*(\mathcal{I}, \tau)$. $\square$

Proposition 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $A \subset X$ is $*$-dense in X if and only if $U \cap A \neq \emptyset$ for any $U \in \tau^* - \{\emptyset\}$.*

Proof. *Necessity.* Let A be $*$-dense in X and let $U \in \tau^* - \{\emptyset\}$. Pick $x \in U$. Then $x \in X = c^*A = A \cup A^*$.

Case 1. $x \in A$.

Then $x \in U \cap A$. So $U \cap A \neq \emptyset$.

Case 2. $x \in A^*$.

Suppose $U \cap A = \emptyset$. Since $X - U$ is $*$-closed in X, $(X - U)^* \subset X - U$. Then $U \subset X - (X - U)^*$. By $x \in U$, $x \notin (X - U)^*$. It follows that $V \cap (X - U) \in \mathcal{I}$ for some $V \in \tau(x)$. By $U \cap A = \emptyset$, $A \subset X - U$. This implies $V \cap A \subset V \cap (X - U)$. Then $V \cap A \in \mathcal{I}$. So $x \notin A^*$, a contradiction. Thus, $U \cap A \neq \emptyset$.

Sufficiency. Suppose $c^*A \neq X$. Put $U = X - c^*A$. Then $U \in \tau^* - \{\emptyset\}$. But $U \cap A = (X - c^*A) \cap A = \emptyset$. This is a contradiction. $\square$

Proposition 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal space and let $A \subset X$. The following are equivalent.*

(1) *$A \in \mathcal{N}^*$;*

(2) *For each $U \in \tau^* - \{\emptyset\}$, $U \not\subset cA$;*

(3) *For each $U \in \tau^* - \{\emptyset\}$, $U - cA \in \tau^* - \{\emptyset\}$;*

(4) *$X - cA$ is $*$-dense in X.*

Proof. (1)$\Longrightarrow$(2) Suppose that $U \subset cA$ for some $U \in \tau^* - \{\emptyset\}$. Since $A \in \mathcal{N}^*$, we have $U = i^*U \subset i^*cA = \emptyset$. Thus, $U = \emptyset$, a contradiction.

(2)$\Longrightarrow$(3) Let $U \in \tau^* - \{\emptyset\}$. By (2), $U \not\subset cA$. Then $U - cA \neq \emptyset$. Since $X - cA \in \tau$ and $\tau \subset \tau^*$, we have $X - cA \in \tau^*$. Note that $U \in \tau^*$. Thus $U - cA = U \cap (X - cA) \in \tau^* - \{\emptyset\}$.

(3)$\Longrightarrow$(4) Let $U \in \tau^* - \{\emptyset\}$. By (3), $U - cA \in \tau^* - \{\emptyset\}$. This implies that $U \cap (X - cA) = U - cA \neq \emptyset$ for any $U \in \tau^* - \{\emptyset\}$. By Proposition 3.5, $X - cA$ is $*$-dense in X.

(4)$\Longrightarrow$(1) Let $X - cA$ be $*$-dense in X. Then $X = c^*(X - cA) = X - i^*cA$. This implies $i^*cA = \emptyset$ and thus $A \in \mathcal{N}^*$. $\square$

Proposition 3.7. *Let $(X, \tau, \mathcal{I})$ be an ideal space and let $A \subset Y \subset X$.*

(1) *If $A \in \mathcal{N}^*(Y)$, then $A \in \mathcal{N}^*(X)$.*

(2) *If $Y \in \tau^*$ and $A \in \mathcal{N}^*(X)$, then $A \in \mathcal{N}^*(Y)$.*

(3) *If Y is $*$-dense in X and $A \in \mathcal{N}^*(X)$, then $A \in \mathcal{N}^*(Y)$.*

Proof. (1) Let $A \in \mathcal{N}^*(Y)$. By Proposition 3.6, $Y - c_YA$ is $*$-dense in Y. So $Y \subset c^*(Y - c_YA) = c^*(Y - cA \cap Y) = c^*(Y - cA)$. Since $X - cA = (Y - cA) \cup (X - Y)$, we have $c^*(X - cA) = c^*(Y - cA) \cup c^*(X - Y) \supset Y \cup (X - Y) = X$. Then $X - cA$ is $*$-dense in X. By Proposition 3.6, $A \in \mathcal{N}^*(X)$.

(2) Let $Y \in \tau^*$ and $A \in \mathcal{N}^*(X)$. For any $W \in \tau^*_Y - \{\emptyset\}$, $W = U \cap Y$ for some $U \in \tau^* - \{\emptyset\}$. Note that $U, Y \in \tau^*$. Then $W \in \tau^*$. Since $A \in \mathcal{N}^*(X)$, by Proposition 3.6, $X - cA$ is $*$-dense in X. By Proposition 3.5, $(X - cA) \cap W \neq \emptyset$. Note that $(Y - c_YA) \cap W = (Y - cA \cap Y) \cap W = (Y - cA) \cap W = ((X - cA) \cap Y) \cap (U \cap Y) = (X - cA) \cap W$. By Proposition 3.5, $Y - c_YA$ is $*$-dense in Y. By Proposition 3.6, $A \in \mathcal{N}^*(Y)$.

