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On $I-$ Lacunary Statistical Convergence of Order α of Sequences of Sets

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Abstract. The idea of $I-$convergence of real sequences was introduced by Kostyrko et al. [*Kostyrko, P. ; Šalát, T. and Wilczyński, W. I−convergence, Real Anal. Exchange 26(2) (2000/2001), 669-686*] and also independently by Nuray and Ruckle [*Nuray, F. and Ruckle, W. H. Generalized statistical convergence and convergence free spaces, J. Math. Anal. Appl. 245(2) (2000), 513–527*]. In this paper we introduce the concepts of Wijsman $I-$lacunary statistical convergence of order α and Wijsman strongly $I-$lacunary statistical convergence of order α, and investigated between their relationship.

1. Introduction

The concept of statistical convergence was introduced by Steinhaus [36] and Fast [15]. Schoenberg [34] established some basic properties of statistical convergence and studied the concept as a summability method. Later on it was further investigated from the sequence space point of view and linked with summability theory by Altın et al. [1], Başarır and Konca [2], Caserta et al. [3], Connor [4], Çakallı [5], Çolak ([8],[9]), Et et al. ([11],[12],[20],[21]), Fridy [17], Gadjiev and Orhan [19], Kolk [22], Mursaleen et al. ([25],[26]), Salat [29], Savaş et al. ([10],[32],[33]) and many others. Nuray and Rhoades [28] extended the notion to statistical convergence of sequences of sets and gave some basic theorems. Ulusu and Nuray [38] defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence.

Let X be a non-empty set. Then a family of sets $I \subseteq 2^X$ (power sets of X) is said to be an *ideal* if I is additive *i.e.* $A, B \in I$ implies $A \cup B \in I$ and hereditary, *i.e.* $A \in I$, $B \subset A$ implies $B \in I$.

A non-empty family of sets $F \subseteq 2^X$ is said to be a *filter* of X if and only if
(*i*) $\phi \notin F$, (*ii*) $A, B \in F$ implies $A \cap B \in F$ and (*iii*) $A \in F$, $A \subset B$ implies $B \in F$.

An ideal $I \subseteq 2^X$ is called *non-trivial* if $I \neq 2^X$.

A non-trivial ideal I is said to be *admissible* if $I \supset \{\{x\} : x \in X\}$.

If I is a non-trivial ideal in $X\left(X \neq \phi\right)$ then the family of sets
$F(I) = \{M \subset X : (\exists A \in I)(M = X \setminus A)\}$ is a filter of X, called the *filter associated with* I.

Let (X, d) be a metric space. For any non-empty closed subset A_k of X, we say that the sequence $\{A_k\}$ is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$. In this case we write $\{A_k\} \in L_\infty$.

Throughout the paper I will stand for a non-trivial admissible ideal of $\mathbb{N}$.

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The idea of $I-$convergence of real sequences was introduced by Kostyrko *et al.* [23] and also independently by Nuray and Ruckle [27] (who called it generalized statistical convergence) as a generalization of statistical convergence. Later on $I-$convergence was studied in ([6],[7],[14],[24],[30], [31],[32],[33],[37],[39]).

2. Main Results

In this section, we will extend the results of Et and Şengül ([13], [35]) to statistical convergence of set sequences, namely; the relationship between the concepts of Wijsman $I-$lacunary statistical convergence of order α and Wijsman strongly $I-$lacunary statistical convergence of order α are given

Definition 2.1. *Let (X,d) be a metric space, θ be a lacunary sequence, $\alpha \in (0,1]$ and $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $I-$lacunary statistical convergent to A of order α (or $S_\theta^\alpha(I_w)-$convergent to A) if for each $\varepsilon > 0$, $\delta > 0$ and $x \in X$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left| \{ k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon \} \right| \geq \delta \right\}$$

belongs to I. In this case, we write $A_k \longrightarrow A\left(S_\theta^\alpha(I_w)\right)$. For $\theta = (2^r)$, we shall write $S^\alpha(I_w)$ instead of $S_\theta^\alpha(I_w)$ and in the special case $\alpha = 1$ and $\theta = (2^r)$ we shall write $S(I_w)$ instead of $S_\theta^\alpha(I_w)$.

