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On ill-conditioning in 3D Similarity Transformation

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Abstract

3D similarity transformations are often used for datum transformation in Geomatic Engineering. The transformations are routinely performed between the point coordinates evaluated from GNSS (Global Navigation Satellite Systems) observations and the point coordinates given in national datum. A mathematical model established among two different 3D coordinate systems contains tree type datum parameters which are translations, rotations and a scale. Coefficients of the parameters reflect large variations from each other's. So, the unknown coefficient matrix in normal equations is to be ill-conditioned. In this study, it is indicated how the ill-condition structure is reduced to acceptable level by means of choosing at different unit of unknown parameters. Finally, the transformation from Kocaeli Metropolitan Municipality GNSS network coordinates to Turkish Geodetic Datum is used as a numerical example.

Keywords: 3D similarity transformation, linear equation system, ill-conditioning

Introduction

Similarity transformations in 2D and 3D are often used for datum transformation in Geomatic Engineering. A mathematical model established among two different 2D or 3D coordinate systems contains tree type datum parameters which are translations, rotations and a scale. The transformation parameters are always chosen as the unknown parameters in a mixed or indirect adjustment models. While the coordinates given in two different orthogonal systems are used as the observations in the mixed, only the transformed coordinates are used in the indirect model. Modeling the mixed adjustment model in 2D and 3D similarity transformations can be found Öztürk and Şerbetçi (1992) and Leick (1995) in detail.

In the study, the indirect model is used because of its simplicity. So, the 3D similarity transformations in the indirect adjustment model are given step by step, explicitly (Kurt, 2007; Hofmann-Wellenhof et al, 2008). Ill-conditioning in the 3D similarity transformations are argued using determinant, spectral and Hadamart condition numbers (Öztürk, 1991; Press et al, 2002). For this propose, 3D similarity transformation software is developed by the author in C++ platform.
Using the software, the theoretical deductions are numerically inspected on the transformation between Kocaeli Metropolitan Municipality GNSS Network (KMM-GN) coordinates and Turkish National Datum (Kurt, 2010). All stages shortly mentioned above are explained following lines in detail.

3D Similarity Transformation Mathematical Models and Their Solutions

3D similarity transformations are usually performed between terrestrial datums, for example from WGS84 to ITRFXX or vice versa). The unknown parameters of these type transformations take small values; some meters (or decimeters), around of zero and 1 for translation parameters
($\mathbf{t} = [a \quad b \quad c]^T$, $\mathbf{t}_0 = [a_0 \quad b_0 \quad c_0]^T$),
rotation parameters
($\boldsymbol{\alpha} = [\alpha \quad \beta \quad \gamma]^T \approx \mathbf{0}^T$) and
a scale parameter
($k \approx 1$) respectively (Figure 1).
Functional model of the transformation is easily established from *Figure 1* by using vector

notations. In the paper, the vector $\mathbf{u}=[u \;\; v \;\; w]^T$ as first (or transforming) coordinate system and the vector $\mathbf{x}=[x \;\; y \;\; z]^T$ as second (or transformed) coordinate system are used. The only two functional models of the transformation are to be designated by using a point (*j*) coordinates known in the both systems as $(\mathbf{u}_j, \mathbf{x}_j)$ (*Figure 1*).

$$\mathbf{x}_j = \mathbf{t} + k\,\mathbf{R}\,\mathbf{u}_j \tag{1}$$

$$\mathbf{x}_j = \mathbf{t}_0 + k\,\mathbf{R}(\mathbf{u}_j - \mathbf{u}_0) \tag{2}$$

$$\mathbf{t}_0 = \mathbf{t} + k\,\mathbf{R}\,\mathbf{u}_0 \tag{3}$$

where the translation vector as $\mathbf{t}$, rotation matrix as $\mathbf{R}$, scale factor as $k = 1 + \mu$, a position vector in the first system ($\mathbf{u}$) as $\mathbf{u}_0$ (to likely be the centroid), the translation vector of $\mathbf{u}_0$ from the origin of second system as $\mathbf{t}_0$ are identified. In the study, Eq.1 and Eq.2 are called as *model-1* and *model-2* in the transformations in order and discussions the condition number are made upon the both models.

