On Idempotents of Completely Bounded Multipliers of the Fourier Algebra $A(G)$

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ABSTRACT. Let $A(G)$ be the Fourier algebra of a locally compact group G and $M_{cb}A(G)$ be the space of completely bounded multipliers of $A(G)$. We give a description of idempotents of $M_{cb}A(G)$ of norm one and then present a necessary condition for an idempotent to be in $M_{cb}A(G)$. In the last part of the paper, we discuss a class of free sets, called L-sets, whose characteristic function is an idempotent of $M_{cb}A(G)$, but not of $B(G)$.

1. INTRODUCTION

In harmonic analysis, we are interested in studying locally compact groups and the Banach algebras associated with these groups. In classical harmonic analysis, the group algebras $L^\infty(G,\mu)$ and $L^1(G,\mu)$ and the measure algebra $M(G)$ (as the multiplier algebra of $L^1(G)$) have been extensively studied (see for example [15]). The group von Neumann algebra $VN(G)$ and the Fourier algebra $A(G)$, are the corresponding dual objects of $L^\infty(G)$ and $L^1(G)$ respectively, in the non-commutative setting. It is still debatable what is the most appropriate dual object of $M(G)$. For this, we have three possible candidates: the Fourier-Stieltjes algebra $B(G)$ and the multiplier algebras $M_{cb}A(G)$ and $MA(G)$. The spaces $A(G)$, $B(G)$, $VN(G)$, $MA(G)$ and $M_{cb}A(G)$ will be defined in Section 2. When G is abelian or amenable it is well known that $B(G)$, $MA(G)$ and $M_{cb}A(G)$ are the same, hence the question really makes sense only in the non-amenable case. For a long time, $B(G)$ has been considered to be the natural dual object of $M(G)$, since $B(G) \cong M(\hat{G})$ when G is abelian. However, many recent developments in the area lead us to believe that $M_{cb}A(G)$ would be the measure algebra's natural counterpart in the non-amenable case. The space of completely bounded multipliers,

$M_{cb}A(G)$ has been studied by many mathematicians including J. E. Gilbert [11] (unpublished manuscript), Jollisaint [18], Haagerup (unpublished manuscript). They showed that this algebra has a nice representation structure (see [25, Theorem 5.1]) and that it has many important applications to operator algebras and harmonic analysis. Recently, Neufang, Ruan and Spronk ([24]), improving a result of Neufang, showed that $M(G)$ and $CB^{\sigma,L^\infty(G)}_{VN(G)}(B(L^2(G)))$ are completely isometrically isomorphic and similarly $M_{cb}A(G)$ and $CB^{\sigma,VN(G)}_{L^\infty(G)}(B(L^2(G)))$ are completely isometrically isomorphic. All these considerations suggest that it would be more appropriate to consider $M_{cb}A(G)$ as the dual object of $M(G)$.

Our goal here is to better understand the completely bounded multipliers, more precisely their idempotent structure.

In 1960, Cohen [4] obtained the characterization of all idempotents of $M(G)$ when G is abelian, a result which won him the Bochner Memorial Prize. Later, in 1986, Host generalized Cohen's characterization, by discovering the general form of idempotents for $B(G)$, for any group G. He showed in [16] that $\chi_\Lambda \in B(G)$ if and only if $\Lambda \subseteq G$ is in the coset ring.

We note that since $M_{cb}A(G)$ is a space of functions from G into $\mathbb{C}$, then its idempotents are characteristic functions of subsets of G. The structure theorem for $M_{cb}A(G)$ ([25, Theorem 5.1]) implies that the set of idempotents of $M_{cb}A(G)$ has a ring structure, which is also true in the case of $B(G)$. However, the idempotent structure of $M_{cb}A(G)$ is more complicated than that of $B(G)$. It is obvious that an idempotent of $B(G)$ is also an idempotent of $M_{cb}A(G)$ since $B(G) \subseteq M_{cb}A(G)$; but this is not the only type of idempotents in $M_{cb}A(G)$. For example, the characteristic function of an L-set is a completely bounded multiplier (see [26]), but it is not in $B(G)$.

