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On Immanants of Jacobi–Trudi Matrices and Permutations with Restricted Position

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Let χ be a character of the symmetric group $\mathscr{S}_n$. The immanant of an $n \times n$ matrix $A = [a_{ij}]$ with respect to χ is $\sum_{w \in S_n} \chi(w)\, a_{1,w(1)} \cdots a_{n,w(n)}$. Goulden and Jackson conjectured, and Greene recently proved, that immanants of Jacobi–Trudi matrices are polynomials with nonnegative integer coefficients. This led one of us (Stembridge) to formulate a series of conjectures involving immanants, some of which amount to stronger versions of the original Goulden–Jackson conjecture. In this paper, we prove some special cases of one of the stronger conjectures. One of the special cases we prove develops from a generalization of the theory of permutations with restricted position which takes into account the cycle structure of the permutations. We also present two refinements of the immanant conjectures, as well as a related conjecture on the number of ways to partition a partially ordered set into chains.

1. INTRODUCTION

Let $A = [a_{ij}]$ be an $n \times n$ matrix, and for each partition λ of n, let χ^λ denote the associated irreducible character of the symmetric group $\mathscr{S}_n$ [JK]. The function

$$\mathrm{Imm}_\lambda(A) = \sum_{w \in \mathscr{S}_n} \chi^\lambda(w) \prod_{i=1}^{n} a_{i,w(i)}$$

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is called the *immanant* of A with respect to λ. When $\lambda=(1^n)$, χ^λ is the sign character and $\mathrm{Imm}_\lambda(A)=\det(A)$, the determinant of A. When $\lambda=(n)$, χ^λ is the trivial character and $\mathrm{Imm}_\lambda(A)=\mathrm{per}(A)$, the permanent of A. For previous work on immanants, see [G], and the references cited there.

Suppose that $\mu=(\mu_1, ..., \mu_n)$ and $\nu=(\nu_1, ..., \nu_n)$ are partitions with $\nu \subseteq \mu$ (i.e., weakly decreasing sequences of integers with $0 \leqslant \nu_i \leqslant \mu_i$ for all i). Define the matrix

$$H_{\mu/\nu}=H_{\mu/\nu}(x)=[h_{\mu_i-\nu_j+j-i}]_{1\leqslant i,j\leqslant n},$$

where $h_k=h_k(x)=h_k(x_1, x_2, ...)$ denotes the kth complete homogeneous symmetric function, i.e., the formal sum of all monomials of degree k in the variables $x=(x_1, x_2, ...)$. Following the usual conventions, we set $h_{-k}=0$ for $k>0$ and $h_0=1$. Under these terms, the matrix $H_{\mu/\nu}$ is called a *Jacobi–Trudi matrix*, named after Jacobi and his student Trudi, who established the special case $\nu=\varnothing$ of the identity

$$\det H_{\mu/\nu}=s_{\mu/\nu}, \tag{1.1}$$

where $s_{\mu/\nu}$ denotes the skew Schur function indexed by the skew shape μ/ν [M]. (When $\nu=\varnothing$, $s_{\mu/\varnothing}$ becomes the ordinary Schur function s_μ.)

In an interesting recent paper, Goulden and Jackson [GJ] investigated the immanants of $H_{\mu/\nu}$. In particular, they conjectured that $\mathrm{Imm}_\lambda H_{\mu/\nu}$ is a *nonnegative* linear combination of monomials, or equivalently, of monomial symmetric functions. This conjecture was subsequently proved by Greene [G], who applied the representation theory of the symmetric group to a reduction of the problem previously given by Goulden and Jackson. Meanwhile, the work of Goulden and Jackson led one of us to formulate a series of conjectures involving immanants [Ste2], some of which amount to stronger versions of the original Goulden–Jackson conjecture (or theorem of Greene). One of these (Conjecture 4.2(a) of [Ste2]) is the conjecture that $\mathrm{Imm}_\lambda H_{\mu/\nu}$ is a nonnegative linear combination of Schur functions, a fact recently proved by Haiman via Kazhdan–Lusztig theory [H].