As an example, consider the following sequence:

$$A_k = \begin{cases} \{3x\}, & k_{r-1} < k < k_{r-1} + \sqrt{h_r} \\ \{0\}, & \text{otherwise.} \end{cases}$$

Let $(\mathbb{R},d)$ be a metric space such that for $x,y \in X$, $d(x,y) = \left|x-y\right|$, $A = \{1\}$, $x > 1$ and $\alpha = 1$. Since

$$\frac{1}{h_r^\alpha} \left| \{ k \in I_r : |d(x,A_k) - d(x,1)| \geq \varepsilon \} \right| \geq \delta$$

belongs to I, the sequences $\{A_k\}$ is Wijsman $I-$lacunary statistical convergent to $\{1\}$ of order α ; that is $A_k \longrightarrow \{1\}\left(S_\theta^\alpha(I_w)\right)$.

Definition 2.2. *Let (X,d) be a metric space, θ be a lacunary sequence, $\alpha \in (0,1]$ and $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is said to be Wijsman strongly $I-$lacunary statistical convergent to A of order α (or $N_\theta^\alpha[I_w]-$convergent to A) if for each $\varepsilon > 0$ and $x \in X$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x,A_k) - d(x,A)| \geq \varepsilon \right\}$$

belongs to I. In this case, we write $A_k \longrightarrow A\left(N_\theta^\alpha[I_w]\right)$. For $\theta = (2^r)$, we shall write $N^\alpha[I_w]$ instead of $N_\theta^\alpha[I_w]$ and in the special case $\alpha = 1$ and $\theta = (2^r)$ we shall write $N[I_w]$ instead of $N_\theta^\alpha[I_w]$.

As an example, consider the following sequence:

$$A_k = \begin{cases} \left\{\frac{xk}{2}\right\}, & k_{r-1} < k < k_{r-1} + \sqrt{h_r} \\ \{0\}, & \text{otherwise.} \end{cases}$$

Let $(\mathbb{R},d)$ be a metric space such that for $x,y \in X$, $d(x,y) = \left|x-y\right|$, $A = \{1\}$, $x > 1$ and $\alpha = 1$. Since

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x,A_k) - d(x,1)| \geq \varepsilon,$$

the sequences $\{A_k\}$ is Wijsman $I-$lacunary statistical convergent to $\{1\}$ of order α ; that is $A_k \longrightarrow \{1\}\left(N_\theta^\alpha[I_w]\right)$.

Theorem 2.3. *$S_\theta^\alpha(I_w) \cap L_\infty$ is a closed subset of L_∞ for $0 < \alpha \leq 1$.*

Proof. Omitted. □

Theorem 2.4. *Let (X,d) be a metric space, $\theta = (k_r)$ be a lacunary sequence and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X, then*

(i) $A_k \to A\left(N_\theta^\alpha[I_w]\right) \Rightarrow A_k \to A\left(S_\theta^\alpha(I_w)\right)$ and $N_\theta^\alpha[I_w]$ is a proper subset of $S_\theta^\alpha(I_w)$,

(ii) $\{A_k\} \in L_\infty$ and $A_k \to A\left(S_\theta^\alpha(I_w)\right) \Rightarrow A_k \to A\left(N_\theta^\alpha[I_w]\right)$,

(iii) $S_\theta^\alpha(I_w) \cap L_\infty = N_\theta^\alpha[I_w] \cap L_\infty$.

