*On Imbeddings into Orlicz Spaces and Some Applications**

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1. Introduction. We prove in this article two theorems concerning the imbedding of certain spaces defined in terms of L_p norms into certain Orlicz spaces. Specificially our results include, for domains $\Omega \subset E^n$ satisfying cone conditions:

(i) the space $W^1_{1,1}(\Omega)$ of strongly differentiable functions with derivatives in the Morrey space $L_{1,1}(\Omega)$ may be continuously imbedded in the Orlicz space $L_{\phi^*}(\Omega)$ where $\phi(t) = e^{|t|} - |t| - 1$,

(ii) the Sobolev spaces $W^k_p(\Omega)$ where $n = kp$ (which are subspaces of $W^1_{1,1}(\Omega)$) may be continuously imbedded in the Orlicz space $L_{\phi^*}(\Omega)$ where $\phi(t) = e^{|t|^{n/(n-1)}} - 1$.

Both results are shown to be sharp in a certain sense. Result (i) leads to a simplified proof of a weak form of a measure theoretic result of John and Nirenberg [6]. This weak form is that used by Moser [11], Serrin [13] and the author [18], [19] to establish Harnack inequalities for weak solutions of elliptic equations. These latter results thus do not depend on the John–Nirenberg lemma. The Harnack inequality result is used by the author [18, 19] to give alternative proofs of the Ladyzhenskaya–Ural'tseva Hölder estimates [9].

Result (ii) fills in a gap in the well known Sobolev imbedding theorems [15], and has various applications to partial differential equations. We consider here the application to eigenvalue or non-uniqueness problems for non-linear elliptic equations as studied by Berger [1, 2], Browder [3, 4], and others. The imbedding theorem (ii) permits also a gap in this work to be sealed.

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2. Preliminaries. Ω is taken to be a bounded domain in Euclidean n space, E^n.

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The *Sobolev space* $W_p^k(\Omega)$ is defined by

$$W_p^k(\Omega) = \{u;\, D^\alpha u \in L_p(\Omega),\ |\alpha| \leqq k\},$$

where derivatives are to be understood in the distributional sense and $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index, $|\alpha| = \sum \alpha_i$. $W_p^k(\Omega)$ is a Banach space under the norm

$$(1) \qquad \|u\|_{W_p^k(\Omega)} = \sum_{|\alpha| \leqq k} \left(\int_\Omega |D^\alpha u|^p \, dx\right)^{1/p}.$$

The space $\dot{W}_p^k(\Omega)$ is the completion of C_0^∞ in $W_p^k(\Omega)$.

A function $u(x) \in L_p(\Omega)$ is said to belong to the *Morrey space* $L_{p,\lambda}(\Omega)$ if there exists a constant K such that

$$(2) \qquad \int_{\Omega \cap S} |u(x)|^p \, dx \leqq K\, |S|^{1-\lambda/n}$$

for every sphere S. $|S|$ denotes the n-dimensional measure of a set S. $L_{p,\lambda}(\Omega)$ is a Banach space under the norm

$$(3) \qquad \|u\|_{L_{p,\lambda}(\Omega)} = \left\{\sup_S |S|^{\lambda/n-1} \int_{\Omega \cap S} |u(x)|^p \, dx\right\}^{1/p}.$$

Note that $L_{p,\lambda}(\Omega) = \{0\}$ for $\lambda < 0$, $L_\infty(\Omega)$ for $\lambda = 0$, $L_p(\Omega)$ for $\lambda \geqq n$.

We define the spaces $W_{p,\lambda}^k(\Omega)$, $k > 0$, by

$$W_{p,\lambda}^k(\Omega) = \{u;\, u \in W_p^k(\Omega),\ D^\alpha u \in L_{p,\lambda}(\Omega) \quad \text{for} \quad |\alpha| = k\}$$

and

$$(4) \qquad \|u\|_{W_{p,\lambda}^k(\Omega)} = \|u\|_{W_p^{k-1}(\Omega)} + \|D^k u\|_{L_{p,\lambda}(\Omega)}.$$

$W_{p,\lambda}^k(\Omega)$ is a Banach space, as is

$$\dot{W}_{p,\lambda}^k(\Omega) = \dot{W}_p^k(\Omega) \cap W_{p,\lambda}^k(\Omega).$$

