# On idempotent discrete uninorms

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## Abstract

This paper is devoted to classify all idempotent uninorms defined on the finite scale $L_n = \{0, 1, \ldots, n\}$, called discrete idempotent uninorms. It is proved that any discrete idempotent uninorm with neutral element $e \in L_n$ is uniquely determined by a decreasing function $g : [0, e] \to [e, n]$ and vice versa. Based on this correspondence, the number of all possible discrete idempotent uninorms on a finite scale of $n + 1$ elements is given depending on $n$.

**Keywords:** Discrete scale, idempotent uninorm, decreasing function, symmetry.

## 1 Introduction

The significance of uninorms is well known in the field of aggregation operators and their applications from both, the theoretical and the practical point of view. Similarly, the study of aggregation operators defined on a finite scale is a field of increasing interest for many reasons (see [3], [5], [10], [11], [12], [14]).

In this sense, the majority of known associative aggregation operators defined on [0,1] have been translated to the discrete case. For instance, smooth t-norms and t-conorms (the counterpart of the continuous ones in [0,1]) are characterized in [15] (see also [14]), nullnorms in [10], copulas in [13], bisymmetric operators in [11], a non-commutative version of nullnorms in [5]. Curiously, the case of uninorms has been partially forgotten in this study. Only the class of uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ (see Definitions 2 and 3) was characterized in [10] and even a non-commutative version of them in [12]. But other known classes of uninorms on [0,1] have not been studied yet. It is clear that representable uninorms on [0,1] have no counterpart in the discrete case, but this is not the case for other classes like idempotent uninorms or uninorms in $\mathcal{U}_{\text{mnx}}$ (those that are locally internal in the region $A(e) = \{(x, y) \mid \min(x, y) < e < \max(x, y)$, see [4]). In general, the characterization of all discrete uninorms with underlying smooth t-norms and t-conorms (like in [7] for uninorms in [0,1]) is still open.

This paper wants to be a first step in the solution of this open problem, that consists in the characterization of all discrete idempotent uninorms, that is, those defined on a finite scale $L_n$ such that $U(x, x) = x$ for all $x \in L_n$. We will see that this kind of uninorms with neutral element $e \in L_n$ is in one to one correspondence with the set of decreasing functions $g : [0, e] \to [e, n]$ with $g(e) = e$, in such a way that each one of these functions uniquely determines an idempotent uninorm on $L_n$ and vice versa. This correspondence allows us to determine the number of all possible idempotent uninorms on $L_n$ depending on $n$.

## 2 Preliminaries

We suppose the reader to be familiar with some basic results on uninorms and their classes that can be found for instance in [6].

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See also the forthcoming paper [2] for a more actualized and complete compilation.

In these preliminaries we recall some known facts on uninorms defined on finite chains, that we will also refer to discrete uninorms. In these cases, the concrete scale to be used is not determinant and the only important fact is the number of elements of the scale (see [15]). Thus, given any positive integer $n$, we will deal from now on with the finite chain

$$L_n = \{0, 1, 2, \dots, n\}.$$

We will use indistinctly the interval notation $L_n = [0, n]$ and also the usual notations $[0, e]$ and $[e, n]$ when $e \in L_n$ for the corresponding subsets of $L_n$.

**Definition 1** *A* uninorm *on $L_n$ is a two-place function $U : L_n^2 \to L_n$ which is associative, commutative, increasing in each place and such that there exists some element $e \in L_n$, called* neutral element*, such that $U(e, x) = x$ for all $x \in L_n$.*

It is clear that the function $U$ becomes a t-norm when $e = n$ and a t-conorm when $e = 0$. For any uninorm on $L_n$ we have $U(n, 0) \in \{0, n\}$ and a uninorm $U$ is called *conjunctive* when $U(n, 0) = 0$ and *disjunctive* when $U(n, 0) = n$. The structure of any discrete uninorm $U$ on $L_n$ with neutral element $0 < e < n$ is always as follows. It is given by a t-norm $T$ on the interval $[0, e]$, by a t-conorm $S$ on the interval $[e, n]$ and it takes values between the minimum and the maximum in all other cases.

Respect to uninorms on $L_n$, only the classes of uninorms in $\mathcal{U}_{\min}$ and uninorms in $\mathcal{U}_{\max}$ have been studied and characterized through some partial smoothness conditions in [10]. Specifically, these kinds of uninorms are as follows.