There is an even stronger version of the Goulden–Jackson conjecture in [Ste2] which remains open. To describe this conjecture, we define

$$F_{\mu/\nu}(x, y)=\sum_{\lambda\vdash n} s_\lambda(y)\,\mathrm{Imm}_\lambda H_{\mu/\nu}(x),$$

where the notation $\lambda \vdash n$ indicates that λ is a partition of n. Note that $F_{\mu/\nu}(x, y)$ is independently a symmetric function of the variables $x=(x_1, x_2, ...)$ and $y=(y_1, y_2, ...)$. Following the notation of [M], let p_λ denote the power-sum symmetric function indexed by λ, and for $w \in \mathscr{S}_n$, let

$\rho(w)$ denote the cycle-type of w, i.e., the partition formed by the lengths of the cycles of w. Since $p_{\rho(w)} = \sum_{\lambda \vdash n} \chi^\lambda(w)\, s_\lambda$ [Sta1, (44)], [M, (7.8)], we have

$$F_{\mu/\nu}(x, y) = \sum_{w \in \mathscr{S}_n} p_{\rho(w)}(y)\, h_{\mu+\delta-w(\nu+\delta)}(x), \tag{1.2}$$

where $\delta = (n-1, n-2, ..., 0)$, $h_\alpha = h_{\alpha_1} \cdots h_{\alpha_n}$, and $w(\alpha) = (\alpha_{w(1)}, ..., \alpha_{w(n)})$. Since $F_{\mu/\nu}$ is homogeneous of degree $N := |\mu| - |\nu|$ with respect to x (where $|\mu|$ denotes $\sum \mu_i$), it has an expansion in terms of Schur functions of degree N in x. For $\theta \vdash N$, let $E^\theta_{\mu/\nu} = E^\theta_{\mu/\nu}(y)$ denote the coefficient of $s_\theta(x)$ in $F_{\mu/\nu}(x, y)$, so that

$$F_{\mu/\nu}(x, y) = \sum_{\theta \vdash N} E^\theta_{\mu/\nu}(y)\, s_\theta(x). \tag{1.3}$$

Note that $E^\theta_{\mu/\nu}$ is homogeneous of degree n.

The following is equivalent to Conjectures 4.1 and/or 4.1′ of [Ste2]. (In fact, $E^\theta_{\mu/\nu}$ is the symmetric function corresponding to the $\mathscr{S}_n$-class function labeled as $\Gamma^\theta_{\mu/\nu}$ in [Ste2].)

1.1. *Conjecture.* $E^\theta_{\mu/\nu}$ is a nonnegative linear combination of h_α's.

We remark that

$$h_\alpha = \sum_\lambda K_{\lambda,\alpha} s_\lambda, \tag{1.4}$$

where $K_{\lambda,\alpha}$ denotes the Kostka number [Sta1, (26)], [M, I.6], so (1.2) and (1.3) imply

$$E^\theta_{\mu/\nu} = \sum_{w \in \mathscr{S}_n} K_{\theta,\mu+\delta-w(\nu+\delta)}\, p_{\rho(w)}. \tag{1.5}$$

It should be emphasized that $K_{\theta,\mu+\delta-w(\nu+\delta)}$ is interpreted as 0 whenever any component of $\mu+\delta-w(\nu+\delta)$ is negative; in view of (1.4), this is consistent with the convention that $h_{-k} = 0$ for $k > 0$.

The fact that the Kostka numbers are nonnegative shows that Conjecture 1.1 would indeed be stronger than the result that $\mathrm{Imm}_\lambda H_{\mu/\nu}$ is a nonnegative linear combination of Schur functions (cf. (1.3) and (1.4)).

