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ON IMBEDDING DIFFERENTIABLE MANIFOLDS IN EUCLIDEAN SPACE*

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1. Introduction

Recently R. Penrose, J. H. C. Whitehead and E. C. Zeeman have obtained powerful results on rectilinear imbeddings of combinatorial manifolds in euclidean n-space R^n [4]. The purpose of the present paper is to show how their methods can be used to obtain information about differentiable imbeddings of smooth manifolds in R^n. We also use results due to J. H. C. Whitehead [6].

Our chief results are the following.

THEOREM 4.6. *A differentiable open n-manifold can be differentiably imbedded in* R^{2n-1}.

THEOREM 4.7. *A differentiable open n-manifold which is parallelizable can be immersed in* R^n.

THEOREM 4.1. *Let M be a closed differentiable n-manifold, and x a point of M. Assume M has a smooth triangulation. If M is* $(m-1)$*-connected,* $2 \leqq 2m \leqq n$*, and if a neighborhood of the* $(n-m)$*-skeleton can be immersed in* R^q, $q \geqq 2n-2m+1$*, then* $M - x$ *can be differentiably imbedded in* R^q.

The power of 4.1 comes from the fact that much is known concerning the immersibility of a neighborhood of a skeleton [1]. For example, if M is $m-1$ connected, a neighborhood of the $(n-m)$-skeleton is always immersible in R^{2n-m}; if M is a π-manifold, M is immersible in R^{n+1}. Some applications are given in 4.2 – 4.5.

The author is indebted to the late Professor J. H. C. Whitehead for many fruitful discussions of the ideas presented here.

2. Terminology

We assume familiarity with the concept of a *differentiable manifold*. Any such manifold possesses a *smooth triangulation* [7], making the

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manifold homeomorphic to a *combinatorial manifold*; for this last and related notions, see [4]. *Differentiable* means of class C^1. A *differentiable imbedding* $f: M \to N$ of one differentiable manifold in another is a differentiable homeomorphism of M onto a submanifold of N such that f^{-1} is differentiable. If, in addition $f(M) = N$, we say f is a *diffeomorphism*. An *immersion* of one differentitable manifold M in another is a differentiable map whose jacobian matrix has rank everywhere equal to dim M. We use R for the real line, R^n for euclidean n-space, R' for the non-negative ray. We consider R^{n-1} as contained in R^n in the usual way, and $R^n = R^{n-1} \times R$. We define *half-n-space* to be $R^{n-1} \times R' = F^n \subset R^n$. We give R^n and F^n their usual differentiable and combinatorial structures. If M is a combinatorial or differentiable manifold, we denote its boundary by ∂M. Thus $\partial F^n = R^{n-1}$. If M is a combinatorial manifold, an *n-element in M* is the image of an n-simplex in M under a piecewise linear imbedding. If M is a differentiable manifold, an *n-disk in M* is the image in M of the unit ball of R^n under a differentiable imbedding. The concepts of a *combinatorial half-n-space in M* and a *differentiable half-n-space in M* are similarly defined, with the restriction that the latter be a closed subset of M. A manifold is *closed* if it is compact and has no boundary. It is *open* if it is not compact and has no boundary. It is *bounded* if it has a boundary. We use int X for the interior of X, and cl (X) for the closure of X.

3. Combinatorial results and their differentiable implications

The main results of this section are 3.3 and 3.7.

THEOREM 3.1. *Let M be a closed $(m-1)$-connected combinatorial n-manifold, $2 \leqq 2m \leqq n$. If U is a neighborhood of the $n-m$ skeleton of rectilinear triangulation of M, there is an n-element E in M such that $M - \text{int } E \subset U$.*

PROOF. This is a corollary of Lemma 2.7 of [4].

THEOREM 3.2. *Let M be an open combinatorial n-manifold. There exists a subcomplex K of dimension $n-1$ with the following property. Let U be a neighborhood of K. There exists a collection $A_1, A_2, \cdots$, of mutually disjoint combinatorial half-n-spaces in M, such that $M - \bigcup \text{int } A_i$ is contained in U. There is a triangulation of M in which every A_i is a subcomplex.*

PROOF. This is essentially proved in the proof of theorem 3.2 of [6].

THEOREM 3.3. *Let M be a closed differentiable n-manifold which is $m-1$ connected, $2 \leqq 2m \leqq n$. Let U be a neighborhood of the $n-m$ skeleton* *of a smooth triangulation of M. There is an n-disk D in M such that $M - \text{int } D \subset U$.*

PROOF. By 3.1, there exists an n-element in M (considered as a combinatorial manifold) such that $M - \text{int } E \subset U$. The interior of E is an open subset of M and is therefore a differentiable manifold. Since E is an n-element, there is a piecewise linear homeomorphism $g: R^n \to \text{int } E$. Since R^n is contractible, by a result of J. Munkres [3, 6.6], there is a diffeomorphism $h: R^n \to \text{int } E$. Since E is compact, there is a compact subset $Q \subset \text{int } E$ such that $M - Q \subset U$. Let B be a ball in R^n so large that $h^{-1}(Q) \subset \text{int } B$. Then $D = h(B)$ is the required disk.