(3) Let Y be $*$-dense in X and $A \in \mathcal{N}^*(X)$. Since $A \in \mathcal{N}^*(X)$, by Proposition 3.6, $X - cA$ is $*$-dense in X. For any $W \in \tau^*_Y - \{\emptyset\}$, $W = U \cap Y$ for some $U \in \tau^* - \{\emptyset\}$. By Proposition 3.5, $(X - cA) \cap U \neq \emptyset$. Then $(X - cA) \cap U \in \tau^* - \{\emptyset\}$. Note that Y is $*$-dense in X. By Proposition 3.5, $(Y - c_YA) \cap W = (Y - cA \cap Y) \cap W = (Y - cA) \cap W = ((X - cA) \cap Y) \cap (U \cap Y) = Y \cap ((X - cA) \cap U) \neq \emptyset$. By Proposition 3.5, $Y - c_YA$ is $*$-dense in Y. By Proposition 3.6, $A \in \mathcal{N}^*(Y)$. $\square$

Proposition 3.8. *Let $(X, \tau, \mathcal{I})$ be an ideal space and let $A, B \subset X$. If $A, B \in \mathcal{N}^*$, then $A \cup B \in \mathcal{N}^*$.*

Proof. Since $A, B \in \mathcal{N}^*$, $i^*cA = i^*cB = \emptyset$. Note that $i^*c(A \cup B) = X - c^*(X - c(A \cup B)) = X - c^*(X - (cA) \cup (cB)) = X - c^*((X - cA) \cap (X - cB))$. Since $X - cA \in \tau$, by Lemma 2.1, we have $X - c^*((X - cA) \cap (X - cB)) \subset X - (X - cA) \cap c^*(X - cB) = cA \cup i^*cB = cA \cup \emptyset = cA$. So $i^*c(A \cup B) \subset cA$. Then, $i^*c(A \cup B) = i^*i^*c(A \cup B) \subset i^*cA = \emptyset$. Thus $A \cup B \in \mathcal{N}^*$. $\square$

Theorem 3.9. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{N}^*$ is an ideal on X.*

Proof. This holds by Remark 3.2 (3) and Proposition 3.8. □

4. $*$-first category and $*$-second category

Recall that a set in a Baire space is said to be a first category set or a meager set, if it can be written as a countable union of nowhere dense sets. We shall also give an analogous notion for $\mathcal{I}$-Baire spaces.

Definition 4.1. Let $(X, \tau, \mathcal{I})$ be an ideal space and let $A \subset X$.

(1) A is called *$*$-first category* or *$*$-meager* in X, if there exists a sequence $\{A_n\}$ consisting of nowhere $*$-dense subsets of X such that $A = \bigcup_{n \in N} A_n$.

(2) A is called *$*$-second category* or *$*$-nonmeager* in X, if A is not $*$-first category in X.

(3) A is called *$*$-residual* or *$*$-comeager* in X, if $X - A$ is $*$-first category in X.

The family of all $*$-first category subsets (resp. all first category subsets) of an ideal space X shall be denoted by $\mathcal{M}^*(X)$ or simply by $\mathcal{M}^*$ (resp. $\mathcal{M}(X)$ or $\mathcal{M}$) when no ambiguity is present.

Remark 4.2. For any ideal space, $\mathcal{N}^* \subset \mathcal{M}^* \subset \mathcal{M}$.

Proposition 4.3. *Let $(X, \tau, \mathcal{I})$ be an ideal space and let $A \subset Y \subset X$.*

(1) If $A \in \mathcal{M}^*(Y)$, then $A \in \mathcal{M}^*(X)$.

(2) If $Y \in \tau^*$ and $A \in \mathcal{M}^*(X)$, then $A \in \mathcal{M}^*(Y)$.

(3) If Y is $*$-dense in X and $A \in \mathcal{M}^*(X)$, then $A \in \mathcal{M}^*(Y)$.

Proof. These hold by Proposition 3.7. □

Corollary 4.4. *Let $(X, \tau, \mathcal{I})$ be an ideal space and let $A \subset Y \subset X$.*

(1) If A is $*$-second category in X, then A is $*$-second category in Y.

(2) If $Y \in \tau^*$ and A is $*$-second category in Y, then A is $*$-second category in X.

(3) If Y is $*$-dense in X and A is $*$-second category in Y, then A is $*$-second category in X.

Proof. These hold by Proposition 4.3. □

Proposition 4.5. *Let $(X, \tau, \mathcal{I})$ be an ideal space and let $A \subset Y \subset X$. If $Y \in \mathcal{M}^*$, then $A \in \mathcal{M}^*$.*

Proof. Let $Y \in \mathcal{M}^*$ and $A \subset Y$. Then $Y = \bigcup_{n \in N} Y_n$ where $Y_n \in \mathcal{N}^*$ for each $n \in N$. So $A = A \cap Y = A \cap (\bigcup_{n \in N} Y_n) = \bigcup_{n \in N} (A \cap Y_n)$. Put $A_n = A \cap Y_n$ for each $n \in N$. Note that each $A_n \subset Y_n$. By Remark 3.2 (3), $A_n \in \mathcal{N}^*$. Consequently, $A = \bigcup_{n \in N} A_n \in \mathcal{M}^*$. □