We pass now to the definitions of Orlicz spaces and classes. Let $\phi(t)$ be a real-valued continuous, convex, even function of the real variable t, satisfying

$$(5) \qquad \lim_{t\to 0} \frac{\phi(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\phi(t)}{t} = \infty.$$

Then the *Orlicz class* $L_\phi(\Omega)$ consists of all functions $u(x)$ such that

$$\int_\Omega \phi(u(x)) \, dx < \infty.$$

The *Orlicz space* $L_{\phi^*}(\Omega)$ may be defined as the linear hull of $L_\phi(\Omega)$ together with the Luxembourg norm

$$(6) \qquad \|u\|_{L_{\phi^*}(\Omega)} = \inf\left\{k;\, \int_\Omega \phi\left(\frac{u}{k}\right) dx \leqq 1\right\}.$$

$L_{\phi^*}(\Omega)$ is a Banach space under (6).

We call $\phi(t)$ a defining function for $L_{\phi^*}(\Omega)$. If for any two defining functions $\phi(t)$, $\psi(t)$ we have for every $\lambda > 0$

$$\lim_{t\to\infty} \frac{\phi(\lambda t)}{\psi(t)} = \infty, \tag{7}$$

then we write $\psi \prec \phi$. Note that this means that

$$L_{\phi^*}(\Omega) \subsetneq L_{\psi^*}(\Omega).$$

If a sequence $u_n(x) \in L_{\phi^*}(\Omega)$ converges in measure and is bounded in $L_{\phi^*}(\Omega)$, then $u_n(x)$ converges in $L_{\psi^*}(\Omega)$ for any $\psi \prec \phi$.

A sequence of functions $u_n(x) \in L_\phi(\Omega)$ is said to be *mean convergent* to $u(x) \in L_{\phi^*}(\Omega)$ if

$$\int_\Omega \phi(u_n - u)\, dx \to 0 \quad \text{as} \quad n \to \infty. \tag{8}$$

Mean convergence is a weaker property than norm convergence although for a large class of Orlicz spaces which includes the L_p spaces, $p > 1$, the two notions are equivalent.

We note also another property of Orlicz spaces. Let $E_\phi(\Omega)$ denote the closure of the bounded functions in $L_{\phi^*}(\Omega)$. Then $E_\phi(\Omega) \subset L_\varphi(\Omega)$. This is a simple consequence of the convexity of $\phi(t)$. The reader is referred to the monograph [7] for a thorough treatment of the elementary properties of Orlicz spaces.

We close this section by presenting two lemmas to be used in the sequel. The domain Ω is said to satisfy a *cone condition* if there is a fixed cone in E^n, k_Ω, such that each point $x \in \Omega$ is the vertex of a cone $k_\Omega(x)$ congruent to k_Ω and contained in Ω.

Lemma 1. *Let Ω satisfy a cone condition, and $u(x) \in W_1^1(\Omega)$. Then almost everywhere in Ω*

$$|u(x)| \leqq C(n, k_\Omega)\left\{\int_\Omega \frac{|\nabla u(\xi)|}{|x - \xi|^{n-1}}\, d\xi + ||u(x)||_{L_1(\Omega)}\right\}, \tag{9}$$

where $C(n, k_\Omega)$ denotes a constant depending on n and k_Ω.

Proof. We have almost everywhere in Ω

$$u(x) - u(y) = \int_0^{|x-y|} u_r\, dr$$

and hence

$$|u(x)| \leqq |u(y)| + \int_0^{|x-y|} |\nabla u|\, dr \quad \text{a.e.}$$

We integrate this inequality with respect to y over the cone $k_\Omega(x)$ to obtain

$$|k_\Omega|\, |u(x)| \leqq \int_{k\Omega(x)} |u(y)|\, dy + \int_{k\Omega(x)} \int_0^{|x-y|} |\nabla u|\, dr \quad \text{a.e.}$$

$$\leqq \|u(x)\|_{L_1(\Omega)} + C(n)r_\Omega^n \int_{k\Omega(x)} \frac{|\nabla u(\xi)|}{|x-\xi|^{n-1}}\, d\xi \quad \text{a.e.},$$

where r_Ω denotes the diameter of k_Ω. The desired inequality (9) then follows. Q.E.D.