Proof. (i) The inclusion part of proof is easy. In order to show that the inclusion $N_\theta^\alpha[I_w] \subseteq S_\theta^\alpha(I_w)$ is proper, let θ be given and we define a sequence $\{A_k\}$ as follows

$$A_k = \begin{cases} \{x^2\}, & k = 1,2,3,...,\left[\sqrt{h_r}\right] \\ \{0\}, & \text{otherwise} \end{cases}.$$

Let $(\mathbb{R}, d)$ be a metric space such that for $x, y \in X$, $d(x,y) = |x - y|$. We have for every $\varepsilon > 0, x > 0$ and $\frac{1}{2} < \alpha \leq 1$,

$$\frac{1}{h_r^\alpha}\left|\{k \in I_r : |d(x,A_k) - d(x,\{0\})| \geq \varepsilon\}\right| \leq \frac{\left[\sqrt{h_r}\right]}{h_r^\alpha},$$

and for any $\delta > 0$ we get

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left|\{k \in I_r : |d(x,A_k) - d(x,\{0\})| \geq \varepsilon\}\right| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{\left[\sqrt{h_r}\right]}{h_r^\alpha} \geq \delta\right\}.$$

Since the set on the right-hand side is a finite set and so belongs to I, it follows that for $\frac{1}{2} < \alpha \leq 1$, $A_k \to \{0\}\left(S_\theta^\alpha(I_w)\right)$.

On the other hand, for $\frac{1}{2} < \alpha \leq 1$ and $x > 0$,

$$\frac{1}{h_r^\alpha}\sum_{k \in I_r}|d(x,A_k) - d(x,\{0\})| = \frac{\left(x^2 - 2x\right)\left[\sqrt{h_r}\right]}{h_r^\alpha} \to 0$$

and for $0 < \alpha < \frac{1}{2}$

$$\frac{\left(x^2 - 2x\right)\left[\sqrt{h_r}\right]}{h_r^\alpha} \to \infty.$$

Hence we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\sum_{k \in I_r}|d(x,A_k) - d(x,\{0\})| \geq 0\right\} = \left\{r \in \mathbb{N} : \frac{\left(x^2 - 2x\right)\left[\sqrt{h_r}\right]}{h_r^\alpha} \geq 0\right\} = \{a, a+1, a+2, ...\}$$

for some $a \in \mathbb{N}$ which belongs to $F(I)$, since I is admissible. So $A_k \nrightarrow \{0\}\left(N_\theta^\alpha[I_w]\right)$.

ii) Suppose that $\{A_k\} \in L_\infty$ and $A_k \to A\left(S_\theta^\alpha(I_w)\right)$. Then we can assume that

$$|d(x,A_k) - d(x,A)| \leq M$$

for each $x \in X$ and all $k \in \mathbb{N}$. Given $\varepsilon > 0$, we get

$$\begin{aligned} \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| &= \frac{1}{h_r^\alpha} \sum_{\substack{k \in I_r \\ |d(x,A_k) - d(x,A)| \geq \varepsilon}} |d(x, A_k) - d(x, A)| \\ &\quad + \frac{1}{h_r^\alpha} \sum_{\substack{k \in I_r \\ |d(x,A_k) - d(x,A)| < \varepsilon}} |d(x, A_k) - d(x, A)| \\ &\leq \frac{M}{h_r^\alpha} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| + \varepsilon. \end{aligned}$$

Hence we have

$$\begin{aligned} &\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq M\delta + \varepsilon \right\} \\ \subset \; &\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta \right\} \in I. \end{aligned}$$

Therefore $A_k \to A\left(N_\theta^\alpha [I_w]\right)$.

iii) Follows from (*i*) and (*ii*). □

Theorem 2.5. *Let $\theta = (k_r)$ be a lacunary sequence and α be a fixed real number such that $0 < \alpha \leq 1$. If $\liminf_r q_r > 1$, then $S^\alpha (I_w) \subset S_\theta^\alpha (I_w)$.*

Proof. Suppose first that $\liminf_r q_r > 1$; then there exists a $\lambda > 0$ such that $q_r \geq 1 + \lambda$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\lambda}{1+\lambda} \Longrightarrow \left(\frac{h_r}{k_r}\right)^\alpha \geq \left(\frac{\lambda}{1+\lambda}\right)^\alpha \Longrightarrow \frac{1}{k_r^\alpha} \geq \frac{\lambda^\alpha}{(1+\lambda)^\alpha} \frac{1}{h_r^\alpha}.$$