In this paper, we begin with the preliminary section two, where we set up the notations and give the necessary definitions. In the third section, we give a description of idempotents of $M_{cb}A(G)$ of norm one. This result generalizes Theorem 2.1 of [17] which describes the idempotents of $B(G)$ of norm one. In section four, we present a necessary condition for an idempotent to be in $M_{cb}A(G)$. This condition provides a quick way of showing that certain idempotents are not completely bounded. In the last section, we discuss a class of free sets, called L-sets, whose characteristic function is an idempotent of $M_{cb}A(G)$, but not of $B(G)$. The main result of the section improves the well known theorem of Leinert (Satz 1, [21]) concerning a special class of lacunary sets, the Leinert sets.

2. Preliminaries

Our standard reference for details on operator spaces and completely bounded maps is the book of Effros and Ruan ([6]). Let G be a locally compact group. Denote by $A(G)$ the *Fourier algebra* of G which consists of all coefficient functions

$$\varphi(s) = \langle \lambda(s)\xi \mid \eta\rangle \quad s \in G,\ \xi,\eta \in L^2(G)$$

of the left regular representation λ on $L^2(G)$. The multiplication in $A(G)$ is the pointwise multiplication (hence commutative) and its norm is given by

$$\|\varphi\|_{A(G)} = \inf\Big\{\|\xi\|\ \|\eta\| \mid \varphi(s) = \langle\lambda(s)\xi|\eta\rangle\Big\}.$$

The *Fourier-Stieltjes algebra* $B(G)$ is defined to be the space of all coefficient functions

$$\psi(s) = \langle\pi(s)\xi|\eta\rangle, \quad s \in G,\ \xi,\eta \in H_\pi,$$

where $\pi : G \to B(H_\pi)$ is a universal representation of G. It is known that $B(G)$ is a commutative algebra and its norm is given by

$$\|\psi\|_{B(G)} = \inf\Big\{\|\xi\|\ \|\eta\| \mid \psi(s) = \langle\pi(s)\xi|\eta\rangle, \\ \pi : G \to B(H_\pi) \text{ a unitary representation}\Big\}.$$

The Fourier algebra $A(G)$ can be isometrically identified with the predual of $VN(G)$ (the *group Von Neumann algebra*) and thus it has a canonical operator space structure which makes $A(G)$ a completely contractive Banach algebra.

A function $\varphi : G \to \mathbb{C}$ is called a *multiplier* of $A(G)$ if $\varphi.\psi \in A(G)$ for any $\psi \in A(G)$. Then, we consider the multiplication map $m_\varphi : A(G) \to A(G)$ defined as

$$m_\varphi(\psi) = \varphi.\psi$$

It is known by the closed graph theorem that $m_\varphi : A(G) \to A(G)$ is bounded on $A(G)$. We let $MA(G)$ be the space of all multipliers of $A(G)$, equipped with the norm $\|\varphi\|_{MA(G)} = \|m_\varphi\|$. A multiplier φ is called *completely bounded* if $\|m_\varphi\|_{cb} < \infty$. We let $M_{cb}A(G)$ be the space of all *completely bounded multipliers* of $A(G)$ equipped with the completely bounded norm.

We recall that $M_{cb}A(G)$ is a dual operator space, since it has a predual $Q(G)$. This predual space, $Q(G)$ is defined to be the completion of $L^1(G)$ under the norm induced from $M_{cb}A(G)^*$, i.e. we may define the $Q(G)$-norm:

$$\|f\|_{Q(G)} = \sup\left\{\left|\int_G f(s)\varphi(s)\,ds\right| : \varphi \in M_{cb}A(G),\ \|\varphi\|_{cb} \le 1\right\}$$

for any $f \in L^1(G)$ (see [5]). We obtain a canonical operator space norm on $Q(G)$ such that $M_{cb}A(G)$ is completely isometric to the operator dual of $Q(G)$ (see [13] and [20]). Let $k_* : L^1(G) \to L^1(G)$ be the isometric anti-isomorphism defined by

$$k_*(f)(t) = f(t^{-1})\Delta(t^{-1}).$$

It has been shown in [24] that there is a complete quotient map

$$m_{id\otimes k_*} : L^1(G) \overset{h}{\otimes} L^1(G) \twoheadrightarrow Q(G) \tag{2.1}$$

such that $m_{id\otimes k_*}(f\otimes g) = f * k_*(g)$.