Corollary 4.6. *Let $(X, \tau, \mathcal{I})$ be an ideal space and let $A \subset Y \subset X$. If A is $*$-second category in X, then Y is $*$-second category in X.*

Proof. This holds by Proposition 4.5. □

Proposition 4.7. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $F_n \in \mathcal{M}^*$ for each $n \in N$, then $\bigcup_{n \in N} F_n \in \mathcal{M}^*$.*

Proof. This is obvious. □

Theorem 4.8. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{M}^*$ is a σ-ideal on X.*

Proof. This holds by Proposition 4.5 and 4.7. □

5. $\mathcal{I}$-Baire spaces

5.1. The concept of $\mathcal{I}$-Baire spaces

Definition 5.1. Let $(X, \tau, \mathcal{I})$ be an ideal space. X is called *$\mathcal{I}$-Baire*, if for any sequence $\{G_n\}$ consisting of open and $*$-dense subsets of X, $\bigcap_{n\in N} G_n$ is dense in X.

Example 5.2. Let $X = N$, $A = \{1, 3, 5, \cdots\}$, $B = X - A$,

$$\tau = \{\emptyset\} \cup \{A \cup M : M \in 2^B\} \text{ and } \mathcal{I} = 2^B.$$

It is easily proved that $(X, \tau, \mathcal{I})$ is an ideal space.

Let $\{G_n\}$ be any sequence consisting of open and $*$-dense subsets of X. We have $\bigcap_{n\in N} G_n \supset A$. Note that $cA = X$. Then $c(\bigcap_{n\in N} G_n) = X$. Thus $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire.

Theorem 5.3. *Let $(X, \tau, \mathcal{I})$ and $(X, \tau, \mathcal{J})$ be two ideal spaces with $\mathcal{I} \subset \mathcal{J}$. If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire, then $(X, \tau, \mathcal{J})$ is $\mathcal{J}$-Baire.*

Proof. This holds by Proposition 3.3. □

Theorem 5.4. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If (X, τ^*) is Baire, then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire.*

Proof. Let $\{G_n\}$ be a sequence of open and $*$-dense subsets of $(X, \tau, \mathcal{I})$. Note that $\tau \subset \tau^*$, and for each $n \in N$, G_n is $*$-dense in (X, τ) if and only if G_n is dense in (X, τ^*). Since (X, τ^*) is Baire, $c^*(\bigcap_{n\in N} G_n) = X$ and thus $c(\bigcap_{n\in N} G_n) = X$. Hence $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire. □

Theorem 5.5. *Let $(X, \tau, \mathcal{I})$ be an ideal space. The following are equivalent.*

(1) *X is $\mathcal{I}$-Baire;*

(2) *Each nonempty $*$-residual subset A of X is dense in X;*

(3) *Each $U \in \tau - \{\emptyset\}$ is $*$-second category in X;*

(4) *$\mathcal{M}^* \subset 2^X - (\tau - \{\emptyset\})$;*

(5) *$iF = \emptyset$ for each $F \in \mathcal{M}^*$.*

Proof. (1)$\Longrightarrow$(2) Suppose that A is $*$-residual in X. Then $X - A = \bigcup_{n\in N} A_n$ where $A_n \in \mathcal{N}^*$. By Remark 3.2 (4), $cA_n \in \mathcal{N}^*$ for each $n \in N$. By Proposition 3.6, each $X - cA_n \in \tau$ is $*$-dense in X. Now

$$A = X - (X - A) = X - \bigcup_{n\in N} A_n = \bigcap_{n\in N}(X - A_n) \supset \bigcap_{n\in N}(X - cA_n).$$

Since X is $\mathcal{I}$-Baire, $c(\bigcap_{n\in N}(X - cA_n)) = X$. Then $cA = X$. Thus A is dense in X.

(2)$\Longrightarrow$(3) Suppose that U is not $*$-second category in X for some $U \in \tau - \{\emptyset\}$. Then $U \in \mathcal{M}^*$.

Case 1. Suppose $U \neq X$. Since $U \in \mathcal{M}^*$, by (2), $X - U$ is $*$-residual in X and then $X - U$ is dense in X. Note that $U \in \tau - \{\emptyset\}$. Then $(X - U) \cap U \neq \emptyset$. This is a contradiction.

Case 2. Suppose $U = X$. By Proposition 4.5, $V \in \mathcal{M}^*(X)$ for any open set $V \subset U$. Now it satisfies the condition of Case 1 and so we omit the remaining proof.

(3)$\Longleftrightarrow$(4) is obvious.

(3)$\Longrightarrow$(5) Let $F \in \mathcal{M}^*$. Then $F = \bigcup_{n\in N} F_n$ where $F_n \in \mathcal{N}^*$. Suppose that $iF \neq \emptyset$. Pick $x \in iF$. Then $x \in U \subset F$ for some $U \in \tau$. Since $F \in \mathcal{M}^*$, by Proposition 4.5, $U \in \mathcal{M}^*$. By (3), U is $*$-second category in X. This is a contradiction.