Our contributions to Conjecture 1.1 are the following:

I. An explicit formula for $E^\theta_{\mu/\nu}$ when μ/ν is a border-strip (i.e., $\mu_{i+1} - \nu_i = 1$ for all i) which transparently proves the validity of Conjecture 1.1 for this case.

II. A formula for $E^{(N)}_{\mu/\nu}$ that we can use to verify Conjecture 1.1 for the special case $\theta = (N)$ whenever the pattern of zeroes in the matrix $H_{\mu/\nu}$ is confined to a "small" region. The case $\theta=(N)$ is particularly interesting since it turns out that $E^{(N)}_{\mu/\nu}$ has a natural description in terms of permutations with restricted position. This suggested to us the idea of developing a generalization of the classical enumerative theory which takes into account the cycle structure of the permutations. (See Section 3.)

III. Some conjectures for expressing $E^{\theta}_{\mu/\nu}$ as a linear combination of the special $E^{(N)}_{*}$'s of II. These conjectures will essentially be generalizations of our solution to I. We also have a conjecture, suggested by the theory we developed for II, on the number of ways to partition a partially ordered set into chains. (See Section 5.)

We should point out that there is a well-known analogue of (1.1) (see [M]) in which the symmetric function h_r is replaced by the elementary symmetric function e_r; namely,

$$\det[e_{\mu_i-\nu_j+j-i}]=s_{\mu'/\nu'},$$

where μ' denotes the conjugate of μ. It would be natural to investigate various positivity questions involving immanants of these matrices as well. However, the map $h_r \mapsto e_r$ extends to an involutory automorphism of the ring of symmetric functions in which $s_\mu \mapsto s_{\mu'}$ [M, (2.7)], so it follows that the Schur function expansions of the immanants of these new matrices can be easily obtained from those of the Jacobi–Trudi matrices (and conversely). Similarly, Conjecture 1.1 is equivalent to the conjecture that the analogues of $E^{\theta}_{\mu/\nu}$ for these new matrices are nonnegative linear combinations of e_α's.

2. The Border-Strip Case

The skew shape μ/ν is a *border-strip* if $\mu_{i+1}-\nu_i=1$ for $1\leqslant i<n$. Geometrically, this means that the Young diagram of μ/ν is rookwise connected and contains no 2×2 subdiagram [M, p. 31]. This is also equivalent to having 1's along the first subdiagonal of $H_{\mu/\nu}$ (i.e., in positions $(2,1), (3,2), ..., (n, n-1)$).

Recall that a *composition* of N is an ordered sequence $\alpha=(\alpha_1, ..., \alpha_l)$ of positive integers with sum N. A *refinement* of α is obtained by replacing each term α_i by a composition of α_i. If β is a refinement of α, we will say conversely that α is a *meld* of β. Note that the compositions of l (the length of α) can be used to index the melds of α. For example, if $l=5$, then the meld indexed by $(2, 1, 2)$ is $(\alpha_1+\alpha_2, \alpha_3, \alpha_4+\alpha_5)$. In general, we will write $\alpha|_\gamma$ for the meld of α indexed by γ.

2.1. LEMMA. *If μ/ν is a border-strip, then*

$$E^{\theta}_{\mu/\nu} = \sum_{\alpha} K_{\theta,(\mu-\nu)|_{\alpha}} p_{\alpha_1} \cdots p_{\alpha_l},$$

summed over all compositions α of n.