Using an (unpublished) result of Gilbert [11], Bożejko and Fendler [2] showed that $M_{cb}A(G)$ is isometrically isomorphic to $B_2(G)$, the space of all *Herz-Schur multipliers* on G. This shows that a function $\varphi : G \to \mathbb{C}$ is contained in $M_{cb}A(G)$ with $\|\varphi\|_{M_{cb}A(G)} \leq 1$ if and only if there exist a Hilbert space K and two bounded maps $\alpha, \beta : G \to K$ such that

$$\varphi(s^{-1}t) = \langle \beta(t)|\alpha(s)\rangle = \alpha^*(s)\beta(t) \quad \forall\, s,\ t \in G \tag{2.2}$$

and

$$\sup_{s\in G}\left\{||\alpha(s)||_K\right\} \sup_{t\in G}\left\{||\beta(t)||_K\right\} \leq 1. \tag{2.3}$$

A short and elegant proof of this result can be found in [18] or [29]. Also, the matricial version of this result holds and can be found for instance, in [24, Lemma 4.1.]. The result can be proved by simply applying Jolissaint's representation argument to the normal completely bounded map

$$[M_{\varphi_{ij}}] : VN(G) \to M_n(VN(G)) \subseteq M_n\Big(B(L_2(G))\Big).$$

It was shown that if $\varphi_{ij} : G \to \mathbb{C}$ are functions on G, $1 \leq i, j \leq n$, then $[\varphi_{ij}] \in M_n(M_{cb}A(G))$ with $\|[\varphi_{ij}]\|_{M_n(M_{cb}A(G))} \leq 1$ if and only if there exist a Hilbert space K and bounded continuous maps $\alpha_i, \beta_j : G \to K$ such that

$$[\varphi_{ij}(s^{-1}t)] = \Big[\langle \beta_j(t)|\alpha_i(s)\rangle\Big] = [\alpha_i(s)^*\beta_j(t)] \quad \forall\, s,\ t \in G \tag{2.4}$$

and

$$\sup\left\{||[\alpha_1(s)\cdots\alpha_n(s)]||_{B(\mathbb{C}^n,K)}\right\} \sup\left\{||[\beta_1(t)\cdots\beta_n(t)]||_{B(\mathbb{C}^n,K)}\right\} \leq 1. \tag{2.5}$$

For any group G, we have that $B(G) \subseteq M_{cb}A(G) \subseteq MA(G)$ and the equality is attained when G is amenable. It is known that when G is the free group of at least two generators (hence non-amenable), the above inclusions are strict ([22], [1], [8]).

3. IDEMPOTENTS OF $M_{cb}A(G)$ OF NORM ONE

Before we present the main result of this section, we need some definitions and facts about completely bounded multipliers when G is a discrete group. Let $U = [U(s,t)]$ and $V = [V(s,t)]$ be two (infinite) matrices indexed by G. Then we can define the *Schur multiplication* of U and V by $[U\cdot V(s,t)] = [U(s,t)\cdot V(s,t)]$ and we call a function $V : G\times G \to \mathbb{C}$ a *Schur multiplier* if there exists $\alpha > 0$ such that $\|V\cdot k\| \leq \alpha\|k\|$ for any finite matrix k. Here, the norm $\|\cdot\|$ is the operator norm on $B(l^2(G))$. We denote by $|||\cdot|||$ the Schur multiplier norm defined as follows:

$$|||V||| = \sup\left\{\frac{\|V\cdot k\|}{\|k\|} \mid k \text{ a finite matrix}\right\}.$$

Given a function $\varphi : G \to \mathbb{C}$, we define $M_\varphi : G\times G \to \mathbb{C}$ by $M_\varphi(s,t) = \varphi(s^{-1}t)$ and we can regard $M_\varphi = [M_\varphi(s,t)]$ as an (infinite) matrix indexed by G. From [2], we see that $\varphi \in M_{cb}A(G)$ if and only if $M_\varphi : G\times G \to \mathbb{C}$ is a Schur multiplier. Then the norm is given by

$$\|\varphi\|_{M_{cb}A(G)} = \sup\left\{\frac{\|M_\varphi\cdot k\|}{\|k\|} \mid k \text{ a finite matrix}\right\}.$$

From the above definition of the norm, it is easy to see that if $0 \neq \varphi \in M_{cb}A(G)$ is an idempotent, then $\|\varphi\|_{M_{cb}A(G)} \geq 1$.