(5)$\Longrightarrow$(1) Let $\{G_n\}$ be a sequence consisting of open and $*$-dense subsets of X. Put $F_n = X - G_n$ ($n \in N$). Note that $X - cF_n = i(X - F_n) = iG_n = G_n$ is $*$-dense in X. By Proposition 3.6, $F_n \in \mathcal{N}^*$ ($n \in N$). Then $\bigcup_{n\in N} F_n \in \mathcal{M}^*$. By (5), $i(\bigcup_{n\in N} F_n) = \emptyset$. Now we have

$$i(\bigcup_{n\in N} F_n) = i(\bigcup_{n\in N}(X - G_n)) = i(X - \bigcap_{n\in N} G_n) = X - c(\bigcap_{n\in N} G_n).$$

Then $c(\bigcap_{n\in N} G_n) = X$. So $\bigcap_{n\in N} G_n$ is dense in X and thus X is $\mathcal{I}$-Baire. □

Proposition 5.6. *Let $(X, \tau, \mathcal{I})$ be $\mathcal{I}$-Baire and let $A \subset X$. If there exists a $*$-dense G_δ-subset G of X such that $G \subset A$, then A is $*$-residual in X.*

Proof. Let $G = \bigcap_{n\in N} G_n \subset A$ with $c^*G = X$ and $G_n \in \tau - \{\emptyset\}$ ($n \in N$). Since G is $*$-dense in X, each $c^*G_n \supset c^*G = X$ and so each G_n is $*$-dense in X. Note that $G_n \in \tau - \{\emptyset\}$. By Proposition 3.6, each $X - G_n \in \mathcal{N}^*$. Since

$$X - G = X - \bigcap_{n\in N} G_n = \bigcup_{n\in N}(X - G_n),$$

we have $X - G \in \mathcal{M}^*$. Since $X - A \subset X - G$, by Proposition 4.5, $X - A \in \mathcal{M}^*$. Thus A is $*$-residual in X. □

5.2. Subspaces of $\mathcal{I}$-Baire spaces

Theorem 5.7. *Let $(X, \tau, \mathcal{I})$ be $\mathcal{I}$-Baire and let $Y \subset X$. If $Y \in \tau - \{\emptyset\}$, then $(Y, \tau_Y, \mathcal{I}_Y)$ is $\mathcal{I}_Y$-Baire.*

Proof. Let $U \in \tau_Y - \{\emptyset\}$. Since $Y \in \tau - \{\emptyset\}$, we have $U \in \tau - \{\emptyset\}$. Since $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire, by Theorem 5.5, U is $*$-second category in X. By Corollary 4.4, U is $*$-second category in Y. By Theorem 5.5, $(Y, \tau_Y, \mathcal{I}_Y)$ is $\mathcal{I}_Y$-Baire. □

Theorem 5.8. *Let $(X, \tau, \mathcal{I})$ be an ideal space. The following are equivalent.*

(1) *X is $\mathcal{I}$-Baire;*

(2) *For any $x \in X$ and $Y \in \tau(x)$, $(Y, \tau_Y, \mathcal{I}_Y)$ is $\mathcal{I}_Y$-Baire;*

(3) *For any $x \in X$, there exists $Y' \in \tau(x)$ such that $(Y', \tau_{Y'}, \mathcal{I}_{Y'})$ is $\mathcal{I}_{Y'}$-Baire.*

Proof. (1)$\Longrightarrow$(2) follows from Theorem 5.7.

(2)$\Longrightarrow$(3) is obvious.

(3)$\Longrightarrow$(1) Suppose that X is not $\mathcal{I}$-Baire. By Theorem 5.5, $G \in \mathcal{M}^*(X)$ for some $G \in \tau - \{\emptyset\}$. Pick $x \in G$. By (3), there exists $Y' \in \tau(x)$ such that $(Y', \tau_{Y'}, \mathcal{I}_{Y'})$ is $\mathcal{I}_{Y'}$-Baire. Now $x \in G \cap Y' \subset X$. Note that $G \in \mathcal{M}^*(X)$ and $Y' \in \tau - \{\emptyset\}$. By Proposition 4.5, $G \cap Y' \in \mathcal{M}^*(X)$. Note that $Y' \in \tau(x)$. Then $Y' \in \tau^* - \{\emptyset\}$. By Proposition 4.3, $G \cap Y' \in \mathcal{M}^*(Y')$. Note that $G \cap Y' \in \tau_{Y'}$. By Theorem 5.5, $(Y', \tau_{Y'}, \mathcal{I}_{Y'})$ is not $\mathcal{I}_{Y'}$-Baire. This is a contradiction. □

Theorem 5.9. *Let $(X, \tau, \mathcal{I})$ be an ideal space and let Y be $*$-residual in X. If X is $\mathcal{I}$-Baire, then $(Y, \tau_Y, \mathcal{I}_Y)$ is $\mathcal{I}_Y$-Baire.*

Proof. Let $(X, \tau, \mathcal{I})$ be $\mathcal{I}$-Baire and let Y be $*$-residual in X. To prove that $(Y, \tau_Y, \mathcal{I}_Y)$ is $\mathcal{I}_Y$-Baire, it suffices to show that any $*$-residual subset of Y is dense in Y.