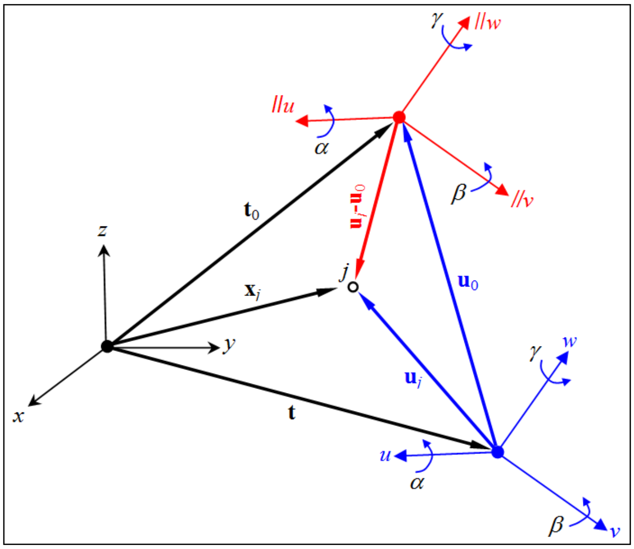

Figure 1: Geometric structures of the two of 3D similarity transformations.
(The blue {and red shifted from blue} as first or transforming system, the black as second or transformed system)

The rotation matrix comprises three orthogonal matrixes around axes of the first coordinate system. The rotation angles (*α*, *β*, *γ*) are positive clockwise as viewed from origin to positive direction around *u*, *v*, *w* axes, which are first, second and third axes, respectively.

Since $\alpha \approx \beta \approx \gamma \approx 0$, the cosines and sinuses of the angles are roughly equal to *1* and *radian* of the angle. Reducing the rotation matrix is obtained as Eq.4.

$$\mathbf{R} = \mathbf{R}_1(\alpha)\,\mathbf{R}_2(\beta)\,\mathbf{R}_3(\gamma) = \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix} \tag{4}$$

By splitting the rotation matrix into two parts, we can write the rotation matrix as combination of $\mathbf{Q}$ and identity ($\mathbf{I}$) matrices.

$$\mathbf{R} = \mathbf{Q} + \mathbf{I} \tag{5a}$$

$$\mathbf{Q} = \begin{bmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & \alpha \\ \beta & -\alpha & 0 \end{bmatrix} \tag{5b}$$

By substituting Eq.5 into Eq.1-2 and using $k = 1 + \mu$ and neglecting $\mu\ \mathbf{Q} \approx \mathbf{0}$, the new forms of Eq.1-2 are acquired as following equations.

$$\mathbf{x}_j - \mathbf{u}_j = \mathbf{t} + \mathbf{Q}\,\mathbf{u}_j + \mu\,\mathbf{u}_j \tag{6}$$

$$\mathbf{x}_j - \mathbf{u}_j = \mathbf{t}_0 - \mathbf{u}_0 + \mathbf{Q}\,(\mathbf{u}_j - \mathbf{u}_0) + \mu\,(\mathbf{u}_j - \mathbf{u}_0) \tag{7}$$

$$\tilde{\mathbf{t}}_0 = \mathbf{t}_0 - \mathbf{u}_0 \tag{8}$$

Arranging Eq.6-7 according to $\mathbf{Q}\,\mathbf{u}_j = \mathbf{D}_j\ \boldsymbol{\alpha}$ and $\mathbf{Q}\,(\mathbf{u}_j - \mathbf{u}_0) = \tilde{\mathbf{D}}_j\ \boldsymbol{\alpha}$, we can obtain the following linear models.