We define a *coset* to be any subset C of G for which there exist an element s in G and a subgroup H of G such that $C = sH$. We let $\Omega(G)$ denote the *open coset ring*, which is the smallest ring of subsets which contains every open coset. In other words, it contains all the complements, the finite unions and intersections of open cosets. It has been shown by Gilbert [28] and Schreiber [10] in the abelian case and Forrest [9] in the non-abelian case that the general form of an element in the coset ring is the following

$$\bigcup_{i=1}^{n}\left(a_iH_i \setminus \bigcup_{j=1}^{m_i} b_{i,j}K_{i,j}\right)$$

where H_i, $K_{i,j}$ are subgroups in G and $a_i, b_{i,j} \in G, m_i, n$ are positive integers.

The beautiful characterization of idempotents of $B(G)$ due to Cohen [4] in the abelian case and Host [16] in the general case states:

Theorem 3.1. *The idempotents of $B(G)$ are the characteristic functions of sets in the open coset ring. In short*

$$\chi_A \in B(G) \Longleftrightarrow A \in \Omega(G).$$

Moreover, we have the following result due to Ilie and Spronk [17] which states:

Theorem 3.2. *The idempotents of $B(G)$ of norm 1 are the characteristic functions of open cosets. In short*

$$\chi_A \in B(G) \Longleftrightarrow A \ \text{ is a coset}.$$

Now we are ready to give the characterization of the idempotents of $M_{cb}A(G)$ of norm one. This result generalizes Theorem 3.2 to the case of completely bounded multipliers of G.

Theorem 3.3. *Let G be a locally compact group and $A \subseteq G$. Then the following are equivalent:*

(i) *A is an open coset in G.*
(ii) $\|\chi_A\|_{M_{cb}A(G)} = 1$
(iii) $\|\chi_A\|_{M_{cb}A(G)} < 2/\sqrt{3}$

Proof. The implication (ii) $\Rightarrow$ (iii) is obvious. One needs to show (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i).

The implication (i) $\Rightarrow$ (ii) follows easily from Theorem 3.2. If A is an open coset, then $\chi_A \in B(G) \subseteq M_{cb}A(G)$. Now, it is well known that $\|\chi_A\|_{M_{cb}A(G)} \leq \|\chi_A\|_{B(G)} = 1$, which implies that $\|\chi_A\|_{M_{cb}A(G)} = 1$.

To prove (iii) $\Rightarrow$ (i) we first assume that G is a discrete group.

Let us start by noting that $\left\| \left| \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right| \right\| \geq 2/\sqrt{3}$. To see this, we take $k = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$. Then $\|k\| = \sqrt{3}$ and $\left\| \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \right\| = 2$, which implies that

$$\left\| \left| \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right| \right\| \geq \frac{1}{\sqrt{3}} \left\| \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \right\| = \frac{1}{\sqrt{3}} \left\| \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \right\| = \frac{2}{\sqrt{3}}.$$

This was first noticed by Livshits [23] and then later proved by Katavolos and Paulsen [19] that the norm is exactly $2/\sqrt{3}$. In the same way we can prove that

$$\left\| \left| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right| \right\| \geq \frac{2}{\sqrt{3}}.$$

Now, let's suppose that $A \subseteq G$ such that $\|\chi_A\|_{M_{cb}A(G)} < 2/\sqrt{3}$. First assume that $e \in A$. We will show that A must be a subgroup of G. To see this, note that $M\chi_A$ is the infinite matrix indexed by G, say $[a(s,t)]_{G\times G}$, which is defined as follows:

$$a(s,t) = \chi_A(s^{-1}t) = \begin{cases} 1, & \text{if } s^{-1}t \in A; \\ 0, & \text{if } s^{-1}t \notin A. \end{cases}$$

Let $s \in A$. Then it is clear that $a(e,e) = a(e,s) = a(s,s) = 1$. Now, if $a(s,e) = 0$, then we created a sub-matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $M\chi_A$ which has norm bigger than or equal to $2/\sqrt{3}$. This would imply that $\|\chi_A\|_{M_{cb}A(G)} \geq 2/\sqrt{3}$ and that is a contradiction of the hypothesis. Hence $a(s,e) = 1$, which means that $s^{-1} \in A$.