Let $A \subset Y$ and $A \in \mathcal{M}^*(Y)$. Then $Y - A$ is $*$-residual in Y. By Proposition 4.3, $A \in \mathcal{M}^*(X)$. Since Y is $*$-residual in X, $X - Y \in \mathcal{M}^*(X)$. By Proposition 4.7, $(X - Y) \cup A \in \mathcal{M}^*(X)$ and then $X - (X - Y) \cup A = Y \cap (X - A)$ is $*$-residual in X. By Theorem 5.5, $Y \cap (X - A)$ is dense in X. Then

$$c_Y(Y - A) = c_Y(Y \cap (X - A)) = c(Y \cap (X - A)) \cap Y = X \cap Y = Y.$$

It follows that $Y - A$ is dense in Y. By Theorem 5.5, $(Y, \tau_Y, \mathcal{I}_Y)$ is $\mathcal{I}_Y$-Baire. □

6. $\mathcal{I}$-Baire spaces, codense ideals and $\mathcal{I}$-dense subsets

6.1. $\mathcal{I}$-Baire spaces and codense ideals

In this subsection, we will characterize $\mathcal{I}$-Baire spaces by means of codense ideals.

Definition 6.1. ([7]) Let $(X, \tau, \mathcal{I})$ be an ideal space. $\mathcal{I}$ is called codense, if $\tau \cap \mathcal{I} = \{\emptyset\}$.

Lemma 6.2. ([16]) *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is codense if and only if $A \subset A^*$ for every $A \in \tau$.*

Lemma 6.3. ([7]) *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. If $A \subset A^*$, then $A^* = cA^* = cA = c^*A$.*

Theorem 6.4. *Let $(X, \tau, \mathcal{I})$ be an ideal space and let $\mathcal{I}$ be codense. Then (X, τ) is Baire if and only if $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire.*

Proof. Necessity. This is obvious.

Sufficiency. Suppose that $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire. Let $\{G_n\}$ be a sequence consisting of open and dense subsets of X. Since $G_n \in \tau$ for each $n \in N$ and $\mathcal{I}$ is codense, by Lemma 6.2, each $G_n \subset G_n^*$. By Lemma 6.3, we have $c^*G_n = cG_n = X$ for each $n \in N$. Since $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire, $\bigcap_{n \in N} G_n$ is dense. Hence (X, τ) is Baire. □

Problem 6.5. *Can the condition " let $\mathcal{I}$ be codense " in Theorem 6.4 be omitted?*

Lemma 6.6. *If $(X, \tau, \mathcal{M}^*)$ is $\mathcal{M}^*$-Baire, then $\mathcal{M}^*$ is codense.*

Proof. Let $(X, \tau, \mathcal{M}^*)$ be $\mathcal{M}^*$-Baire. By Theorem 5.5, every $U \in \tau - \{\emptyset\}$ is $*$-second category in X and then $\tau \cap \mathcal{M}^* = \{\emptyset\}$. This implies that $\mathcal{M}^*$ is codense. □

Theorem 6.7. *Let X be a space. The following are equivalent.*

(1) *(X, τ) is Baire;*
(2) *$(X, \tau, \{\emptyset\})$ is $\{\emptyset\}$-Baire;*
(3) *$(X, \tau, \mathcal{N}^*)$ is $\mathcal{N}^*$-Baire;*
(4) *$(X, \tau, \mathcal{M}^*)$ is $\mathcal{M}^*$-Baire;*
(5) *$(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire for any ideal $\mathcal{I}$ on X.*

Proof. (1) $\Longrightarrow$ (2) is obvious.

(2)$\Longrightarrow$(3)$\Longrightarrow$(4) hold by Theorem 3.9, Remark 4.2, Theorem 4.8 and Theorem 5.3.

(1) $\Longrightarrow$ (5) $\Longrightarrow$ (4) are obvious.

(4)$\Longrightarrow$(1) holds by Theorem 6.4 and Lemma 6.6. □

6.2. $\mathcal{I}$-Baire spaces and $\mathcal{I}$-dense subsets

Definition 6.8. ([6]) Let $(X, \tau, \mathcal{I})$ be an ideal space. $A \subset X$ is called *$\mathcal{I}$-dense*, if $A^* = X$.

Remark 6.9. (1) Every $\mathcal{I}$-dense set is $*$-dense. However, $*$-dense sets need not be $\mathcal{I}$-dense (see [6]).

(2) If $\mathcal{I}$ is codense, then by Lemma 6.2 and 6.3, $\mathcal{I}$-denseness, $*$-denseness and denseness are equivalent.

Definition 6.10. ([6]) An ideal space $(X, \tau, \mathcal{I})$ is called *$\mathcal{I}$-resolvable*, if X has two disjoint $\mathcal{I}$-dense subsets.