Proof. Consider the description of $E^{\theta}_{\mu/\nu}$ in (1.5). A little thought shows that when μ/ν is a border-strip, one has $\mu+\delta-w(\nu+\delta)\geqslant 0$ if and only if the cycles of w are $(a_i, a_i-1, ..., a_{i-1}+1)$ for $i=1, ..., l$, where $0=a_0< a_1<\cdots<a_l=n$ is some increasing sequence of integers. (This fact was also used by Goulden and Jackson [GJ, Sect. 5].) Therefore, let us define $\alpha_i=a_i-a_{i-1}$, so that $\alpha=(\alpha_1, ..., \alpha_l)$ is a composition of n. There will be one nonzero term in $\mu+\delta-w(\nu+\delta)$ corresponding to each of the l cycles of w. Using the fact that $\mu_{j+1}-\nu_j=1$, it follows that the ith such nonzero term will be

$$\mu_{a_{i-1}+1}-\nu_{a_i}+\alpha_i-1=\sum_{j=a_{i-1}+1}^{a_i}(\mu_j-\nu_j).$$

In short, $(\mu-\nu)|_{\alpha}$ is the subsequence of positive terms in $\mu+\delta-w(\nu+\delta)$. Apply (1.5). ∎

Let us consider the special case $\mu/\nu=(1^n)/\varnothing$ in more detail. We write J_θ as an abbreviation for $E^{\theta}_{\mu/\nu}$ in this case. Since the Kostka number $K_{\theta,(\alpha_1, ..., \alpha_l)}$ does not depend on the order of the terms α_i, we may rewrite the expansion of Lemma 2.1 in the form

$$J_\theta=\sum_{\alpha\vdash n}\frac{\ell(\alpha)!}{m_1(\alpha)!\, m_2(\alpha)!\cdots}K_{\theta,\alpha}p_\alpha, \tag{2.1}$$

where $m_i(\alpha)$ denotes the number of parts of α equal to i, and $\ell(\alpha)$ denotes the total number of (positive) parts of α.

It follows that if we regard J_θ as the characteristic $\mathrm{ch}(\psi^\theta)$ of a certain class function ψ^θ of $\mathscr{S}_n$ (following the terminology of [M, I.7]), then

$$\psi^\theta(\alpha)=\ell(\alpha)!\,\alpha_1\cdots\alpha_l K_{\theta,\alpha}. \tag{2.2}$$

The property that J_θ is a nonnegative (integer) linear combination of Schur functions is equivalent to ψ^θ being a true character (not merely a virtual character) of $\mathscr{S}_n$. Similarly, having J_θ be a nonnegative (integer) linear combination of h_α's is equivalent to ψ^θ being the character of a permutation representation whose transitive components are isomorphic to the action of $\mathscr{S}_n$ on the cosets of Young subgroups $\mathscr{S}_\alpha=\mathscr{S}_{\alpha_1}\times\cdots\times\mathscr{S}_{\alpha_l}$ (cf. [Ste2, Sect. 4]).

We do not know of a "natural" representation of $\mathscr{S}_n$ with character ψ^θ, except when $\theta = (n)$. In this case, one has $K_{\theta,\alpha} = 1$ for all $\alpha \vdash n$, and so (2.1) and (2.2) become

$$J_{(n)} = \sum_{\alpha \vdash n} \frac{\ell(\alpha)!}{m_1(\alpha)!\, m_2(\alpha)! \cdots} p_\alpha$$

$$\psi^{(n)}(\alpha) = \ell(\alpha)!\, \alpha_1 \cdots \alpha_l.$$

This character of $\mathscr{S}_n$ and the corresponding symmetric function $J_{(n)}$ were studied previously in [B], [Sta3], [Ste1]. In particular, from the work of DeConcini and Procesi, one knows that $\psi^{(n)}$ is the character of an action of $\mathscr{S}_n$ on the cohomology ring of a certain toric variety; see [Sta3, p. 529]. Recently, Dolgachev and Lunts [DL] showed that the fact that $\psi^{(n)}$ is a permutation character of the sort described above can be deduced from the geometry of this toric variety.

For the sake of completeness, we include a direct, elementary proof that $J_{(n)}$ is indeed a nonnegative (integer) linear combination of h_α's. Similar proofs can be found in [GJ, Sect. 5], [Ste1, Sect. 3].