If $A_k \to A\left(S^\alpha (I_w)\right)$, then for every $\varepsilon > 0$, for each $x \in X$, and for sufficiently large r, we have

$$\begin{aligned} \frac{1}{k_r^\alpha} |\{k \leq k_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| &\geq \frac{1}{k_r^\alpha} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ &\geq \frac{\lambda^\alpha}{(1+\lambda)^\alpha} \frac{1}{h_r^\alpha} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|. \end{aligned}$$

For $\delta > 0$, we have

$$\begin{aligned} &\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta \right\} \\ \subseteq \; &\left\{ r \in \mathbb{N} : \frac{1}{k_r^\alpha} |\{k \leq k_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \frac{\delta\lambda^\alpha}{(1+\lambda)^\alpha} \right\} \in I. \end{aligned}$$

This completes the proof. □

Theorem 2.6. *Let $\theta = (k_r)$ be a lacunary sequence and the parameters α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then $N_\theta^\alpha [I_w] \subseteq N_\theta^\beta [I_w]$ and the inclusion is strict.*

Proof. The inclusion part of proof is easy. To show that the inclusion is strict define $\{A_k\}$ such that for $(\mathbb{R}, d)$, $x > 1$ and $A = \{0\}$,

$$A_k = \begin{cases} \{3x+5\}, & k_{r-1} < k < k_{r-1} + \sqrt{h_r} \\ \{0\}, & \text{otherwise} \end{cases}.$$

Then $\{A_k\} \in N_\theta^\beta [I_w]$ for $\frac{1}{2} < \beta \leq 1$ but $\{A_k\} \notin N_\theta^\alpha [I_w]$ for $0 < \alpha \leq \frac{1}{2}$. □

Theorem 2.7. *Let $\theta = (k_r)$ be a lacunary sequence and the parameters α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then $S_\theta^\alpha(I_w) \subseteq S_\theta^\beta(I_w)$ and the inclusion is strict.*

Proof. The inclusion part of proof is easy. To show that the inclusion is strict define $\{A_k\}$ such that for $X = \mathbb{R}^2$

$$A_k = \begin{cases} (x,y) \in \mathbb{R}^2, x^2 + (y-1)^2 = k^2, & \text{if } k \text{ is square} \\ \{(0,0)\}, & \text{otherwise} \end{cases}.$$

Then $\{A_k\} \in S_\theta^\beta(I_w)$ for $\frac{1}{2} < \beta \leq 1$ but $\{A_k\} \notin S_\theta^\alpha(I_w)$ for $0 < \alpha \leq \frac{1}{2}$. □

Theorem 2.8. *Let the parameters α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then $S^\beta(I_w) \subseteq N^\alpha[I_w]$.*

Proof. For any sequence $\{A_k\}$ and $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{n^\alpha}\sum_{k=1}^{n} |d(x,A_k) - d(x,A)| &\geq \frac{1}{n^\alpha} |\{k \leq n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| \ \varepsilon \\ &\geq \frac{1}{n^\beta} |\{k \leq n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| \ \varepsilon \end{aligned}$$

and so

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\sum_{k=1}^{n} |d(x,A_k) - d(x,A)| \geq \delta\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n^\beta} |\{k \leq n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| \geq \frac{\delta}{\varepsilon}\right\} \in I.$$

This gives that $S^\beta(I_w) \subseteq N^\alpha[I_w]$. □

Theorem 2.9. *Let $\theta = (k_r)$ be a lacunary sequence and α be a fixed real number such that $0 < \alpha \leq 1$. If $\lim_{r\to\infty} \inf \frac{h_r^\alpha}{k_r} > 0$ then $S(I_w) \subseteq S_\theta^\alpha(I_w)$.*