Lemma 6.11. ([6]) *Let $(X, \tau, \mathcal{I})$ be an ideal space. If X is $\mathcal{I}$-resolvable, then $\mathcal{I}$ is codense.*

Theorem 6.12. *Let $(X, \tau, \mathcal{M}^*)$ be an ideal space. If X is $\mathcal{M}^*$-resolvable, then X is $\mathcal{M}^*$-Baire.*

Proof. Since $(X, \tau, \mathcal{M}^*)$ is $\mathcal{M}^*$-resolvable, by Lemma 6.11, $\mathcal{M}^*$ is codense. So $\tau \cap \mathcal{M}^* = \{\emptyset\}$. It follows that $U \notin \mathcal{M}^*(X)$ for any $U \in \tau - \{\emptyset\}$. By Theorem 5.5, X is $\mathcal{M}^*$-Baire. □

Theorem 6.13. *Let $(X, \tau, \mathcal{M})$ be an ideal space. Then X is $\mathcal{M}$-resolvable if and only if X has two disjoint dense $\mathcal{I}$-Baire subspaces (i.e., $X = A \cup B$, where $A \cap B = \emptyset$, $cA = cB = X$, A and B are respectively $\mathcal{M}^*_A$-Baire and $\mathcal{M}^*_B$-Baire).*

Proof. This holds by Theorem 3.3 in [6] and Theorem 6.7. □

Definition 6.14. ([3]) An ideal space $(X, \tau, \mathcal{I})$ is called *$\mathcal{I}$-separable*, if X has a countable $\mathcal{I}$-dense subset.

Theorem 6.15. *Let $(X, \tau, \mathcal{M})$ be an ideal space. Then X is $\mathcal{M}$-separable if and only if X has a countable dense $\mathcal{M}^*_Y$-Baire subspace $(Y, \tau_Y, \mathcal{M}^*_Y)$.*

Proof. This holds by Theorem 2.10 in [3] and Theorem 6.7. □

7. Some properties of $\mathcal{I}$-Baire spaces

7.1. Mapping properties of $\mathcal{I}$-Baire spaces

Lemma 7.1. ([21]) *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a mapping. Then $f(\mathcal{I}) = \{f(A) : A \in \mathcal{I}\}$ is an ideal on Y.*

Definition 7.2. ([13]) A subset A of an ideal space $(X, \tau, \mathcal{I})$ is called *semi-$\mathcal{I}$-open*, if $A \subset c^*iA$.

The family of all semi-$\mathcal{I}$-open sets in an ideal space $(X, \tau, \mathcal{I})$ shall be denoted by $SIO(X, \tau)$.

Definition 7.3. ([14]) An ideal space $(X, \tau, \mathcal{I})$ is called *semi-$\mathcal{I}$-complete*, if $\tau^* = SIO(X, \tau)$.

Definition 7.4. ([9]) A mapping $f : (X, \tau) \to (Y, \sigma)$ is called *feebly open*, if for any $U \in \tau - \{\emptyset\}$, $if(U) \neq \emptyset$.

Definition 7.5. ([13]) A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called *semi-$\mathcal{I}$-continuous*, if $f^{-1}(V) \in SIO(X, \tau)$ for any $V \in \sigma$.

Lemma 7.6. ([15]) *Let $(X, \tau, \mathcal{I})$ be an ideal space and let $\emptyset \neq A \subset X$. Then $A \in SIO(X, \tau)$ if and only if there exists $U \in \tau - \{\emptyset\}$ such that $U \subset A \subset c^*U$.*

Lemma 7.7. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be semi-$\mathcal{I}$-continuous. Then $f(i^*cA) \subset cf(A)$ for any $A \subset X$.*

Proof. Let $A \subset X$ and $B = f(A)$. Since $Y - cB \in \sigma$ and f is semi-$\mathcal{I}$-continuous, $f^{-1}(Y - cB) \in SIO(X, \tau)$. Then $f^{-1}(Y - cB) \subset c^*if^{-1}(Y - cB)$ and so $X - f^{-1}(cB) \subset c^*i(X - f^{-1}(cB))$. This implies $f^{-1}(cB) \supset X - c^*i(X - f^{-1}(cB)) = i^*(X - i(X - f^{-1}(cB))) = i^*cf^{-1}(cB)$. It follows that $i^*cf^{-1}(B) \subset f^{-1}(cB)$. Then

$$f(i^*cA) \subset f(i^*cf^{-1}(f(A))) \subset f(f^{-1}(cf(A))) \subset cf(A).$$

Consequently, $f(i^*cA) \subset cf(A)$ for any $A \subset X$. □

Theorem 7.8. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be semi-$\mathcal{I}$-continuous and feebly open. If $(X, \tau, \mathcal{I})$ is semi-$\mathcal{I}$-complete and $\mathcal{I}$-Baire, then $(Y, \sigma, f(\mathcal{I}))$ is $f(\mathcal{I})$-Baire.*

Proof. Suppose that $(Y, \sigma, f(\mathcal{I}))$ is not $f(\mathcal{I})$-Baire. Then there exists $B \in \sigma - \{\emptyset\}$ such that $B \in \mathcal{M}^*(Y)$. Put $B = \bigcup_{n \in N} B_n$ where $B_n \in \mathcal{N}^*(Y)$. Then $\emptyset \neq f^{-1}(B) = f^{-1}(\bigcup_{n \in N} B_n) = \bigcup_{n \in N} f^{-1}(B_n)$.

Claim. $f^{-1}(B_n) \in \mathcal{N}^*(X)$ $(n \in N)$.