2.2. PROPOSITION. *We have*

$$\sum_{n \geqslant 0} J_{(n)} t^n = \left(1 - \sum_{n \geqslant 1} p_n t^n\right)^{-1} = \frac{\sum_{n \geqslant 0} h_n t^n}{1 - \sum_{n \geqslant 1} (n-1)\, h_n t^n}.$$

Proof. For the first equality, consider

$$\left(1 - \sum_{n \geqslant 1} p_n t^n\right)^{-1} = \sum_{l \geqslant 0} \left(\sum_{n \geqslant 1} p_n t^n\right)^l = \sum_{l \geqslant 0} \left(\sum_{a_i \geqslant 1} p_{a_1} \cdots p_{a_l} t^{a_1 + \cdots + a_l}\right),$$

and apply Lemma 2.1, using $\mu/\nu = (1^n)/\varnothing$, $\theta = (n)$, and $K_{\theta,(a_1, ..., a_l)} = 1$.

The second equality (which Goulden and Jackson attribute to I. Gessel) is an immediate consequence of the identity

$$\left(\sum_{n \geqslant 1} p_n t^n\right)\left(\sum_{n \geqslant 0} h_n t^n\right) = \sum_{n \geqslant 0} n h_n t^n,$$

which in turn is equivalent to [M, (2.10)]. ∎

A corollary of this result is the following explicit expansion of $J_{(n)}$ as a nonnegative linear combination of h_α's,

$$J_{(n)} = \sum (a_1 - 1) \cdots (a_l - 1)\, h_{a_0} h_{a_1} \cdots h_{a_l},$$

summed over all "compositions" $(a_0, ..., a_l)$ of n in which a_0 is allowed to be zero.

We now return to analysis of the general border-strip μ/ν.

Recall that a *standard Young tableau* (SYT) of shape $\theta \vdash N$ may be regarded as a placement of the numbers 1, 2, ..., N without repetition into the cells of the Young diagram of θ so that every row and column is increasing. For example,

$$\begin{matrix} 1 & 2 & 3 & 5 \\ 4 & 7 & 8 & \\ 6 & 9 & & \end{matrix}$$

is an SYT of shape $\theta = (4, 3, 2)$. For any such tableau T, the *descent set* $D(T)$ is defined to be the set of all i $(1 \leqslant i < N)$ such that $i+1$ appears in a row lower than the row of i. In the above example, we have $D(T) = \{3, 5, 8\}$.

For any composition $\alpha = (\alpha_1, ..., \alpha_l)$ of N, let $S(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, ..., \alpha_1 + \cdots + \alpha_{l-1}\}$, a subset of $\{1, 2, ..., N-1\}$. The following simple result appears in Prop. 4.1 of [Sta2].

2.3. Lemma. *If α is any composition of N, then the number of SYT T of shape θ satisfying $D(T) \subseteq S(\alpha)$ is the Kostka number $K_{\theta,\alpha}$.*

The following is the main result of this section.

2.4. Theorem. *If μ/ν is a border-strip, then*

$$E^{\theta}_{\mu/\nu} = \sum_T J_{(\alpha_1)} \cdots J_{(\alpha_l)}, \tag{2.3}$$

where T ranges over all SYT of shape θ with $D(T) \subseteq S(\mu - \nu)$, and $\alpha = \alpha(T)$ is defined by $D(T) = S((\mu - \nu)|_\alpha)$.

Proof. By Lemmas 2.1 and 2.3, we have

$$E^{\theta}_{\mu/\nu} = \sum_\beta \sum_T p_{\beta_1} \cdots p_{\beta_k}, \tag{2.4}$$

where the inner sum ranges over all SYT T of shape θ with $D(T) \subseteq S((\mu - \nu)|_\beta)$. If we interchange the order of summation, we will obtain an outer sum over SYT with $D(T) \subseteq S(\mu - \nu)$. Each such T will have the property that $D(T) = S((\mu - \nu)|_\alpha)$ for some composition α of n, so the inner sum will range over all refinements β of α.