Proof. Let (X,d) be a metric space, $\theta = (k_r)$ be a lacunary sequence and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X. If $\lim_{r\to\infty} \inf \frac{h_r^\alpha}{k_r} > 0$, then we can write

$$\begin{aligned} \{k \leq k_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\} &\supset \{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\} \\ \frac{1}{k_r} |\{k \leq k_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| &\geq \frac{1}{k_r} |\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| \\ &= \frac{h_r^\alpha}{k_r}\frac{1}{h_r^\alpha} |\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}|. \end{aligned}$$

So

$$\begin{aligned} &\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| \geq \delta\right\} \\ \subseteq \ &\left\{r \in \mathbb{N} : \frac{1}{k_r} |\{k \leq k_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| \geq \delta\frac{h_r^\alpha}{k_r}\right\} \end{aligned}$$

which implies that $S(I_w) \subseteq S_\theta^\alpha(I_w)$. □

Theorem 2.10. *Let (X,d) be a metric space and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X. If $\theta = (k_r)$ is a lacunary sequence with $\lim\sup \frac{(k_j - k_{j-1})^\alpha}{k_{r-1}^\alpha} < \infty$ $(j = 1,2,...,r)$, then $A_k \to A\left(S_\theta^\alpha(I_w)\right)$ implies $A_k \to A(S^\alpha(I_w))$.*

Proof. If $\limsup \frac{(k_j - k_{j-1})^\alpha}{k_{r-1}^\alpha} < \infty$, then without any loss of generality, we can assume that there exists a $0 < B_j < \infty$ such that $\frac{(k_j - k_{j-1})^\alpha}{k_{r-1}^\alpha} < B_j$, $(j = 1, 2, ..., r)$ for all $r \geq 1$. Suppose that $A_k \to A\left(S_\theta^\alpha (I_w)\right)$ and for $\varepsilon, \delta, \delta_1 > 0$ define the sets

$$C = \left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left| \{ k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| < \delta \right\}$$

and

$$T = \left\{ r \in \mathbb{N} : \frac{1}{n^\alpha} \left| \{ k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| < \delta_1 \right\}.$$

It is obvious from our assumption that $C \in F(I)$, the filter associated with the ideal I. Further observe that

$$A_i = \frac{1}{h_i^\alpha} \left| \{ k \in I_i : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| < \delta$$

for all $i \in C$. Let $n \in N$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\begin{aligned}
\frac{1}{n^\alpha} \left| \{ k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| &\leq \frac{1}{k_{r-1}^\alpha} \left| \{ k \leq k_r : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| \\
&= \frac{1}{k_{r-1}^\alpha} \left| \{ k \in I_1 : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| + ... \\
&\quad + \frac{1}{k_{r-1}^\alpha} \left| \{ k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| \\
&= \frac{k_1^\alpha}{k_{r-1}^\alpha} \frac{1}{h_1^\alpha} \left| \{ k \in I_1 : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| \\
&\quad + \frac{(k_2 - k_1)^\alpha}{k_{r-1}^\alpha} \frac{1}{h_2^\alpha} \left| \{ k \in I_2 : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| \\
&\quad + ... + \frac{(k_r - k_{r-1})^\alpha}{k_{r-1}^\alpha} \frac{1}{h_r^\alpha} \left| \{ k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| \\
&\leq \sup_{i \in C} A_i . \frac{k_1^\alpha + (k_2 - k_1)^\alpha + ... + (k_r - k_{r-1})^\alpha}{k_{r-1}^\alpha} \\
&\leq \sup_{i \in C} A_i (B_1 + B_2 + ... + B_r) < \delta \sum_{j=1}^{r} B_j.
\end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{\sum_{j=1}^{r} B_j}$ and in view of the fact that $\cup \{ n : k_{r-1} < n < k_r, r \in C \} \subset T$ where $C \in F(I)$. This completes the proof of the theorem. □

In [16], it is defined that the lacunary sequence $\theta' = (s_r)$ is called a lacunary refinement of the lacunary sequence $\theta = (k_r)$ if $(k_r) \subseteq (s_r)$. In [18], the inclusion relationship between S_θ and $S_{\theta'}$ is studied.