Suppose $f^{-1}(B_n) \notin \mathcal{N}^*(X)$ for some $n \in N$. Then $i^*cf^{-1}(B_n) \neq \emptyset$. Put $A_n = f^{-1}(B_n)$. Note that $(X, \tau, \mathcal{I})$ is semi-$\mathcal{I}$-complete and $i^*cA_n \in \tau^*$, we have $\emptyset \neq i^*cA_n \subset c^*ii^*cA_n \subset c^*iccA_n = c^*icA_n$. Then $icA_n \neq \emptyset$. Since f is feebly open, $if(icA_n) \neq \emptyset$. Note that $if(icA_n) \subset i^*f(i^*cA_n)$. Since f is semi-$\mathcal{I}$-continuous, by Lemma 7.7, we have

$$\emptyset \neq i^*f(i^*cA_n) \subset i^*cf(A_n) \subset i^*cB_n.$$

Then $B_n \notin \mathcal{N}^*(Y)$, a contradiction. Thus $f^{-1}(B_n) \in \mathcal{N}^*(X)$ $(n \in N)$ and consequently, $f^{-1}(B) \in \mathcal{M}^*(X)$.

Since f is semi-$\mathcal{I}$-continuous, $\emptyset \neq f^{-1}(B) \in SIO(X, \tau)$. By Lemma 7.6, $U \subset f^{-1}(B) \subset c^*U$ for some $U \in \tau - \{\emptyset\}$. Note that $f^{-1}(B) \in \mathcal{M}^*(X)$, by the claim. So, we have $U \in \mathcal{M}^*(X)$ by Proposition 4.5. By Theorem 5.5, $(X, \tau, \mathcal{I})$ is not $\mathcal{I}$-Baire. This is a contradiction. □

Definition 7.9. ([8]) A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called *$*$-closed*, if $f(A)$ is $*$-closed in Y for every $*$-closed subset A of X.

Theorem 7.10. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a $*$-closed and continuous injection. If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire, then $(Y, \sigma, f(\mathcal{I}))$ is $f(\mathcal{I})$-Baire.*

Proof. Let $\{V_n\}$ be a sequence of open and $*$-dense subsets of Y. Put $V_n = f(U_n)$ $(n \in N)$. Since f is injective, $f^{-1}(V_n) = f^{-1}(f(U_n)) = U_n$. Since f is $*$-closed and each V_n is $*$-dense in Y, $f(c^*U_n) \supset c^*f(U_n) = c^*V_n = Y$ and then $c^*U_n = X$. Thus U_n is $*$-dense in X $(n \in N)$. Moreover, since f is continuous, $U_n = f^{-1}(V_n) \in \tau - \{\emptyset\}$ $(n \in N)$.

Now, $\{U_n\}$ is a sequence of open and $*$-dense subsets of X. Since $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire, we obtain that $c(\bigcap_{n\in N} U_n) = X$. Note that f is continuous. Thus,

$$Y = f(X) = f(c(\bigcap_{n\in N} U_n)) \subset cf(\bigcap_{n\in N} U_n) \subset c(\bigcap_{n\in N} f(U_n)) = c(\bigcap_{n\in N} V_n).$$

This implies $c(\bigcap_{n\in N} V_n) = Y$. Hence $(Y, \sigma, f(\mathcal{I}))$ is $f(\mathcal{I})$-Baire. □

7.2. Topological sums

Lemma 7.11. ([4]) *If every $\mathcal{I}_\alpha$ is an ideal on X_α, then $\{\bigcup_{\alpha\in\Gamma} I_\alpha : I_\alpha \in \mathcal{I}_\alpha\}$ is an ideal of $\bigcup_{\alpha\in\Gamma} X_\alpha$.*

Let $\{(X_\alpha, \tau_\alpha, \mathcal{I}_\alpha) : \alpha \in \Gamma\}$ be a family of pairwise disjoint ideal spaces, i.e., $X_\alpha \cap X_\beta = \emptyset$ for $\alpha \neq \beta$.

Put

$$X = \bigcup_{\alpha\in\Gamma} X_\alpha,$$

$$\tau = \{A \subset X : A \cap X_\alpha \in \tau_\alpha \text{ for each } \alpha \in \Gamma\}$$

and

$$\mathcal{I} = \{\bigcup_{\alpha\in\Gamma} I_\alpha : I_\alpha \in \mathcal{I}_\alpha\}.$$

It is easy to prove that τ is a topology on X and X_α is clopen in X for any $\alpha \in \Gamma$, and hence each X_α is $*$-closed and $*$-open in X.

By Lemma 7.11, $(X, \tau, \mathcal{I})$ is an ideal space, which is called the topological sum of $\{(X_\alpha, \tau_\alpha, \mathcal{I}_\alpha) : \alpha \in \Gamma\}$. We denote it by $\bigoplus_{\alpha\in\Gamma} X_\alpha$.

Theorem 7.12. *Let $(X, \tau, \mathcal{I})$ be the topological sum of $\{(X_\alpha, \tau_\alpha, \mathcal{I}_\alpha) : \alpha \in \Gamma\}$. Then X is $\mathcal{I}$-Baire if and only if X_α is $\mathcal{I}_\alpha$-Baire for any $\alpha \in \Gamma$.*

Proof. *Necessity.* Let $(X, \tau, \mathcal{I})$ be $\mathcal{I}$-Baire. Note that X_α is clopen in X for any $\alpha \in \Gamma$ and $\mathcal{I}_{X_\alpha} = \mathcal{I}_\alpha$. By Theorem 5.7, X_α is $\mathcal{I}_\alpha$-Baire for any $\alpha \in \Gamma$.