Let s, $t \in A$. Then, $a(e,e) = a(e,t) = a(s,e) = 1$. Now, if $a(s,t) = 0$, then using the sub-matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and an argument similar with the one above we conclude that $a(s,t) = 1$, which means $st \in A$.

In conclusion A is a subgroup of G.

To complete the proof, let's suppose that $A \subseteq G$ such that $\|\chi_A\|_{M_{cb}A(G)} < 2/\sqrt{3}$, but $e \notin A$. Fix an element $s \in A$ and let $B = s^{-1}A$. It is obvious that

$e \in B$ and we also note that $\|\chi_B\|_{M_{cb}A(G)} = \|\chi_A\|_{M_{cb}A(G)} < 2/\sqrt{3}$. Then, by the argument above, B is a subgroup, which implies that A is a coset.

Let now G be a locally compact group and G_d be the group G endowed with the discrete topology. The fact that the preceding result holds for any G follows easily from Corollary 6.3 of [29] which says that the inclusion $M_{cb}A(G) \hookrightarrow M_{cb}A(G_d)$ is a complete isometry.

$\square$

4. A necessary condition for an idempotent to be in $M_{cb}A(G)$

We start this section with some functorial properties of $M_{cb}A(G)$. The following Proposition is similar to the one proven by Nico Spronk in [29, Cor. 6.3] and uses an idea of Bożejko and Fendler's [3, Lemma 1.2]:

Proposition 4.1. *If H is an open subgroup of G, then the map which sends φ in $M_{cb}A(H)$ to the function $\tilde{\varphi}$ in $M_{cb}A(G)$ that takes values of φ on H and 0 otherwise, is a complete isometry. Also, the restriction map $\varphi \mapsto \varphi|_H$ is a complete quotient from $M_{cb}A(G)$ onto $M_{cb}A(H)$.*

Proof. It is known from [24] that the map $m_{id\otimes k_*} : L^1(G) \overset{h}{\otimes} L^1(G) \twoheadrightarrow Q(G)$ given by (2.1) is a complete quotient. Recall the fact that if H is an open subgroup of G, then we have an isometric injection of $L^1(H)$ into $L^1(G)$. Since the maximum matrix norm is the canonical matrix norm on L^1, we deduce that $L^1(H)$ injects completely isometrically into $L^1(G)$. Now using the fact that the Haagerup tensor product is injective, we have the following commutative diagram

$$\begin{array}{ccc} L^1(H) \overset{h}{\otimes} L^1(H) & \xrightarrow{\text{c. isometry}} & L^1(G) \overset{h}{\otimes} L^1(G) \\ \text{c. quotient}\downarrow & & \text{c. quotient}\downarrow \\ Q(H) & \xrightarrow{T} & Q(G). \end{array}$$

Here T maps

$$\varphi \mapsto \tilde{\varphi}$$

where

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & x \in H \\ 0, & x \notin H. \end{cases}$$

From the diagram it is clear that T is a complete contraction. One can easily check that the adjoint map $T^* : M_{cb}A(G) \longrightarrow M_{cb}A(H)$ is the restriction map which sends $\varphi \mapsto \varphi|_H$, hence T^* is also a complete contraction.

Next, we claim that the map from $M_{cb}A(H)$ to $M_{cb}A(G)$ which sends $\varphi \mapsto \tilde{\varphi}$ is a complete contractive injection. Let $\varphi \in M_{cb}A(H)$. It follows that there exist a Hilbert space K and bounded and continous maps $\alpha, \beta : H \longrightarrow K$ such that $\varphi(s^{-1}t) = \langle \beta(t)|\alpha(s)\rangle$. (see relations (2.1) and (2.2)). Now, since H is open (hence also closed) in G, we deduce that $G = \bigcup_{x\in E} xH$ where E is a coset representative of G. Let $\{e_x\}_{x\in E}$ be an orthonormal basis for $l^2(E)$.