$$\mathbf{x}_j - \mathbf{u}_j = \mathbf{t} + \mathbf{D}_j\ \boldsymbol{\alpha} + \mu\,\mathbf{u}_j \tag{9}$$

$$\mathbf{x}_j - \mathbf{u}_j = \tilde{\mathbf{t}}_0 + \tilde{\mathbf{D}}_j\ \boldsymbol{\alpha} + \mu\,\tilde{\mathbf{u}}_j \tag{10}$$

$$\mathbf{D}_j = \begin{bmatrix} 0 & -w_j & v_j \\ w_j & 0 & -u_j \\ -v_j & u_j & 0 \end{bmatrix},\ \tilde{\mathbf{D}}_j = \begin{bmatrix} 0 & -\tilde{w}_j & \tilde{v}_j \\ \tilde{w}_j & 0 & -\tilde{u}_j \\ -\tilde{v}_j & \tilde{u}_j & 0 \end{bmatrix},\ \tilde{\mathbf{u}}_j = \mathbf{u}_j - \mathbf{u}_0 = \begin{bmatrix} \tilde{u}_j \\ \tilde{v}_j \\ \tilde{w}_j \end{bmatrix} = \begin{bmatrix} u_j - u_0 \\ v_j - v_0 \\ w_j - w_0 \end{bmatrix} \tag{11}$$

If Eq.9 and 10 are arranged with respect to the transformation parameters again, the functional model of 3D similarity transformations are derived for solving the unknowns by means of common points given the both coordinate systems (Öztürk and Şerbetçi, 1992; Leick, 1995; Kurt, 2007; Hofmann-Welenhof et al, 2008).

$$j = 1,2,\ldots,p$$

$$\boldsymbol{\Delta}_j = \begin{bmatrix} \mathbf{I} & \mathbf{D}_j & \mathbf{u}_j \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \boldsymbol{\alpha} \\ \mu \end{bmatrix} \qquad \textit{Model 1} \tag{12}$$

$$\boldsymbol{\Delta}_j = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{D}}_j & \tilde{\mathbf{u}}_j \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{t}}_0 \\ \boldsymbol{\alpha} \\ \mu \end{bmatrix} \qquad \textit{Model 2} \qquad \mathbf{t} = \mathbf{u}_0 + \tilde{\mathbf{t}}_0 - k\,\mathbf{R}\,\mathbf{u}_0 \tag{13}$$

$$\boldsymbol{\Delta}_j = \begin{bmatrix} \mathbf{I} & c_{asec}\mathbf{D}_j & c_{ppm}\mathbf{u}_j \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \boldsymbol{\alpha}_{asec} \\ \mu_{ppm} \end{bmatrix} \qquad \textit{Model 3} \qquad \boldsymbol{\delta} = \begin{bmatrix} \mathbf{t} & \boldsymbol{\alpha} & \mu \end{bmatrix}^T \tag{14}$$

$$\boldsymbol{\Delta}_j = \begin{bmatrix} \mathbf{I} & c_{asec}\tilde{\mathbf{D}}_j & c_{ppm}\tilde{\mathbf{u}}_j \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{t}}_0 \\ \boldsymbol{\alpha}_{asec} \\ \mu_{ppm} \end{bmatrix} \qquad \textit{Model 4} \qquad \boldsymbol{\delta} = \begin{bmatrix} \tilde{\mathbf{t}}_0 & \boldsymbol{\alpha} & \mu \end{bmatrix}^T \tag{15}$$

$$\mathbf{A}_j\boldsymbol{\delta} = \boldsymbol{\Delta}_j \qquad \boldsymbol{\Delta}_j = \mathbf{x}_j - \mathbf{u}_j \tag{16}$$

Where $c_{a\,sec} = \pi \times 648e-3$ and $c_{ppm} = 1e-6$. Using least squares solution to the combined linear model including the coefficient matrixes of all common points (*j*), it can be obtained following equations to estimate the transformation parameters and their variance –covariance matrix.