We remark here that if $u(x) \in \dot{W}_1^1(\Omega)$ and Ω is unrestricted, then

$$|u(x)| \leqq \frac{1}{w_n} \int_\Omega \frac{|\nabla u(\xi)|}{|x-\xi|^{n-1}}\, d\xi \quad \text{a.e.} \tag{10}$$

Lemma 2. *Let Ω be convex and for $u(x) \in W_1^1(\Omega)$ set*

$$u_\Omega = \frac{1}{|\Omega|} \int_\Omega u\, dx. \tag{11}$$

Then

$$\|u - u_\Omega\|_{L_1(\Omega)} \leqq C(n)\, |\Omega|^{1/n} \frac{(\operatorname{diam} \Omega)^n}{|\Omega|} \|u_x\|_{L_1(\Omega)}. \tag{12}$$

Lemma 2 is a Poincaré type lemma and is proved for example in [16] and for spheres in [13].

3. The imbedding theorems. We prove in this section the two theorems mentioned in the introduction. The first theorem is

Theorem 1. *Let Ω satisfy a cone condition. Then the space $W_{1,1}^1(\Omega)$ may be continuously imbedded in the Orlicz space $L_{\phi^*}(\Omega)$, where $\phi(t) = e^{|t|} - |t| - 1$. Furthermore for any $u(x) \in W_{1,1}^1(\Omega)$ we have*

$$\|u(x)\|_{L_{\phi^*}(\Omega)} \leqq C(n, k_\Omega, \operatorname{diam} \Omega)\, \|u(x)\|_{W_{1,1}^1(\Omega)}. \tag{13}$$

Proof. It suffices to show there is a constant C depending on n, k_Ω, diam Ω such that

$$\int_\Omega e^{b|u|} - b\,|u| - 1)\, dx \leqq 1, \tag{14}$$

where $b = 1/C\ \|u\|_{W_{1,1}^1(\Omega)}$. Let us set $K = \|u\|_{W_{1,1}^1(\Omega)}$, $d = \operatorname{diam} \Omega$. To achieve (14) we estimate first $\|u\|_{L_q(\Omega)}$ linearly in terms of q. The exponential estimate (14) then follows by considering its power series expansion.

Let $f(x) \in L_p(\Omega)$, $p \geqq 1$, and let q be the Hölder conjugate of p, i.e. $1/p + 1/q = 1$. We now estimate, using Lemma 1,

$$\int_\Omega |f(x)u(x)|\, dx \leqq C\left\{\int_\Omega |f(x)|\, dx\left(\int_\Omega \frac{|\nabla u(\xi)|}{|x-\xi|^{n-1}}\, d\xi + K\right)\right\}$$

$$\leqq C\left\{\left(\int_\Omega \int_\Omega \frac{|\nabla u(\xi)|}{|x-\xi|^{n-1/q}}\, d\xi\, dx\right)^{1/q}\right.$$

$$\left.\cdot\left(\int_\Omega |f|^p\, dx \int_\Omega \frac{|\nabla u(\xi)|}{|x-\xi|^{n-1-1/q}}\, d\xi\right)^{1/p} + K\int_\Omega |f(x)|\, dx\right\}$$

by Hölder's inequality. Now

$$\int_\Omega \int_\Omega \frac{|\nabla u(\xi)|}{|x-\xi|^{n-1/q}}\, d\xi\, dx \leqq \int_\Omega |\nabla u(\xi)|\, d\xi \int_\Omega \frac{dx}{|x-\xi|^{n-1/q}}$$
$$\leqq C(n)\, d^{n-1+1/q} qK.$$

Also, putting for $x \in \Omega$,

$$v_x(\rho) = \int_{\Omega \cap S_x(\rho)} |\nabla u(\xi)|\, d\xi \leqq K\rho^{n-1}, \tag{15}$$

where $S_x(\rho)$ denotes the sphere of radius ρ centered at x, we have

$$\int_\Omega \frac{|\nabla u(\xi)|}{|x-\xi|^{n-1-1/q}}\, d\xi \leqq \int_0^d \frac{v_x'(\rho)}{\rho^{n-1-1/q}}\, d\rho$$
$$\leqq d^{1/q}K + (n-1)\int_0^d \frac{v_x(\rho)}{\rho^{n-1/q}} d\rho$$
$$\leqq d^{1/q}(1 + (n-1)q)K.$$