Define $\tilde{\alpha}, \tilde{\beta} : G \longrightarrow l^2(E) \otimes K$ by:

$$\tilde{\alpha}(s) = e_x \otimes \alpha(s_0) \quad \text{if } s \in xH \text{ and } s_0 = x^{-1}s,,$$
$$\tilde{\beta}(t) = e_x \otimes \beta(t_0) \quad \text{if } t \in xH \text{ and } t_0 = x^{-1}t$$

It is easy now to check that

$$\langle \tilde{\beta}(t)|\tilde{\alpha}(s)\rangle = \begin{cases} \langle e_x \otimes \beta(t_0)|e_x \otimes \alpha(s_0)\rangle, & s = xs_0,\ t = xt_0,\ s_0, t_0 \in H \\ \langle e_x \otimes \beta(t_0)|e_y \otimes \alpha(s_0)\rangle, & s = xs_0,\ t = yt_0,\ s_0, t_0 \in H \end{cases}$$
$$= \begin{cases} \varphi(s^{-1}t), & s^{-1}t \in H \\ 0, & s^{-1}t \notin H \end{cases}$$
$$= \tilde{\varphi}(s^{-1}t).$$

This proves the claim that the map $\varphi \mapsto \tilde{\varphi}$ is a contraction. To prove that the map is a complete contraction we use the matricial form of the representation theorem (see relations (2.4) and (2.5)) and a similar argument as above.

Since the composition of the two completely contractive maps

$$M_{cb}A(H) \longrightarrow M_{cb}A(G) \longrightarrow M_{cb}A(H)$$
$$\varphi \mapsto \tilde{\varphi} \qquad \varphi \mapsto \varphi|_H$$

is the identity map we conclude that $\varphi \mapsto \tilde{\varphi}$ is a complete isometry and $\varphi \mapsto \varphi|_H$ is a complete quotient, which completes the proof of the proposition. □

Combining Proposition 4.1 and Theorem 3.1, we can derive a necessary condition for an idempotent to belong to $M_{cb}A(G)$.

Corollary 4.2. *If $A \subseteq G$ such that there exists an open amenable subgroup H of G with $A \cap H$ not in the coset ring, then $\chi_A \notin M_{cb}A(G)$.*

Proof. Assume that $\chi_A \in M_{cb}A(G)$. From Proposition 4.1 we see that $\chi_A|_H \in M_{cb}A(H)$ for any open subgroup H of G, which means that $\chi_{A\cap H} \in M_{cb}A(H)$. Since H is amenable, we have $M_{cb}A(H) = B(H)$. Therefore by Theorem 3.1, $\chi_{A\cap H} \in B(H)$ and hence $A \cap H$ is in the coset ring. This completes the proof of the corollary. □

Example 4.3. Let $\mathbb{F}_2$ be the free group on two generators, say a and b. Then the characteristic functions of the following subsets are not in $M_{cb}A(G)$.

(i) Let $S = \{e, a, a^2, a^3, \ldots\}$; then we have that $\chi_S \notin M_{cb}A(G)$. To see this, choose the abelian subgroup $Z = \langle a \rangle$ and note that $S \cap Z = S$ is not in the coset ring of Z since it is isomorphic to $\mathbb{N}$. In the same way we can prove that the characteristic function of the subset of $\mathbb{F}_2$ consisting of all the words starting with the letter a is not a completely bounded multiplier.

(ii) If T is the set of words in $\mathbb{F}_2$ starting with either a or a^{-1}, then $\chi_T \notin M_{cb}A(G)$. To see this, let $Z = \langle ab \rangle$ and note that $T \cap Z = \{ab, (ab)^2, (ab)^3, \ldots\}$ which is not in the coset ring of Z since it is isomorphic to $\mathbb{N}$.

(iii) Let $F_2^+ = \{a^n b^m | n, m \in \mathbb{N}\}$; then $\chi_{F_2^+} \notin M_{cb}A(G)$.

5. Leinert sets

Note that if A is in the open coset ring, then $\chi_A \in B(G) \subseteq M_{cb}A(G)$. The natural question would be whether there are other sets which are not in the coset ring but whose characteristic function is a completely bounded multiplier. When G is discrete and non-amenable, the answer is positive, since we have a "lacunary" type of sets called *L-sets*, that satisfy the property. These sets are not in the coset ring unless they are finite. They were first introduced by Bożejko [1] following the definition of a *Leinert set* introduced by Leinert in early 1970's.

Throughout this section G will be a discrete group. We start with the definitions of the above terms.