Sufficiency. Let $\{G_n\}$ be a sequence of open and $*$-dense subsets of X. Put $G = \bigcap_{n\in N} G_n$. For any $n \in N$ and $\alpha \in \Gamma$, we denote

$$G_{n\alpha} = G_n \cap X_\alpha \textit{ and } G_\alpha = \bigcap_{n\in N} G_{n\alpha}.$$

Then $G = \bigcup_{\alpha\in\Gamma} G_\alpha$. Now, to prove that $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Baire, it suffices to show that G is dense in X.

Since $G_n \in \tau - \{\emptyset\}$, we have $G_{n\alpha} \in \tau_\alpha$ for any $n \in N$ and $\alpha \in \Gamma$. Since $X_\alpha \in \tau$, by Lemma 2.1, $c^*G_{n\alpha} \supset c^*G_n \cap X_\alpha = X \cap X_\alpha = X_\alpha$. Note that $G_{n\alpha} \subset X_\alpha$ and X_α is $*$-closed in X. Then $c^*G_{n\alpha} = X_\alpha$ and so $c^*_{X_\alpha} G_{n\alpha} = X_\alpha$. Thus for any $\alpha \in \Gamma$, $\{G_{n\alpha}\}$ is a sequence of open and $*$-dense subsets of X_α. Since X_α is $\mathcal{I}_\alpha$-Baire, $cG_\alpha \cap X_\alpha = c_{X_\alpha} G_\alpha = X_\alpha$. This implies $cG_\alpha \supset X_\alpha$. Note that $G \supset G_\alpha$ for any $\alpha \in \Gamma$. Then $cG \supset cG_\alpha \supset X_\alpha$, which implies $cG \supset \bigcup_{\alpha\in\Gamma} X_\alpha = X$. Thus X is $\mathcal{I}$-Baire. □

References

[1] A. Acikgoz, S. Yuksel, Decompositions of some forms of continuity, Commun. Fac. Sci. Univ. Ank., Series A 56 (2007) 21–32.
[2] S. Chakrabarti, H. Dasgupta, On m-Baire spaces, Inter. Math. Forum 6(2) (2011) 95–101.
[3] Z. Cai, D. Zheng, Z. Li, H. Chen, $\mathcal{I}$-separability on ideal topological spaces, J. Adv. Res. Pure Math. 3(4) (2011) 85–91.
[4] J. Dontchev, On Hausdorff spaces via topological ideals and $\mathcal{I}$-irresolute functions, Annals New York Acad. Sci., General Topology Appl. 767 (1995) 28–38.
[5] J. Dontchev, M. Ganster, T. Noiri, Unified operation approach of generalized closed sets via topological ideals, Math. Japon. 49 (1999) 395–401.
[6] J. Dontchev, M. Ganster, D. Rose, Ideal resolvability, Topology Appl. 93 (1999) 1–16.
[7] V.R. Devi, D. Sivaraj, T.T. Chelvam, Codense and completely codense ideals, Acta Math. Hungar. 108 (2005) 197–205.
[8] E. Ekici, On $\mathcal{I}$-Alexandroff and $\mathcal{I}_g$-Alexandroff ideal topological spaces, Filomat 25:4 (2011) 99–108.
[9] Z. Frolik., Remarks concerning the invariance of Baire spaces under mappings, Czech. Math. J. 11 (1961) 381–384.
[10] G. Gruenhage, D. Lutzer, Baire and Volterra spaces, Proc. Amer. Math. Soc. 128 (2000) 3115–3124.
[11] E. Hayashi, Topologies defined by local properties, Math. Ann. 156 (1964) 205–215.
[12] E. Hatir, A. Keskin, T. Noiri, A note on strong β-$\mathcal{I}$-sets and strongly β-$\mathcal{I}$-continuous functions, Acta Math. Hungar. 108 (2005) 87–94.
[13] E. Hatir, T. Noiri, On decompositions of continuity via idealization, Acta Math. Hungar. 96 (2002) 341–349.
[14] E. Hatir, T. Noiri, On Hausdoff spaces via ideals and semi-$\mathcal{I}$-irresolute, European J. Pure Appl. Math. 2 (2009) 172–181.
[15] E. Hatir, T. Noiri, On semi-$\mathcal{I}$-open sets and semi-$\mathcal{I}$-continuous functions, Acta Math. Hungar. 107 (2005) 345–353.
[16] D. Janković, T.R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly 97 (1990) 295–310.
[17] K. Kuratowski, Topology, Academic Press, New York, 1966.
[18] A. Keskin, T. Noiri, S. Yuksel, Idealization of a decomposition theorem, Acta Math. Hungar. 102 (2004) 269–277.
[19] R.A. McCoy, Baire spaces and hyperspaces, Pacific J. Math. 58 (1975) 133–142.
[20] M.N. Mukherjee, B. Roy, R. Sen, On extensions of topological spaces in terms of ideals, Topology Appl. 154 (2007) 3167–3172.
[21] R.L. Newcomb, Topologies which are compact modulo an ideal, Ph.D. thesis, University of Cal. at Santa Barbara, 1967.
[22] R. Vaidyanathaswamy, The localisation theory in set topology, Proc. Indian Acad. Sci. 20 (1945) 51–61.