Hence

$$||u||_{L_q(\Omega)} \leqq \sup_{f \in L_p(\Omega)} \frac{\int_\Omega |f(x)u(x)|\, dx}{||f||_p}$$
$$\leqq C_1(n)\, d^{n/q} qK, \qquad q > 1,$$

and thus

$$\int_\Omega \sum_{j=1}^N \frac{(b\,|u|)^j}{j!}\, dx \leqq d^n \sum_{j=1}^N \frac{(bC_1Kj)^j}{j!}.$$

It is readily seen that the series

$$\sum_{j=1}^\infty \frac{(aj)^j}{j!}$$

converges provided $a < 1/e$.

Hence choosing $b < 1/eC_1K$ guarantees by Fatou's lemma

$$\int_\Omega (e^{\mu|u|} - 1)\, dx \leqq A\, d^n \tag{16}$$

where A is a fixed constant.

Modifying b in terms of Ad^n we may arrive at the desired estimate (14). Q.E.D.

The dependence of the estimate (16) on $||u||_{L_1(\Omega)}$ may be removed in the case of a convex domain if we replace u by u_Ω and use Lemma 2. As a corollary we would then have the usual form of the conclusion of the John–Nirenberg lemma.

Corollary. *Let Ω be convex and suppose $u_x \in L_{1,1}(\Omega)$. Then there exist constants A and B depending on n and Ω such that*

$$\int_\Omega (e^{A|u-u_\Omega|/K} - 1) \leqq B, \tag{17}$$

where $K = ||u_x||_{L_{1,1}(\Omega)}$.

If Ω is a cube or a sphere, we may take A independent of Ω and B a constant times $|\Omega|$. The John–Nirenberg lemma is stronger than Theorem 1 in that it establishes for cubes an equivalence between $L_{\phi^*}(\Omega)$ and the space $\mathcal{L}_{1,0}(\Omega)$ of functions of bounded mean oscillation, i.e. functions $u(x)$ for which there exists a constant K such that

$$\int_{\Omega'} |u - u_{\Omega'}| \, dx \leqq K \, |\Omega'| \tag{18}$$

for any parallel subcube Ω' of Ω.

We see by Hölder's inequality that Theorem 1 will hold for the spaces $W^1_{p,\lambda}(\Omega)$ where $p = \lambda$, in particular for $W^1_n(\Omega) = W^1_{n,n}(\Omega)$. In the latter case we obtain in fact a better result, namely result (ii) of the introduction, which we now treat.

Theorem 2. *Let Ω satisfy a cone condition. Then the Sobolev spaces $W^k_p(\Omega)$ where $n = kp$ may be continuously imbedded in the Orlicz space $L_{\phi^*}(\Omega)$ where*

$$\phi(t) = e^{|t|^{n/(n-1)}} - 1.$$

Furthermore $W^k_p(\Omega)$ may be continuously imbedded in the sense of mean convergence in any Orlicz class $L_\psi(\Omega)$ where $\psi(t) \leqq \phi(\lambda\, t)$ for some $\lambda > 0$. The imbedding into $L_{\psi^*}(\Omega)$ for any $\psi \prec \phi$ is compact.

Proof. Our proof is a variant of that of Theorem 1. As before, we estimate for $f(x) \in L_p(\Omega)$

$$\begin{aligned} &\int_\Omega |f(x)u(x)| \, dx \\ &\quad \leqq C\left\{\int_\Omega |f(x)| \, dx\left(\int_\Omega \frac{|\nabla u(\xi)|}{|x-\xi|^{n-1}} \, d\xi + ||u(x)||_{L_1(\Omega)}\right)\right\} \\ &\quad \leqq C\left\{\left(\int_\Omega \int_\Omega \frac{|f(x)|}{|x-\xi|^{n-1/q}} \, d\xi \, dx\right)^{1-1/n}\left(\int_\Omega |\nabla u(\xi)|^n \, d\xi \int_\Omega \frac{|f(x)| \, dx}{|x-\xi|^{(n-1)/q}}\right)^{1/n} \right. \\ &\qquad \left. + ||u(x)||_{L_1(\Omega)} \int_\Omega |f(x)| \, dx\right\}. \end{aligned}$$