Definition 5.1. (1) A set $\Lambda \subseteq G$ is called a *Leinert set* if $l^\infty(\Lambda) \subseteq MA(G)$;

(2) A set $\Lambda \subseteq G$ is called an *L-set (strong 2-Leinert set)* if $l^\infty(\Lambda) \subseteq M_{cb}A(G)$;

(3) A set $\Lambda \subseteq G$ satisfies the *Leinert condition* if for all $n \in \mathbb{N}$ and for all $\{x_i\}_{i=1}^{2n} \in \Lambda$ with $x_i \neq x_{i+1}$ we have that

$$x_1 x_2^{-1} x_3 x_4^{-1} \cdots x_{2n-1} x_{2n}^{-1} \neq e.$$

Example 5.2. Let $\mathbb{F}_\infty$ be the free group of (countably) infinitely many generators.

(1) Denote by $E = \{x_1, x_2, \ldots\}$, the set of all generators of $\mathbb{F}_\infty$. It is well-known that E is an L-set and also it satisfies Leinert condition. ([14] and [21]).

(2) Denote by $E^2 = \{x_1^2, x_1 x_2, \ldots\}$. Then E^2 is a Leinert set (Haagerup [12]), but not an L-set (Bozejko [1]) and also doesn't satisfy the Leinert condition.

Remark 5.3. Both χ_E and χ_{E^2} are idempotents of $M_{cb}A(G)$.

Remark 5.4. If a discrete group admits an infinite Leinert set, then the group is non-amenable.

It is easy to see that if $\Lambda \subseteq G$ is an L-set, this implies Λ is also a Leinert set. In [21], Leinert showed that any set in G that satisfies the Leinert condition is a Leinert set. Using Pisier's theorem (Theorem 3.3 in [26]) and a combinatorial approach we can prove a stronger version of his result. This is done in Theorem 5.11. Before stating the theorem, we need the following definitions and properties concerning the so-called *chainable* matrices. These results (Properties 5.6 to 5.8) along with their proofs can be found in Section 5.3.1 of [27], but we expose them here for the convenience of the reader.

Definition 5.5. A matrix $A = [a_{ij}]$, $i = 1, \dots, m$ and $j = 1, \dots, n$ with entries 0 and 1 is called:

1. *chainable* if
   (a) it has no zero rows or columns, and
   (b) for any pair of elements $a_{rt} = 1$ and $a_{pq} = 1$, there exists a sequence of elements $a_{i_1 j_1} = a_{i_2 j_2} = \dots = a_{i_s j_s} = 1$ such that $i_1 = r$, $j_1 = t$, $i_s = p$, $j_s = q$ and $i_k = i_{k+1}$ or $j_k = j_{k+1}$ for $k = 1, \dots, s-1$.

   This sequence of elements is called a *chain*. A closed chain is called a *cycle*.
2. *minimal chainable* if A is chainable and if for any i and j for which $a_{ij} = 1$ the matrix $A - E_{ij}$ is not chainable. Here E_{ij} stands for the matrix with the only 1 located at the ith row and jth column and the rest are 0's.

Property 5.6. *If an $m \times n$ matrix A with only 0 and 1 entries has neither zero rows nor zero columns, then there exist permutation matrices P and P_1 such that*

$$PAP_1 = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_N \end{bmatrix}$$

where $A_1, A_2, \dots, A_N$ are chainable matrices.

Property 5.7. *Let a matrix $A = [a_{ij}]$, $i = 1, \dots, m$ and $j = 1, \dots, n$ with 0 and 1 entries be chainable. Then the matrix A is minimal chainable if and only if it has exactly $m + n - 1$ of 1's.*

Property 5.8. *A matrix $A = [a_{ij}]$, $i = 1, \dots, n$ and $j = 1, \dots, n$ with 0 and 1 entries is minimal chainable if and only if it has no cycles and the matrix $A + E_{ij}$ has a cycle for each (ij) such that $a_{ij} = 0$.*

Proposition 5.9. *Let A be a $n \times n$ matrix with 0 and 1 entries and exactly $2n$ of 1's. Then A has a cycle.*