Theorem 2.11. *Suppose $\theta' = (s_r)$ is a lacunary refinement of the lacunary sequence $\theta = (k_r)$. Let $I_r = (k_{r-1}, k_r]$ and $J_r = (s_{r-1}, s_r]$, $r = 1, 2, 3, ...$ If there exists $\epsilon > 0$ such that for $0 < \alpha \leq \beta \leq 1$,*

$$\frac{|J_j|^\beta}{|I_i|^\alpha} \geq \epsilon \text{ for every } J_j \subseteq I_i .$$

Then $A_k \to A\left(S_\theta^\alpha (I_w)\right)$ implies $A_k \to A\left(S_{\theta'}^\beta (I_w)\right)$, i.e., $S_\theta^\alpha (I_w) \subseteq S_{\theta'}^\beta (I_w)$.

Proof. For any $\varepsilon > 0$, and every J_j, we can find I_i such that $J_j \subseteq I_i$; then we have

$$\begin{aligned}\frac{1}{\left|J_j\right|^{\beta}}\left|\left\{k \in J_j : \left|d(x,A_k) - d(x,A)\right| \geq \varepsilon\right\}\right| &= \left(\frac{|I_i|^{\alpha}}{\left|J_j\right|^{\beta}}\right)\left(\frac{1}{|I_i|^{\alpha}}\right)\left|\left\{k \in J_j : \left|d(x,A_k) - d(x,A)\right| \geq \varepsilon\right\}\right| \\ &\leq \left(\frac{|I_i|^{\alpha}}{\left|J_j\right|^{\beta}}\right)\left(\frac{1}{|I_i|^{\alpha}}\right)\left|\left\{k \in I_i : \left|d(x,A_k) - d(x,A)\right| \geq \varepsilon\right\}\right| \\ &\leq \left(\frac{1}{\epsilon}\right)\left(\frac{1}{|I_i|^{\alpha}}\right)\left|\left\{k \in I_i : \left|d(x,A_k) - d(x,A)\right| \geq \varepsilon\right\}\right|,\end{aligned}$$

and so

$$\begin{aligned}&\left\{r \in \mathbb{N} : \frac{1}{\left|J_j\right|^{\beta}}\left|\left\{k \in J_j : \left|d(x,A_k) - d(x,A)\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ \subseteq\ &\left\{r \in \mathbb{N} : \left(\frac{1}{|I_i|^{\alpha}}\right)\left|\left\{k \in I_i : \left|d(x,A_k) - d(x,A)\right| \geq \varepsilon\right\}\right| \geq \delta\epsilon\right\} \in I.\end{aligned}$$

The proof completes immediately. □

Theorem 2.12. *Suppose $\theta = (k_r)$ and $\theta' = (s_r)$ are two lacunary sequences. Let $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $r = 1,2,3,...$, and $I_{ij} = I_i \cap J_j$, $i,j = 1,2,3,....$ If there exists $\epsilon > 0$ such that for $0 < \alpha \leq \beta \leq 1$,*

$$\frac{\left|I_{ij}\right|^{\beta}}{|I_i|^{\alpha}} \geq \epsilon \text{ for every } i,j = 1,2,3,..., \text{provided } I_{ij} \neq \varnothing.$$

Then $A_k \to A\left(S_{\theta}^{\alpha}(I_w)\right)$ implies $A_k \to A\left(S_{\theta'}^{\beta}(I_w)\right)$, i.e., $S_{\theta}^{\alpha}(I_w) \subseteq S_{\theta'}^{\beta}(I_w)$.