Note here, however, our different use of Hölder's inequality. We now estimate, similarly to the integrals in the proof of Theorem 1,

$$\int_\Omega \int_\Omega \frac{|f(x)|}{|x-\xi|^{n-1/q}} \, dx \, d\xi \leqq C(n) \, ||f||_{L_p(\Omega)} \, |\Omega|^{(1/q)(1+1/n)} \, q$$

$$\int_\Omega \frac{|f(x)|}{|x-\xi|^{(n-1)/q}}\,dx \leqq ||f||_{L_p(\Omega)}\, ||\, |x-\xi|^{(1-n)/q}||_{L_q(\Omega)}$$
$$\leqq C(n)\, ||f||_{L_p(\Omega)}\, |\Omega|^{1/nq}.$$

Hence, collecting terms we obtain

$$(17) \qquad ||u||_{L_q(\Omega)} \leqq C\, |\Omega|^{1/q}\, (q^{1-1/n}\, ||u_x||_{L_n(\Omega)} + ||u||_{L_1(\Omega)})$$

and thus

$$(18) \qquad \int_\Omega |u|^{nq/(n-1)}\, dx \leqq |\Omega|\, (C_1 q\, ||u||_{W_p^k(\Omega)})^q,$$

where C_1 depends on k_Ω, n, $|\Omega|$.

Considering the power series expansion of $\phi(t) = e^{|t|^{n/(n-1)}}$ we thus obtain

$$(19) \qquad \int_\Omega (e^{\mu|t|^{n/(n-1)}} - 1)\, dx \leqq A\, |\Omega|,$$

where $b < 1/[eC_1\, ||u||_{W_p^k(\Omega)}]$ and A is a constant. From (19) follows the estimate

$$(20) \qquad ||u||_{L_{\phi^*}(\Omega)} \leqq C(n, k_\Omega, |\Omega|)\, ||u||_{W_p^k(\Omega)}, \qquad \phi = e^{|t|^{n/(n-1)}} - 1.$$

The imbedding and its continuity in the sense of norm convergence are thus established.

To show the second statement of the theorem we note that since the bounded functions are dense in $W_p^k(\Omega)$ we must have $W_p^k(\Omega) \subset E_\phi(\Omega)$ by the continuity of the imbedding. Hence $W_p^k(\Omega) \subset L_\psi(\Omega)$ for any $\psi(t) \leqq \phi(\lambda\, t)$ for some $\lambda > 0$. Furthermore since convergence in norm implies convergence in the mean, the imbedding into $L_\psi(\Omega)$ is continuous in the mean.

The compactness of the imbedding into $L_{\psi^*}(\Omega)$ for any $\psi \prec \phi$ is true since a bounded sequence in $W_p^k(\Omega)$ contains a subsequence converging in measure by Rellich's theorem and by (20) is bounded in $L_{\phi^*}(\Omega)$. Hence by the criterion mentioned in the preliminaries, the sequence converges in $L_{\psi^*}(\Omega)$ if $\psi \prec \phi$. The proof of Theorem 2 is thus complete. Q.E.D.

The imbeddings of Theorems 1 and 2 will, of course, be true for the spaces $\dot{W}^1_{1,1}(\Omega)$, $\dot{W}_p^k(\Omega)$, $n = kp$, under no restrictions on Ω. In these cases the proofs and the various constants appearing in them simplify in that inequality (9) may be replaced by (10).

We mention here that the imbedding of $\dot{W}_p^k(\Omega)$, $n = kp$, into $L_{\phi^*}(\Omega)$ where $\phi = e^{|t|} - 1$ follows immediately from the Sobolev inequality [5]. Pöhozaev [12] has stated part of Theorem 2 for the case $n = 2$.