**Definition 2** *A binary operation $U : L_n^2 \longrightarrow L_n$ is a uninorm in $\mathcal{U}_{\min}$ with neutral element $0 < e < n$ if and only if there is a t-norm $T$ on $[0, e]$ and a t-conorm $S$ on $[e, n]$ such that $U$ is given by*

$$U(x, y) = \begin{cases} T(x, y) & \text{if } x, y \in [0, e] \\ S(x, y) & \text{if } x, y \in [e, n] \\ \min(x, y) & \text{elsewhere.} \end{cases}$$

**Definition 3** *A binary operation $U : L_n^2 \longrightarrow L_n$ is a uninorm in $\mathcal{U}_{\max}$ with neutral element $0 < e < n$ if and only if there is a t-norm $T$ on $[0, e]$ and a t-conorm $S$ on $[e, n]$ such that $U$ is given by*

$$U(x, y) = \begin{cases} T(x, y) & \text{if } x, y \in [0, e] \\ S(x, y) & \text{if } x, y \in [e, n] \\ \max(x, y) & \text{elsewhere.} \end{cases}$$

## 3 Discrete idempotent uninorms

In this paper we want to characterize all discrete idempotent uninorms. To do this, let us give first a discrete counterpart of the Czogala-Drewniak theorem (see [1]) in the following way.

**Theorem 1** *Let $U : L_n^2 \to L_n$ be an idempotent, associative, increasing binary operator with neutral element $e \in L_n$ with $0 < e < n$. Then, there exists a decreasing function $g : L_n \to L_n$ with $g(e) = e$ such that $U(x, y) =$*

$$\begin{cases} \min(x, y) & \text{if } y \le g(x) \text{ and } g(x) > 0 \\ \min(x, y) \text{ or } \max(x, y) & \text{if } y = g(x) = 0 \\ \max(x, y) & \text{elsewhere.} \end{cases} \quad (1)$$

Note that in the case of [0,1] the Czogala-Drewniak theorem was improved in [9] where it is proved that any idempotent, associative, increasing binary operator $U$ with neutral element $e$ must be in fact commutative except perhaps in points $(x, g(x))$ for $x \in [0, 1]$ such that $g^2(x) = x$. This is not true in the discrete case as Example 1 shows (what actually happens is that commutativity can fail not only in points $(x, g(x))$, but also in points $(x, g(x)+1)$ or $(x, g(x) - 1)$ for some $x \in [0, n]$).

**Example 1** *If we take $n = 4$, let us consider the following function $g$*

$$g(x) = \begin{cases} 2 & \text{if } x \le 2 \\ 4 - x & \text{otherwise} \end{cases}$$

*We can observe function $g$ in Figure 1. Let $F$ be the operator given by:*

$$F(x, y) = \begin{cases} \min(x, y) & \text{if } y \le g(x) \\ \max(x, y) & \text{otherwise} \end{cases}$$

*We have that $F$ is an idempotent, associative, increasing binary operator with neutral element $e = 2$. However, $F$ is not commutative for instance in the points $(0, 3)$ and $(3, 0)$.*

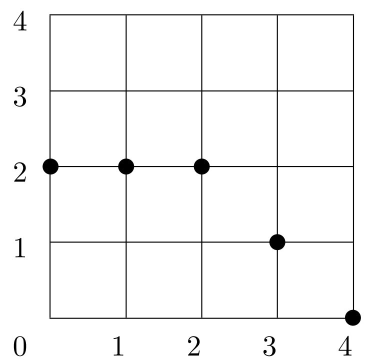



Figure 1: Drawn points represent function $g$ of Example 1.

Now, in order to get our main goal, we introduce some definitions that can be understood as discrete counterparts of those given in [8] for the interval [0,1].

**Definition 4** *Given any decreasing function $g : L_n \to L_n$, we define its* completed graph *as the set:*

$$F_g = \{(0, y) \mid y \geq g(0)\} \cup$$

$$\{(x, y) \in [0, n-1] \times [0, n] \mid g(x+1) \leq y \leq g(x)\} \cup$$

$$\{(n, y) \mid y \leq g(n)\}$$

**Definition 5** *Let $F$ be a subset of $L_n^2$. $F$ is said to be* Id-symmetrical *if it satisfies:*

$$(x, y) \in F \iff (y, x) \in F.$$

Note that the above definition captures the idea of a subset $F$ of $L_n^2$ to be symmetric with respect to the diagonal. On the other hand, if we want to give a similar definition for a decreasing function $g : L_n \to L_n$, the idea should be that the region under the function was Id-symmetrical. Note that following this idea, the graph of $g$ given by $\{(x, g(x)) \mid x \in L_n\}$ can not be considered (see example 2).