Proof. Let $\theta'' = \theta' \cup \theta$. Then θ'' is a lacunary refinement of the lacunary sequence θ', also θ. Then interval sequence of θ'' is $\left\{I_{ij} = I_i \cap J_j : I_{ij} \neq \varnothing\right\}$. From Theorem 2.11, the condition in Theorem 2.12: $\frac{\left|I_{ij}\right|^{\beta}}{|I_i|^{\alpha}} \geq \epsilon$, for every $i,j = 1,2,3,...$,provided $I_{ij} \neq \varnothing$ yields that $A_k \to A\left(S_{\theta}^{\alpha}(I_w)\right)$implies $A_k \to A\left(S_{\theta''}^{\beta}(I_w)\right)$. Since θ'' is also a lacunary refinement of the lacunary sequence θ', we have that $A_k \to A\left(S_{\theta''}^{\alpha}(I_w)\right)$implies $A_k \to A\left(S_{\theta'}^{\beta}(I_w)\right)$. The proof follows immediately. □

Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let α and β be positive real numbers such that $0 < \alpha \leq \beta \leq 1$. Now we shall give some general inclusion relations between the sets of $S_{\theta}^{\alpha}(I_w)$ −convergent sequences and $N_{\theta}^{\alpha}[I_w]$ −summable sequences for different $\alpha' s$ and $\theta' s$ which also include Theorem 2.4, Theorem 2.6, Theorem 2.7 and Theorem 2.8 as a special case.

Theorem 2.13. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let α and β be such that $0 < \alpha \leq \beta \leq 1$,*

(i) If

$$\lim_{r\to\infty} \inf \frac{h_r^{\alpha}}{\ell_r^{\beta}} > 0 \tag{1}$$

then $S_{\theta'}^{\beta}(I_w) \subseteq S_{\theta}^{\alpha}(I_w)$,

(ii) If

$$\lim_{r\to\infty}\frac{\ell_r}{h_r^{\beta}}=1 \tag{2}$$

then $S_{\theta}^{\alpha}(I_w)\subseteq S_{\theta'}^{\beta}(I_w)$.

Proof. (*i*) Let (X,d) be a metric space, $\theta=(k_r)$ be a lacunary sequence and A, A_k (for all $k\in\mathbb{N}$) be non-empty closed subsets of X. Suppose that $I_r\subset J_r$ for all $r\in\mathbb{N}$ and let (1) be satisfied. For given $\varepsilon>0$ we have

$$\{k\in J_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\supseteq\{k\in I_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\},$$

and so

$$\frac{1}{\ell_r^{\beta}}\left|\{k\in J_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|\geq\frac{h_r^{\alpha}}{\ell_r^{\beta}}\frac{1}{h_r^{\alpha}}\left|\{k\in I_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|.$$

Hence

$$\begin{aligned}&\left\{r\in\mathbb{N}:\frac{1}{h_r^{\alpha}}\left|\{k\in I_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|\geq\delta\right\}\\ \subseteq\ &\left\{r\in\mathbb{N}:\frac{1}{\ell_r^{\beta}}\left|\{k\in J_r : |d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|\geq\delta\frac{h_r^{\alpha}}{\ell_r^{\beta}}\right\}\in I\end{aligned}$$

for all $r\in\mathbb{N}$, where $I_r=(k_{r-1},k_r]$, $J_r=(s_{r-1},s_r]$, $h_r=k_r-k_{r-1}$, $\ell_r=s_r-s_{r-1}$. Now taking the limit as $r\to\infty$ in the last inequality and using (1) we get $S_{\theta'}^{\beta}(I_w)\subseteq S_{\theta}^{\alpha}(I_w)$.

(*ii*) Omitted. □

Theorem 2.14. *Let* $\theta=(k_r)$ *and* $\theta'=(s_r)$ *be two lacunary sequences such that* $I_r\subseteq J_r$ *for all* $r\in\mathbb{N}$, α *and* β *be fixed real numbers such that* $0<\alpha\leq\beta\leq 1$. *Then we have*

(*i*) *If* (1) *holds then* $N_{\theta'}^{\beta}[I_w]\subset N_{\theta}^{\alpha}[I_w]$,

(*ii*) *If* (2) *holds and* $\{A_k\}\in L_\infty$ *then* $N_{\theta}^{\alpha}[I_w]\subset N_{\theta'}^{\beta}[I_w]$.