$$\hat{\boldsymbol{\delta}} = \mathbf{N}^{-1} \sum_{j=1}^{p} \mathbf{A}_j^T \boldsymbol{\Delta}_j \qquad \mathbf{N} = \sum_{j=1}^{p} \mathbf{A}_j^T \mathbf{A}_j \tag{17a}$$

$$\boldsymbol{\Sigma}_{\hat{\boldsymbol{\delta}}} = \hat{\sigma}_0^2 \, \mathbf{N}^{-1} \qquad \hat{\sigma}_0^2 = \frac{\sum_{j=1}^{p} \hat{\varepsilon}_j^T \hat{\varepsilon}_j}{3p-7} \tag{17b}$$

$$\hat{\varepsilon}_j = \mathbf{A}_j \hat{\boldsymbol{\delta}} - \boldsymbol{\Delta}_j \tag{17c}$$

The common point number as p, the unknown parameters and coefficient matrix of corresponding model (1, 2, 3, 4) as $\boldsymbol{\delta}$ and $\mathbf{A}_j$, the right side vector as $\boldsymbol{\Delta}_j$, the norming elements of corresponding parameters as $c_{a\,sec}$ and c_{ppm}, the coefficient matrix of normal equations as $\mathbf{N}$ are used from Eq.12 to Eq.17.

Condition number of a matrix

Condition number is defined as a measure to determine ill-condition level of a linear equation system. In other words, the small variations on the observations (or right side vector) and approximate values of the unknowns cannot change the estimated parameters in the linear models. There are various methods to compute the number. Only two of them are used in the study. Those are spectral condition numbers (C_S) and Hadamard condition number (C_H). Taking a symmetric positive definite matrix as $\mathbf{N}_{u \times u} = (n_{jk})$ (being similar to a coefficient matrix in normal equation), the both numbers are calculated following formulas.

$$C_S = \frac{\lambda_{max}}{\lambda_{min}} < 1000 \qquad \mathbf{S}^T \mathbf{N} \mathbf{S} = \boldsymbol{\Lambda} \tag{18}$$

$$C_H = \frac{|det\,\mathbf{N}|}{\prod_{j=1}^{u} \theta_j} > 0.010 \qquad \theta_j = \sqrt{\sum_{k=1}^{u} n_{jk}^2} \tag{19}$$

Where, $\mathbf{S}$ is the eigenvector matrix and $\lambda_{max} = max(\boldsymbol{\Lambda})$, $\lambda_{min} = min(\boldsymbol{\Lambda})$ are extreme values of eigenvalues of the unknown coefficient matrix. Optimum number of the both is equal to *1*. The estimated parameters are very consistent if the condition numbers take their optimum values. From long term experiments, the parameters are accepted as consistent if $C_S < 1000$ or $C_H > 0.010$ (Öztürk, 1991; Press et al, 2002). In the study the both of them are used to measure condition level because their ill-condition levels are changed to different directions although their optimum values are the same.

Numerical Example

Kocaeli Metropolitan Municipality (KMM) gave out by contract to Kutlubey Harita (Ankara/Turkey) to make large scale base map by using numerical photogrammetric method. Technical controls with respect to the national specifications carried out by the author for the geodetic studies and by Ozan Arslan (Associate Professor in Kocaeli University, Geomatic Department) for photogrammetric studies. The first two of four technical repots were prepared for the horizontal and the vertical control studies. The numerical example for the scope is chosen from 1st Technical Report (Kurt, 2010). The report includes the controlling GNSS network adjustment and transformation of the network point coordinates to national datum and computation of point velocities with respect to the national specifications.