On the other hand, Lemma 2.1 also implies that $J_{(n)} = \sum p_{\beta_1} \cdots p_{\beta_k}$, where β ranges over all compositions of n, and more generally,

$$J_{(\alpha_1)} \cdots J_{(\alpha_l)} = \sum_\beta p_{\beta_1} \cdots p_{\beta_k},$$

where the sum ranges over all refinements of α. Thus, the amount contributed to (2.4) by a given SYT T with $D(T)=S((\mu-\nu)|_\alpha)$ will be $J_{(\alpha_1)}\cdots J_{(\alpha_l)}$, which proves (2.3). ∎

As a consequence of Proposition 2.2 and this result, we may deduce that $E^\theta_{\mu/\nu}$ is a nonnegative linear combination of h_α's whenever μ/ν is a border-strip; i.e.,

2.5. COROLLARY. *Conjecture* 1.1 *is valid whenever* μ/ν *is a border-strip.*

2.6. PROPOSITION. *Let* t *be a positive integer. If Conjecture* 1.1 *is valid for a given skew shape* μ/ν *(and all* θ*), then Conjecture* 1.1 *is also valid for* $t*(\mu/\nu)$, *where*

$$t*(\mu/\nu)=(t\mu+(t-1)\,\delta)/(t\nu+(t-1)\,\delta).$$

We remark that border-strips are closed under the $*$-operator, so this result does not by itself extend the domain to which Corollary 2.5 applies.

Proof. Let R denote the ring of "functions" which are independently symmetric in the variables $x=(x_1, x_2, ...)$ and $y=(y_1, y_2, ...)$. Since the symmetric functions $h_i(x)$ and $h_i(y)$ for $i\geqslant 1$ are algebraically independent and generate R (as an algebra over $\mathbf{Z}$), there is a unique algebra endomorphism $\gamma=\gamma_t\colon R\to R$ satisfying $\gamma(h_i(x))=h_{ti}(x)$ and $\gamma(h_i(y))=h_i(y)$ for all $i\geqslant 1$. One easily checks that if we apply γ to each entry of the matrix $H_{\mu/\nu}(x)$, then we obtain $H_{t*(\mu/\nu)}$. In particular, $\gamma(s_\theta(x))=s_{t*\theta}(x)$ for all θ. Now from the definition of $F_{\mu/\nu}(x, y)$ there follows

$$\gamma(F_{\mu/\nu}(x, y))=F_{t*(\mu/\nu)}(x, y).$$

Now apply γ to (1.3). Since the Littlewood–Richardson rule implies that any skew Schur function is a nonnegative (integral) linear combination of ordinary Schur functions [M, I.9], it follows that $E^\theta_{t*\mu/\nu}$ is a nonnegative (integral) linear combination of $E^\phi_{\mu/\nu}$'s. ∎

3. PERMUTATIONS WITH RESTRICTED POSITION

In the following, we write $[n]$ as an abbreviation for $\{1, 2, ..., n\}$ and regard $[n]^2=[n]\times[n]$ as an $n\times n$ chessboard. The classical theory of permutations with restricted position is concerned with the number of ways to place n non-attacking rooks on some subset B of $[n]^2$. Each such placement corresponds to a permutation $w\in\mathcal{S}_n$, viz., $w(i)=j$ if a rook occupies position (i, j). We then call w a *B-compatible* permutation. Note that the number of legal placements of n rooks on B, or equivalently, the number of B-compatible permutations, is the permanent of the 0–1 matrix $[a_{ij}]$ with $a_{ij}=1$ for $(i, j)\in B$.