We demonstrate next the sharpness of the two theorems. Consider first the function

$$u(x) = \log \frac{1}{r}$$

defined in the unit circle $S = S_0(1)$ where r denotes the radial coordinate.

Then it is easy to see that $u(x) \in W^1_{1,1}(S)$ but $u(x) \notin L_\phi(S)$ where $\phi = e^{|t|} - |t| - 1$. This example shows that $W^1_{1,1}(\Omega)$ may not be imbedded into Orlicz classes $L_\phi(\Omega)$ where $\phi = e^{b|t|} - b\,|t| - 1$, $b > 0$ and a fortiori not into Orlicz spaces $L_{\psi^*}(\Omega)$ where $\psi \succ \phi$. The example incidentally shows that the bounded functions are not dense in $W^1_{1,1}(\Omega)$.

We illustrate the sharpness of Theorem 2 by showing that $W^1_n(\Omega)$ may not be imbedded in Orlicz spaces $L_{\phi^*}(\Omega)$ where

$$\phi = e^{|t|^\gamma} - 1 \quad \text{and} \quad \gamma > \frac{n}{n-1},$$

i.e. we show that the exponent $n/(n-1)$ is optimal. Let us consider the function

$$u(x) = \left(\log \frac{1}{r}\right)^{\gamma'} \quad \text{where} \quad \frac{n-1}{n} > \gamma' > \frac{1}{\gamma}.$$

The function $u(x) \notin L_{\phi^*}(\Omega)$, $\phi = e^{|t|^\gamma} - 1$, however

$$\int_{S_0(1)} |u_x|^n \, dx = w_n \int_0^1 \frac{dr}{r\left(\log \frac{1}{r}\right)^{-n(\gamma'-1)}}$$

$$< \infty \quad \text{since} \quad -n(\gamma' - 1) > 1.$$

We conclude this section by indicating schematically the set of imbeddings for $W^k_{p,\lambda}(\Omega)$ spaces which include those given by Theorems 1 and 2, the inequalities of Sobolev and Morrey ([10] pp. 78-81) and some estimates mentioned in [17]. Ω is assumed to satisfy a cone condition or else $W^k_{p,\lambda}(\Omega)$ is replaced by $\dot{W}^k_{p,\lambda}(\Omega)$. The imbeddings for $k > 1$ are obtained by iteration of those for $k = 1$ and hence we restrict ourselves to this case. We then have

$$W^1_{p,\lambda}(\Omega) \to \begin{cases} L_{p^*,\lambda}(\Omega)\text{-weak} & \text{if } p < \lambda, \ \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{\lambda} \\ L_{p^*}(\Omega) & \text{if } p < \lambda = n \\ L_{\phi^*}(\Omega) & \text{if } p = \lambda, \ \phi = e^{|t|} - |t| - 1 \\ L_{\phi^*}(\Omega) & \text{if } p = \lambda = n, \ \phi = e^{|t|^{n/(n-1)}} - 1 \\ C_\mu(\Omega) & \text{if } p > \lambda, \ \mu = 1 - \dfrac{\lambda}{p}. \end{cases}$$

The arrows above indicate continuous imbeddings. $L_{p,\lambda}(\Omega)$-weak is defined by replacing the strong L_p norm in the definition of $L_{p,\lambda}(\Omega)$ by the weak L_p norm. Note that $L_{q,\lambda}(\Omega) \supset L_{p,\lambda}(\Omega)$-weak for any $q < p$. $C_\mu(\Omega)$ denotes the space of functions, Hölder continuous in Ω with exponent μ.

4. Eigenvalue problems for non-linear elliptic equations. Berger [1, 2], Browder [3, 4] consider the problem of minimizing a functional

$$g(u) = \int_\Omega G(x, u, Du, \cdots, D^m u) \, dx$$

over the class of functions in $W_p^m(\Omega)$, $p > 1$, subject to the restriction

$$f(u) = \int_\Omega F(x, u, Du, \cdots, D^{m-1}u)\, dx = C$$

where C is some positive constant. The minimizing function $u(x)$ and the corresponding infinum $\lambda = g(u(x))$ furnish a solution of the eigenvalue-eigenfunction problem

$$Au + \lambda Bu = 0, \qquad 0 \not\equiv u \in \dot{W}_p^m(\Omega),$$

where A and B are the Euler–Lagrange operators associated with g and f respectively.