Hence, we will adopt the following definition:

**Definition 6** *A decreasing function $g : L_n \to L_n$ is said to be* Id-symmetrical *if so is its completed graph $F_g$.*

**Example 2** *Now, we will see an Id-symmetrical decreasing function $g : L_n \to L_n$ such that its graph is not Id-symmetrical.*

*Let $n = 4$, and $g : L_4 \to L_4$ defined by*

$$g(x) = \begin{cases} 4 & \text{if } x = 0 \\ 2 & \text{if } x \in \{1, 2\} \\ 0 & \text{otherwise.} \end{cases}$$

*Note that the graph of $g$ is the set*

$$\{(0, 4), (1, 2), (2, 2), (3, 0), (4, 0)\}$$

*whereas its completed graph $F_g$ is given by*

$$F_g = \{(0, 4), (0, 3), (0, 2), (1, 2),$$

$$(2, 2), (2, 1), (2, 0), (3, 0), (4, 0)\}.$$

*It is clear that $F_g$ is Id-symmetrical and so is $g$, whereas the graph of $g$ is not. We can observe function $g$, and its completed graph $F_g$ in Figure 2.*

**Lemma 1** *Let $g : L_n \to L_n$ be a decreasing function. The following statements are equivalent:*

*i)* *$g$ is Id-symmetrical.*

*ii)* *$g(x) = 0$ for all $x > g(0)$ and the set $F_{g,0}$ is Id-symmetrical where $F_{g,0} =$*

$$\{(x, y) \in [0, g(0)]^2 \mid g(x+1) \leq y \leq g(x)\}.$$

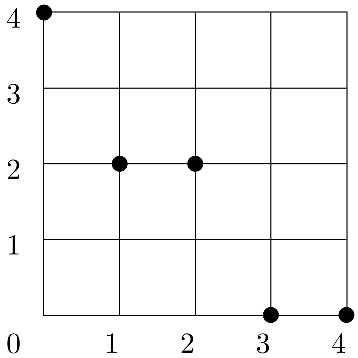
*iii)* *$g(x) = n$ for all $x < g(n)$ and the set $F_{g,n}$ is Id-symmetrical where $F_{g,n} =$*

$$\{(x, y) \in [g(n), n]^2 \mid g(x+1) \leq y \leq g(x)\}.$$

**Lemma 2** *Let $g : L_n \to L_n$ be a decreasing function with $g(0) < n$. The following statements are equivalent:*

*i)* *$g$ is Id-symmetrical.*

*ii)* *$g(x) = 0$ for all $x > g(0)$ and $g(g(x)) \geq x$ for all $x \in [0, g(0)]$.*



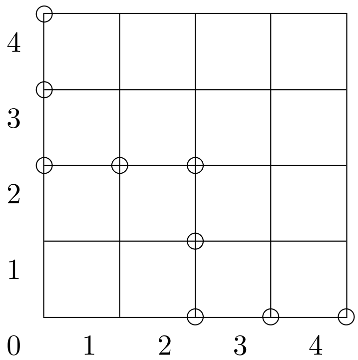




Figure 2: Drawn points represent function $g$ of example 2 (left) and its completed graph $F_g$ (right).

*iii)* $g(x) = 0$ *for all* $x > g(0)$ *and for all* $(x, y) \in [0, g(0)]^2$ *it is verified*

$$y \leq g(x) \quad \Longleftrightarrow \quad x \leq g(y).$$

**Lemma 3** *Take* $e \in L_n$ *with* $0 < e < n$ *and let* $g : [0, e] \to [e, n]$ *be a decreasing function such that* $g(e) = e$. *Then there is one and only one Id-symmetrical extension of* $g$, $\overline{g} : L_n \to L_n$, *that is given by* $\overline{g}(x) =$

$$\begin{cases} g(x) & \text{if } x \leq e \\ \max\{z \in [0, e] \mid g(z) \geq x\} & \text{if } e \leq x \leq g(0) \\ 0 & \text{if } x > g(0) \end{cases} \tag{2}$$

Now, we can give the characterization theorem of idempotent discrete uninorms.