Table 1: KMM-GN10 Statistics (Kurt, 2010)

Names	***Symbols***	***Free Adjustment***	***Constraint Adjustment***
# of base	*b*	*6294*	*6038*
# of outlier	*o*	*256*	-
# of base	*b-o*	*6038*	*6038*
# of point	*p*	*1780*	*1780*
# of fixed point	*s*	*0*	*8*
# of observation	*n=3b*	*18114*	*18114*
# of unknown	*u=3(p-s)*	*5340*	*5316*
# of datum defect	*d*	*3*	*0*
degree of freedom	*f=n-u+d*	*12777*	*12798*
a priory RMS	σ_0	*±1.00 cm*	*±1.00 cm*
a posteriori RMS	$\hat{\sigma}_0$	***±0.85 cm***	***±0.96 cm***

In this report, the GNSS network (hereinafter referred as to KMM-GN10) constituted from 1780 points was adjusted as a free network at the epoch 2010.42 and then the adjusted coordinates in the network were transformed to the national datum by using 8 common points. The common points in the both (the free and national) datums were tested for any outliers by means of the 3D similarity transformation and it wasn't found any outlier. Correspondence of the common point coordinates in the national datum to the free network datum was inspected by using the constraint adjustment with the common points (*Table 1*).

In this paper, the 8 common point 3D similarity transformation shortly mentioned in the previous paragraph is chosen, and it is focused on the ill-conditioning problem in the transformation. Two transformations are carried out: the first of the two from the KMM-GN10 coordinates to the ITRF05 (ITRF at reference epoch 2005) in the *Figure 2a* and the second from the KMM-GN10 coordinates to the ITRF10 (ITRF at epoch 2010.42) in the *Figure 2b*.

Since the first transformation ($\hat{\sigma}_0$ =*±3.12cm*) gives an interesting result better than the second ($\hat{\sigma}_0$ =*±4.09cm*), it is included in the paper content (*Figure 2*). From the result, it can be noted that the point velocities estimated from the global network (the A-level network) are not identical the local velocities sufficiently

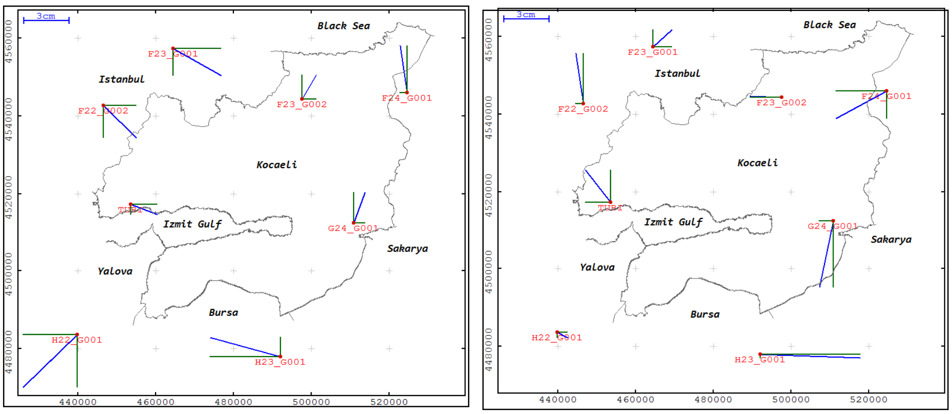

Figure 2: Transformations (a) between KMMGN10 and ITRF05 and (b) between KMMGN10 and ITRF2010

In Figure 2b, the H23_G001 point having the maximum errors of the transformations was removed from the common point set (*Figure 2*). Why the transformation carried out by remaining 7 common points did not significantly improve the transformation result ($\hat{\sigma}_0$ =*±3.62cm*), it was decided to use the all common point in the transformation (Kurt, 2010).