Proof. Omitted. □

Theorem 2.15. *Let* $\theta=(k_r)$ *and* $\theta'=(s_r)$ *be two lacunary sequences such that* $I_r\subseteq J_r$ *for all* $r\in\mathbb{N}$, α *and* β *be fixed real numbers such that* $0<\alpha\leq\beta\leq 1$. *Then*

(*i*) *Let* (1) *holds, if a sequence is strongly* $N_{\theta'}^{\beta}[I_w]$ *−summable to* A, *then it is* $S_{\theta}^{\alpha}(I_w)$ *−statistically convergent to* A,

(*ii*) *Let* (2) *holds and* $\{A_k\}$ *be a bounded sequence, if a sequence is* $S_{\theta}^{\alpha}(I_w)$ *−statistically convergent to* A *then it is strongly* $N_{\theta'}^{\beta}[I_w]$ *−summable to* A.

Proof. (*i*) Omitted.

(ii) Suppose that $S_{\theta}^{\alpha}(I_w)-\lim A_k=A$ and $\{A_k\}\in L_\infty$. Then there exists some $M>0$ such that $|d(x,A_k)-d(x,A)|\le M$ for all k, then for every $\varepsilon>0$ we may write

$$\begin{aligned}\frac{1}{\ell_r^{\beta}}\sum_{k\in J_r}|d(x,A_k)-d(x,A)| &= \frac{1}{\ell_r^{\beta}}\sum_{k\in J_r-I_r}|d(x,A_k)-d(x,A)|+\frac{1}{\ell_r^{\beta}}\sum_{k\in I_r}|d(x,A_k)-d(x,A)|\\ &\le \left(\frac{\ell_r-h_r}{\ell_r^{\beta}}\right)M+\frac{1}{\ell_r^{\beta}}\sum_{k\in I_r}|d(x,A_k)-d(x,A)|\\ &\le \left(\frac{\ell_r-h_r^{\beta}}{\ell_r^{\beta}}\right)M+\frac{1}{\ell_r^{\beta}}\sum_{k\in I_r}|d(x,A_k)-d(x,A)|\\ &\le \left(\frac{\ell_r}{h_r^{\beta}}-1\right)M+\frac{1}{h_r^{\beta}}\sum_{\substack{k\in I_r\\ |d(x,A_k)-d(x,A)|\ge\varepsilon}}|d(x,A_k)-d(x,A)|\\ &\quad+\frac{1}{h_r^{\beta}}\sum_{\substack{k\in I_r\\ |d(x,A_k)-d(x,A)|<\varepsilon}}|d(x,A_k)-d(x,A)|\\ &\le \left(\frac{\ell_r}{h_r^{\beta}}-1\right)M+\frac{M}{h_r^{\alpha}}\left|\{k\in I_r:|d(x,A_k)-d(x,A)|\ge\varepsilon\}\right|+\frac{\ell_r}{h_r^{\beta}}\varepsilon\end{aligned}$$

and so

$$\begin{aligned}&\left\{r\in\mathbb{N}:\frac{1}{\ell_r^{\beta}}\sum_{k\in J_r}|d(x,A_k)-d(x,A)|\ge\delta\right\}\\ \subseteq\ &\left\{r\in\mathbb{N}:\frac{1}{h_r^{\alpha}}\left|\{k\in I_r:|d(x,A_k)-d(x,A)|\ge\varepsilon\}\right|\ge\frac{\delta}{M}\right\}\in I,\end{aligned}$$

for all $r\in\mathbb{N}$. Using (2) we obtain that $N_{\theta'}^{\beta}[I_w]-\lim A_k=A$, whenever $S_{\theta}^{\alpha}(I_w)-\lim A_k=A$. □

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