**Theorem 2** *Let* $U : L_n^2 \to L_n$ *be a binary operator with neutral element* $e \in L_n$ *with* $0 < e < n$. *Then,* $U$ *is an idempotent uninorm if and only if there exists a decreasing function* $g : [0, e] \to [e, n]$ *with* $g(e) = e$ *such that*

$$U(x, y) = \begin{cases} \min(x, y) & \text{if } y \leq \overline{g}(x) \text{ and} \\ & x \leq \overline{g}(0) \\ \max(x, y) & \text{elsewhere.} \end{cases} \tag{3}$$

*where* $\overline{g}$ *is the unique Id-symmetrical extension of* $g$, *given by equation (2).*

**Remark 1** *It is clear that a similar study could be done taking function* $g$ *as follows*

$$g(x) = \min\{z \in L_n \mid U(x, z) = \max(x, z)\}$$

*for all* $x \in [e, n]$, *and* $\overline{g}$ *its unique Id-symmetrical extension to* $L_n$. *Thus, a similar characterization should be obtained with the difference that, in this case, uninorm* $U$ *should be given by*

$$U(x, y) = \begin{cases} \max(x, y) & \text{if } y \geq \overline{g}(x) \text{ and} \\ & x \geq \overline{g}(n) \\ \min(x, y) & \text{elsewhere.} \end{cases}$$

Now, we can introduce some examples of families of idempotent uninorms.

**Example 3** *If we take the following function* $g : [0, e] \to [e, n]$

$$g(x) = \begin{cases} n & \text{if } x < e \\ e & \text{if } x = e \end{cases}$$

*its unique Id-symmetrical extension is given by*

$$\overline{g}(x) = \begin{cases} n & \text{if } x < e \\ e & \text{otherwise} \end{cases}$$

*and then the corresponding uninorm obtained by (3) is the idempotent uninorm in* $\mathcal{U}_{\min}$.

*Similarly, considering the constant function* $g : [0, e] \to [e, n]$ *given by* $g(x) = e$ *for all* $x \in [0, e]$, *its unique Id-symmetrical extension is given by*

$$\overline{g}(x) = \begin{cases} e & \text{if } x \leq e \\ 0 & \text{if } x > e \end{cases}$$

*and we obtain the idempotent uninorm in* $\mathcal{U}_{\max}$.

*An example for* $n = 5$ *and* $e = 3$ *can be viewed in figure 3.*

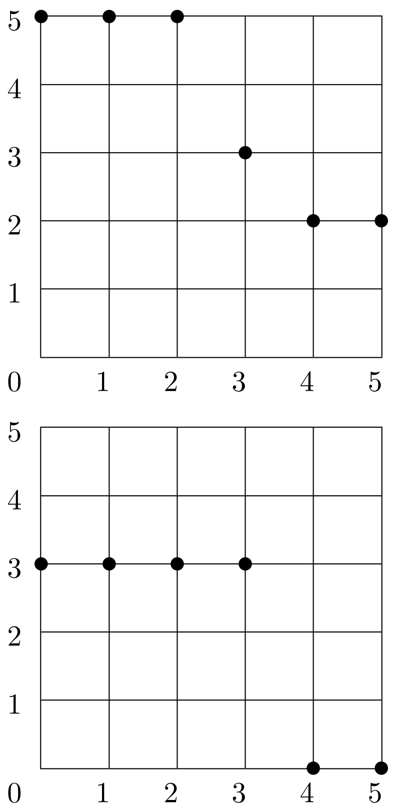



Figure 3: Drawn points represent function $\overline{g}$ for uninorm $U$ in $\mathcal{U}_{\min}$ (left) and $\mathcal{U}_{\max}$ (right) of example 3.

**Example 4** *In the case of* $[0,1]$ *some important examples of idempotent uninorms are obtained by taking as the function $g$ an involutive negation. In our case, there is only one involutive negation on $L_n$ which is given by $N(x) = n - x$. However, this negation has a fixed point only when $n$ is even and then the fixed point is $n/2$. Thus, when $n$ is even the function*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y \leq n-x \\ \max(x,y) & \text{if } y > n-x \end{cases} \quad (4)$$

*is an idempotent uninorm that arises from the function $g : [0, n/2] \to [n/2, n]$ given by $g(x) = n - x$ for all $x \in [0, n/2]$, that is, the restriction of $N$ to $[0, n/2]$.*

*When $n$ is odd, function $U$ is also an idempotent uninorm but in this case it arises from the function $g : [0, \frac{n+1}{2}] \to [\frac{n+1}{2}, n]$ given by*

$$g(x) = \begin{cases} n - x & \text{if } x < \frac{n+1}{2} \\ \frac{n+1}{2} & \text{if } x = \frac{n+1}{2} \end{cases}$$

*In Figure 4 we can see the necessary functions $g$ (in fact its Id-symmetrical extensions) to derive the idempotent uninorm given by* (4) *in the cases $n = 4$ and $n = 5$.*