Table 2: The unknown parameter statistics and their condition numbers of the different transformation models (The bold numbers are acceptable level for consistent solution)

Parameters	*Symbols*	*Model 1 (Eq.12)*	*Model 2 (Eq.13)*	*Model 3 (Eq.14)*	*Model 4 (Eq.15)*
Determinant (!=0 and ~±1)	[**N**]	3.97e+042	3.97e+042	**5.16e-002**	**5.16e-002**
Spectral (<1.00e+3)	C_S	2.38e+018	1.67e+009	5.60e+007	**5.98e+002**
Hadamard (>1.00e-2)	C_H	5.12e-039	**6.93e-001**	8.07e-023	**6.93e-001**
# of common points	*p*	8	8	8	8
Degree of freedom	*f*	17	17	17	17
RMS	$\hat{\sigma}_0$ [cm]	±4.09cm	±4.09cm	±4.09cm	±4.09cm
Centroid of the first system	u_0 [m]	0.0000	4195618.9374	0.0000	4195618.9374
	v_0 [m]	0.0000	2397732.1678	0.0000	2397732.1678
	w_0 [m]	0.0000	4148951.1801	0.0000	4148951.1801
Translations (**t**)	*a* [m]	-3.8259 ±2.6878	-0.0960 ±0.0145	-3.8259 ±2.6878	-0.0960 ±0.0145
	b [m]	1.6322 ±3.1939	0.0636 ±0.0145	1.6322 ±3.1939	0.0636 ±0.0145
	c [m]	3.2100 ±2.9231	0.0242 ±0.0145	3.2100 ±2.9231	0.0242 ±0.0145
Rotations ($\boldsymbol{\alpha}$)	α [asec]	0.0467 ±0.0974	0.0467 ±0.0974	0.0467 ±0.0974	0.0467 ±0.0974
	β [asec]	-0.1232 ±0.0981	-0.1232 ±0.0981	-0.1232 ±0.0981	-0.1232 ±0.0981
	γ [asec]	0.1195 ±0.0902	0.1195 ±0.0902	0.1195 ±0.0902	0.1195 ±0.0902
Scale factor (μ)	μ [ppm]	-0.0328 ±0.3535	-0.0328 ±0.3535	-0.0328 ±0.3535	-0.0328 ±0.3535

Since the transformation between KMM-GN10 and ITRF2010 is convenient to the national specification and the statistical consistency of the 8 common constrained adjustments into the free adjustment is clearly followed in the *Table 1*, the 8 common point transformations from KMM-GN10 to ITRF2010 are selected for the paper content.

The normal equation coefficient matrixes of the four models are used for computing the condition number which demonstrates the coherence level of parameters to their linear model (*Table 2*). According to *Table 2*, the most consistent model is the Model 4. All condition numbers for the Model 4 are satisfied in their consistency threshold ranges. From the table, it is also viewed that the determinant of the Model 4 reached to 1, which is optimum value of the determinant.

The estimated unknowns for all models are same, because the both systems used in the transformation are very closed each other's geometrically. The closeness of the two systems is easily seen from the Model 1-3 columns in *Table 2*, e.g. their origin differences in *±3m* and rotation angles in *±0.1asec* and scale factor in *1e7*. If the both coordinate systems farther from each other, the ill-conditioning would cause deviations of the estimated parameters.

Conclusions

Spectral and Hadamart condition numbers and determinant are very sensitive measures to determine the ill-conditioning level of a linear model when all of them are used in together. Their same optimum values are equal to 1.

When the determinant is reached to zero, the coefficient matrix of normal equation is to reach the singularity. The singularity (det=0) can use to check whether the model is suitable or not to estimate unknown parameters. A greater absolute value of the determinant can indicate an ill-conditioning problem. After that, one should compute the two condition number defined previous paragraph to check ill-conditioning level of the normal equation

matrix because their ill-conditioning levels are change different directions, getting greater for spectral while getting smaller for Hadamart condition numbers.
In this contribution, it is proposed an easy way to overcome the ill-conditioning problem in 3D similarity transformation. The way is to change the unit of the rotation and scale parameters as *asec* (arc second) and *ppm* (per per million) respectively during the establishment of linear model. The suggestion is tested on the transformation between KMM-GN10 and national datum successfully as mentioned in the numerical example part of the paper.

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