The functions $F(x, \xi)$, $G(x, \xi)$ and their derivatives are required to satisfy certain polynomial structure conditions depending on the exponent p. Ellipticity (or monotonicity) conditions are imposed on the operator A. The proofs proceed by the standard direct methods of the calculus of variations, the Sobolev imbedding theorem being the key tool. The constant C serves to normalize the eigenfunction $u(x)$. Under such a normalization, Browder [4], Berger [2] show the existence of a countable number of eigenvalues λ_n and associated eigenfunctions $u_n(x)$ if in addition the functionals $f(u)$, $g(u)$ are assumed even. The latter results depend on the application of the Ljusternik–Schnirelman notion of category (see [14]) and the characterization of successive eigenvalues in terms of a variant of the Courant max-min principle.

Theorem 2 enables the polynomial structure on the functions $F(x, \xi)$, $G(x, \xi)$ and their derivatives to be generalized and thus the above results will hold for wider classes of equations. In order to illustrate clearly this point, we consider the special case where A is linear and $Bu = B(x, u)$. This case will extend results of Levinson [9], Pohozaev [12] for equations $\Delta u + \lambda f(x, u) = 0$.

Let us set for $u(x) \in W_2^m(\Omega)$, Ω an arbitrary bounded domain,

$$Au = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)\, D^\beta u) \tag{21}$$

so that

$$G(x, \xi) = \sum_{|\alpha|, |\beta| \leq m} a_{\alpha\beta}(x) \xi^\alpha \xi^\beta \tag{22}$$

The coefficients $a_{\alpha\beta}(x)$ are assumed real, continuous in $\bar{\Omega}$ and to satisfy the condition of strong ellipticity

$$\sum_{|\alpha| = |\beta| = m} a_{\alpha\beta} \xi^\alpha \xi^\beta \geqq \delta\, |\xi|^2 \;\forall\; \xi \in R^n, \quad \text{for some} \quad \delta > 0. \tag{23}$$

The function $B(x, u)$ is assumed to be jointly continuous in $x \in \Omega$, $u \in R$, $uB(x, u) > 0$ for $u \neq 0$ and

$$\sup_{x \in \Omega} B(x, u) \prec \begin{cases} |u|^{(n+2m)/(n-2m)} & n > 2m \\ e^{|u|^{n/(n-1)}} & n = 2m, \end{cases} \tag{24}$$

We now have

Theorem 3. *Under the above assumptions on A and B, the boundary value problem*

$$Au + \lambda B(x, u) = 0, \qquad u \in \dot{W}_2^m(\Omega) \tag{25}$$

possesses a generalized eigenfunction $u(x)$ normalized by $f(u) = C$ for any $C > 0$, and characterized as a solution of the variational problem inf $\{g(u); f(u) = C\}$. Furthermore if $B(x, u)$ is an odd function of u, there are a countable number of eigenvalues $\lambda_N \to \infty$ with associated eigenfunctions $u_N(x)$ normalized by $f(u_N) = C$.

Proof. The case $n = 2m$ is not included in the work of Berger, Browder. From [2], we see that the conclusions of Theorem 3 will hold in this case provided the functional $g(u)$ is weakly continuous in $\dot{W}_2^m(\Omega)$. The latter is a consequence of the compactness of the imbedding of $\dot{W}_2^m(\Omega)$ into $L_{\psi^*}(\Omega)$ for $\psi \prec e^{|u|^{n/(n-1)}} - 1$.
Q.E.D.

We remark that it is clear that if the coefficients in (25) and Ω are smooth the eigenfunctions $u_N(x)$ will be accordingly smooth, *e.g.* $a_{\alpha\beta}(x)$, $B(x, u) \in C^\infty(\Omega)$, $C^\infty(\Omega \times R)$, $\partial\Omega \in C^\infty$ implies $u_N(x) \in C^\infty(\bar{\Omega})$.

In the cases $n > 2m$ the growth assumptions (24) are in fact necessary for the validity of Theorem 3. This is shown in Pohozaev [12]. It would be of interest to consider the sharpness of these assumptions in the case $n = 2m$.

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