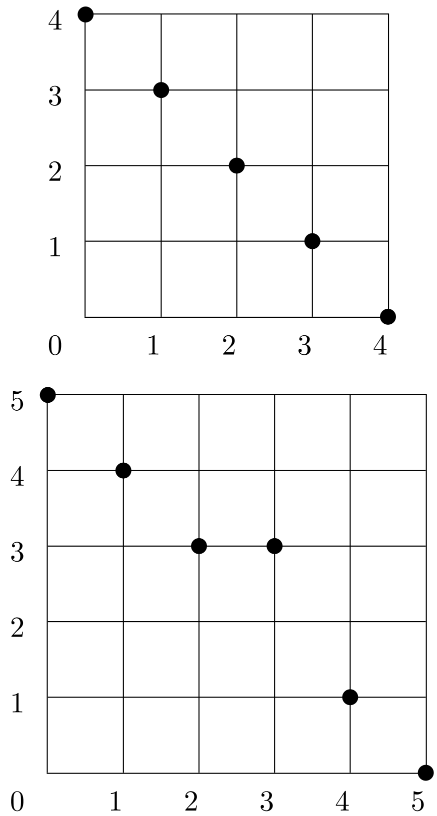



Figure 4: Drawn points represent functions $g$ of example 4 from which uninorm (4) derives, in cases $n = 4$ and $n = 5$.

**Example 5** *Finally, suppose that we want to deal with the same uninorms as above but taking the maximum on the points $\{(x, n-x) \mid x \in L_n\}$. That is,*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < n-x \\ \max(x,y) & \text{if } y \geq n-x. \end{cases} \quad (5)$$

*It is clear that, taking the approach given in Remark 1, exactly the same functions $g$ given in the previous example apply here to obtain (5). However, with the approach developed in* *the paper we can also obtain uninorms of type (5). When $n$ is even, the necessary function $g : [0, n/2] \to [n/2, n]$ should be*

$$g(x) = \begin{cases} n - x - 1 & \text{if } x < \frac{n}{2} \\ \frac{n}{2} & \text{if } x = \frac{n}{2}. \end{cases}$$

*On the other hand, when $n$ is odd the necessary function $g : [0, (n-1)/2] \to [(n-1)/2, n]$ should be simply $g(x) = n - x - 1$ for all $x \leq (n-1)/2$. Again, these necessary functions $g$ extended to their Id-symmetrical extensions in the cases $n = 4$ and $n = 5$ can be viewed in Figure 5.*

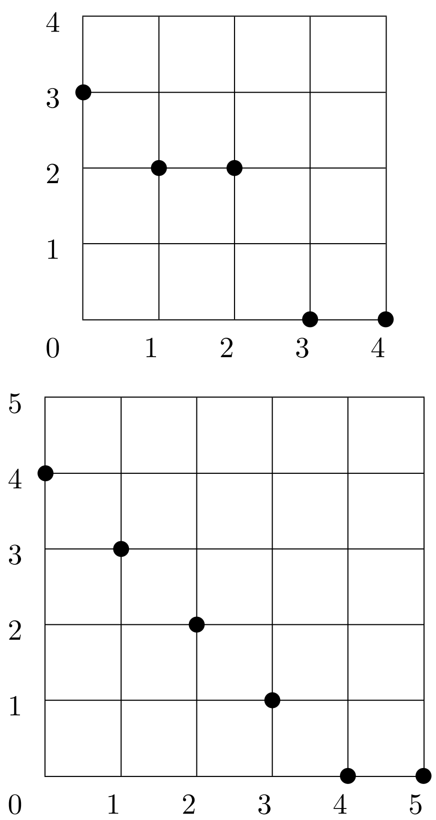



Figure 5: Drawn points represent the Id-symmetrical extensions of functions $g$ of example 5 from which uninorm (4) derives, in cases $n = 4$ and $n = 5$.

As a consequence of the one-to-one correspondence between decreasing functions $g : [0, e] \to [e, n]$ with $g(e) = e$ and discrete idempotent uninorms with neutral element $e$ we can easily count the total number of this kind of discrete uninorms.

**Theorem 3** *Let $n$ be a natural number with $n \geq 2$ and let $e$ be such that $0 < e < n$. Then*

1. *The number of discrete idempotent uninorms on $L_n$ with neutral element $e$ is given by*
$$N_{e,n} = \binom{n}{e}$$
2. *The total number of discrete idempotent uninorms on $L_n$ with neutral element $e$ such that $0 \leq e \leq n$ is given by*
$$N_n = \sum_{e=0}^{n} N_{e,n} = \sum_{e=0}^{n} \binom{n}{e} = 2^n$$

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