LEMMA 3.4. *Let M be an open differentiable n-manifold with a smooth triangulation. There exists a subcomplex K of dimension $n-1$ with the following property. If U is a neighborhood of K, there exist a collection $B_1, B_2, \cdots$ of mutually disjoint differentiable half-n-spaces in M such that $M - \bigcup \text{int } B_i$ is contained in U. Each B_i is contained in an open set V_i of M such that $V_i \cap V_j$ is empty for $i \neq j$.*

PROOF. Let K be the $(n-1)$-dimensional subcomplex of M described in 3.2. Let U be a neighborhood of K. Let $A_1, A_2, \cdots$ be combinatorial half n-spaces in M as in 3.2. For each i let V_i be a neighborhood of A_i such that $V_i \cap V_j$ is empty for $i \neq j$. We shall find a differentiable half-n-space $B_i \subset V_i$ such that $V_i - \text{int } B_i \subset U$, and doing this for $i = 1, 2, \cdots$ prove the lemma. It is enough to consider a single i. Replacing V_i by M, $V_i \cap U$ by U, and A_i by A, we must prove

LEMMA 3.5. *Let M be an open differentiable n-manifold and A a combinatorial half-n-space in M. Let U be an open set in M such that $M - \text{int } A \subset U$. There is a differentiable half-n-space B in M such that $M - \text{int } B \subset U$.*

PROOF. Let $f: F^n \to M$ be a piecewise linear imbedding such that $f(F^n) = A$. Let d be a metric on M. Let $\delta: F^n \to R$ be a continuous function such that $\delta(x, 0) = 0$, $\delta(x, y) > 0$ if $y > 0$. (Recall that $F^n = R^{n-1} \times R'$.) We apply [3, 6.2] and conclude that there exists a diffeomorphism $g: \text{int } F^n \to \text{int } A$ such that $d(g(x), f(x)) < \delta(x)$ for every $x \in F^n$. Using the properties of δ and the paracompactness of $R^{n-1} = \partial F^n$, we choose a positive differentiable function $\alpha: R^{n-1} \to R$ such that if $(x, y) \in F^n$ and $0 \leqq y \leqq \alpha(x)$, then $g(x, y) \in U$. Define $h: F^n \to M$ by $h(x, y) = g(x, y + \alpha(x))$. Then h is a differentiable imbedding and $M - \text{int } h(F^n) \subset U$. Thus $B = h(F^n)$ is the required differentiable half-n-space.

LEMMA 3.6. *Let B be a differentiable half-n-space in an open differentiable manifold M. Let V be a neighborhood of B. There is a diffeomor-*

phism $f: M \to M - B$ *such that if* $x \in M - V$, $f(x) = x$.

PROOF. Let $h_0: F^n \to B$ be a diffeomorphism. We may find a neighborhood U of F^n in R^n and a differentiable imbedding $h: U \to M$ such that cl $(h(U)) \subset V$, and $h \mid F^n = h_0$; see e.g., [2]. The lemma follows from the easily proved fact that there is a diffeomorphism $g: R^n \to R^n - F^n$ such that $g(x) = x$ for $x \in R^n - U$.

THEOREM 3.7. *Let* M *be an open differentiable* n*-manifold with a smooth triangulation. Let* U *be a neighborhood of the* $n-1$ *skeleton. There is a differentiable imbedding* $f: M \to U$.

PROOF. Let K be the $(n-1)$-dimensional subcomplex of M of 3.4. Let $B_1, B_2, \cdots$, and $V_1, V_2, \cdots$ be as in 3.4, taking U of 3.7 for U of 3.4. By 3.6, for each $i = 1, 2, \cdots$, there is a diffeomorphism $f_i: M \to M - B_i$ which leaves $M - V_i$ fixed. Define $f: M \to M - \bigcup B_i$ by

$$f(x) = \begin{cases} x & \text{if } x \in M - \bigcup V_i\,, \\ f_i(x) & \text{if } x \in V_i\,. \end{cases}$$

Since the V_i are disjoint, f is well defined. Since $M - \bigcup B_i \subset U$, f is as required.

It is perhaps worth noting that the analogue of 3.7 for M compact and bounded can be proved without using [6] or J. H. C. Whitehead's "regular neighborhood" theory. This can be done by successively "pushing in" the n-simplices of M, starting with those that meet ∂M, until all the n-simplices have been deformed into a neighborhood of an $(n-1)$-dimensional subcomplex.

4. Applications

From now on *all manifolds and imbeddings will be differentiable.*

THEOREM 4.1. *Let* M *be a closed* $(m-1)$*-connected* n*-manifold,* $2 \leqq 2m \leqq n$, *with a smooth triangulation; let* x *be a point of* M. *If a neighborhood of the* $n-m$ *skeleton can be immersed in* R^q *for* $q \geqq 2n - 2m + 1$, *then* $M - x$ *can be imbedded in* R^q.