Proof. Let's first suppose that A doesn't have any zero rows or columns. Then, by Property 5.6 there exist permutation matrices P and P_1 such that

$$PAP^{-1} = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_N \end{bmatrix}$$

where the matrices $A_1, \ldots, A_N$ are chainable. Suppose now that A_i, $i = 1, \ldots, n$ have dimensions $r_i \times s_i$ with $\sum_{i=1}^{N} r_i = n$ and $\sum_{i=1}^{N} s_i = n$. Next, we show that among these matrices there exists an A_{i_0} which has at least $r_{i_0} + s_{i_0}$ of 1's. Suppose this is not true, i.e. all A_i's have at most $r_i + s_i - 1$ of 1's. Then the total number of 1's in the matrix A would be $\sum_{i=1}^{N}(r_i + s_i - 1) = 2n - N < 2n$, since $N \geq 1$. This is a contradiction with the hypothesis, hence, there exists at least one chainable matrix, say A_{i_0} of dimension $r_{i_0} \times s_{i_0}$ with at least $r_{i_0} + s_{i_0}$ of 1's. Without loss of generality, we can assume that A_{i_0} has exactly $r_{i_0} + s_{i_0}$ of 1's. By Property 5.7 the matrix A_{i_0} is not minimal chainable, hence by definition, there exists $a_{kl} = 1$ such that $A_{i_0} - E_{kl}$ is chainable. Now, Property 5.7 implies that the matrix $A_{i_0} - E_{kl}$ is minimal chainable since it has exactly $r_{i_0} + s_{i_0} - 1$ of 1's.

It follows from Property 5.8 that $(A_{i_0} - E_{kl}) + E_{kl} = A_{i_0}$ has a cycle. In conclusion, A has a cycle.

Let's suppose now that A has zero rows and/or columns. In this case, we eliminate all the zero rows and columns and we obtain a matrix of dimension $r \times s$, with $r, s \leq n$ and $2n$ of 1's. Observe that $r + s \leq 2n$ since A has at least one zero row or column. By the same argument as the preceding one, we obtain that the matrix A has a cycle and the Lemma follows. $\square$

Remark 5.10. If the matrix A of Proposition 5.9 has more than $2n$ of 1's, then the conclusion still holds.

Theorem 5.11. *Let G be a discrete group and $A \subseteq G$ a set satisfying the Leinert condition. Then A is an L-set.*

Proof. Let $A \subseteq G$ be a subset satisfying the Leinert condition. Let

$$R_A = \{(s,t) \in G \times G \mid s^{-1}t \in A\}$$

Suppose A is not an L-set. Then, by Pisier's theorem (Thm. 3.3, [26]), for all $C > 0$ there exist two finite sets E and F in G with $|E| = |F| = N$ such that

$$|R_A \cap (E \times F)| > CN.$$

Take $C = 2$. Then there exists $E, F \subseteq G$ with $|E| = |F| = N$ such that:

$$|R_A \cap (E \times F)| > 2N$$

Note first that $2 < N$ since $2N < |R_A \cap (E \times F)| \leq |E \times F| = N^2$. In this way we've constructed an $N \times N$ matrix, denoted by $[a_{s,t}]_{E \times F}$ such that:

$$a_{s,t} = \begin{cases} 1, & \text{if } s^{-1}t \in A\,; \\ 0, & \text{if } s^{-1}t \notin A. \end{cases}$$

Since $|R_A \cap (E \times F)| > 2N$, it follows that $[a_{s,t}]$ is a $N \times N$ matrix with 0 and 1 entries, which has at least $2N$ of 1's. By Proposition 5.9 and the above remark, we can find a cycle. Suppose this cycle is $a_{s_1,t_1}, a_{s_2,t_1}, a_{s_2,t_2} \dots a_{s_n,t_n}, a_{s_1,t_n}$. Since these elements are all equal to 1, then $s_1^{-1}t_1, s_2^{-1}t_1, \dots, s_1^{-1}t_n \in A$. Also note that

$$s_1^{-1}t_1(s_2^{-1}t_1)^{-1}s_2^{-1}t_2 \dots s_n^{-1}t_n(s_1^{-1}t_n)^{-1} = e.$$

Therefore the set A doesn't satisfy the Leinert condition, which proves our theorem. □

Remark 5.12. Note that the converse of the above theorem is not true. To see that, consider the free group $\mathbb{F}_\infty$ of infinitely many generators and let $E = \{x_1, x_2, \dots\}$ be the set of all its generators. Let $E_n^2 = \bigcup_{i=1}^n x_i E$. It is clear that E_n^2 doesn't satisfy the Leinert condition, however it is an L-set, since it is a finite union of translations of an L-set.

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