PROOF. Let M_{n-m} denote the $n-m$ skeleton. Let V be a neighborhood of M_{n-m} and $f: V \to R^q$ an immersion. Since $2(n-m) < q$, there is an approximation g to f which is also an immersion and which imbeds M_{n-m}; see Theorem 2(e) of [9]. A compactness argument shows that there is a smaller neighborhood U of M_{n-m} such that $g: U \to R^q$ is an imbedding. Let D be an n-disk in M such that $M - \text{int } D \subset U$; this is possible by 3.3. Thus $g: M - D \to R^q$ is an imbedding. It is easy to construct a diffeomorphism $h: M \to M$ such that $h(x) \in \text{int } D$. Let $v: M - h(x) \to M - D$ be a diffeomorphism. Then $gvh: M - x \to R^n$ is the required imbedding.

COROLLARY 4.2. *$M - x$ can be imbedded in R^{2n-m}.*

PROOF. By 4.1, it suffices to immerse a neighborhood of M_{n-m} in R^{2n-m}. This can be done by [1, 6.2].

A *π-manifold* is one which is imbeddable in some R^q with a trivial normal bundle. It is well known that a parallelizable manifold is a π-manifold [8].

COROLLARY 4.3. *If M is as in* 4.1 *and if $M - x$ is a π-manifold, $M - x$ is imbeddable in $R^{2n-2m+1}$.*

PROOF. By 4.1, it suffices to immerse $M - x$ in $R^{2n-2m+1}$. By 6.5 of [1], $M - x$ is immersible in $R^{n+1} \subset R^{2n-2m+1}$.

COROLLARY 4.4.[1] *If M is a closed homotopy n-sphere* (i.e., *M is $n - 1$ connected*), *$M - x$ is imbeddable in R^{n+2} if n is odd, and in R^{n+1} if n is even.*

PROOF. $M-x$ is parallelizable and $(m-1)$-connected, where $2m=n-1$ or n, and 4.3 applies.

The *double* of a bounded manifold M is the manifold N obtained by identifying two copies of M along their boundaries. We consider M as a submanifold of N in an obvious way. One can obtain information about a bounded compact manifold by applying the preceding results to the double. For example,

THEOREM 4.5.[1] *Let M be a compact bounded n-manifold whose double is N. Suppose M is contractible. Then M is imbeddable in R^{n+2} or R^{n+1}, and N in R^{n+3} or R^{n+2}, according to whether n is odd or even.*

PROOF. Let x be an element of N. It is easy to see that N is a homotopy n-sphere, so $N - x$, and hence $M \subset N - x$, is imbeddable in R^q, where $q = n + 2$ if n is odd, $q = n + 1$ if n is even. Thus $M \subset R^q \subset R^{q+1}$. We may deform M in R^{q+1} so that $M \subset F^{q+1}$, $M \cap R^q = \partial M$, and the rays in F^{q+1} normal to R^q are tangent to M. The double of F^{q+1} is R^{q+1}, and the double N of M is thus naturally contained in R^{q+1}. The imbedding of N can easily be made differentiable; it is already so except along ∂M, where the tangent planes to the two halves of N coincide.

THEOREM 4.6. *A non-closed n-manifold M can be imbedded in R^{2n-1}.*

PROOF. It is enough to assume M is open. Whitney [10] proves that M can be immersed in R^{2n-1}. Reasoning as in the beginning of the proof of 4.1, we see that a neighborhood U of the $(n - 1)$-skeleton (of a smooth triangulation) can be imbedded. Let $g: U \to R^{2n-1}$ be such an imbedding.

[1] In the case $n \geqq 5$, these results are superseded by the recent solutions of the generalized Poincaré conjecture by Smale [12] and Stallings [11]. Using the results of either of these works, it can be shown that $M - x$ of 4.4, and M of 4.5, can be imbedded in R^n.

By 3.7 there is an imbedding $f: M \rightarrow U$. Then $gf: M \rightarrow R^{2n-1}$ is the required imbedding.

THEOREM 4.7. *A non-closed parallelizable n-manifold M is immersible in* R^n.

PROOF. Again it suffices to take M open. It follows from 5.7 of [1] that a neighborhood of the $(n-1)$-skeleton of a smooth triangulation can be immersed in R^n. (Although the hypothesis of the reference requires dim $M < n$, this is not used in the proof if L of the reference has dimension less than n.) Applying 3.7 proves 4.7.

This was proved for $n = 3$ by J. H. C. Whitehead [6], assuming merely orientability instead of parallelizability.

The following theorem does not seem to be in the literature. It is usually "proved" by saying, "push the singularities to infinity". It was pointed out to the author by J. H. C. Whitehead.

THEOREM 4.8. *If M is a non-closed n-manifold there exists a differentiable function on M without critical points.*

PROOF. We may assume M is open. A theorem due to M. Morse states that there exists a differentiable map $f: M \rightarrow R$ with only isolated critical points; see e.g., [5, p. 72]. We may assume none of these points lies on the $(n-1)$-skeleton of a smooth triangulation of M. The conclusion follows from 3.7.

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