In this section, we consider not just the number of B-compatible permutations, but also their distribution by cycle-type. Thus define the *cycle indicator* $Z[B]$ of B to be the symmetric function

$$Z[B]=\sum_{w} p_{\rho(w)}=\sum_{w} p_1^{m_1(w)}p_2^{m_2(w)}\cdots,$$

where $m_i(w)$ is the number of i-cycles of w, and the sum is extended over all B-compatible permutations w. For example, if $B=[n]^2$, then every permutation is B-compatible, and we get the well-known formula (e.g., see Example I.9 on p. 20 of [M])

$$Z[B]=\sum_{w\in\mathscr{S}_n} p_{\rho(w)}=n!\,h_n. \tag{3.1}$$

Let us now recall from the classical theory of permutations with restricted position (e.g., [R, Chap. 7]) that the Principle of Inclusion–Exclusion may be used to prove

$$\eta(B)=\sum_{k}(-1)^k\,(n-k)!\,r_k(\bar{B}), \tag{3.2}$$

where $\eta(B)$ denotes the number of B-compatible permutations, and $r_k(\bar{B})$ denotes the number of ways to place k non-attacking rooks on the complementary board $\bar{B}$. The factor $(n-k)!$ represents the number of ways to extend a set of k non-attacking rooks to a full set of n non-attacking rooks. In order to extend this reasoning to compute $Z[B]$ instead of merely $\eta(B)=Z[B](p_i\to 1)$, we need to determine the cycle indicator for the set of permutations that extend a given set of k rooks (i.e., a "partial permutation"). However, unlike the classical case, these cycle indicators do not depend only on n and k.

To describe our solution to this problem, let $S\subset[n]^2$ denote a placement of (at most n) non-attacking rooks, and define

$$Z_{\supseteq S}=\sum_{w} p_{\rho(w)},$$

summed over all $w\in\mathscr{S}_n$ with $w(i)=j$ for all $(i, j)\in S$. In the following, it will be convenient to regard S (or indeed, any subset of $[n]^2$) as a directed graph with vertex set $[n]$. In these terms, S must be a union of disjoint directed paths and cycles. We define the *type* of S to be the pair of partitions $(\alpha;\beta)$ such that (1) the numbers of vertices in the directed paths of S are $\alpha_1, \alpha_2, \ldots$, and (2) the lengths of the cycles of S are $\beta_1, \beta_2, \ldots$. Note that $|\alpha|+|\beta|=n$ and that the number of isolated vertices in S is $m_1(\alpha)$, the multiplicity of 1 in α.

The following result shows that the cycle indicators $Z_{\supseteq S}$ are closely related to the "forgotten" symmetric functions f_λ [M, p. 15]. These symmetric functions are usually defined by setting

$$f_\lambda = \omega(m_\lambda),$$

where m_λ denotes the monomial symmetric function of type λ, and ω denotes the automorphism of the ring of symmetric functions such that $\omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda$ [M, I.2]. These symmetric functions are considered forgotten since explicit descriptions of them are hard (but not impossible) to find; see [Db].

3.1. LEMMA. *If $S \subset [n]^2$ is of type $(\alpha; \beta)$ then*

$$Z_{\supseteq S} = (-1)^{|\alpha| - \ell(\alpha)} m_1(\alpha)!\, m_2(\alpha)! \cdots f_\alpha p_\beta.$$

Proof. Clearly, each j-cycle of S contributes a factor of p_j to $Z_{\supseteq S}$, so we may henceforth assume $\beta = \varnothing$ and $|\alpha| = n$. Now for any $\lambda \vdash n$, the coefficient of p_λ in $Z_{\supseteq S}$ is the number of $w \in \mathscr{S}_n$ that contain S as a subgraph and have type $(\varnothing; \lambda)$. Since there are a total of $n!/m_1(\alpha)!\, m_2(\alpha)! \cdots$ graphs of type $(\alpha; \varnothing)$, this coefficient can also be expressed as

$$\frac{m_1(\alpha)!\, m_2(\alpha)! \cdots}{n!} I_\alpha^\lambda, \tag{3.3}$$

where I_α^λ denotes the number of inclusions $S_1 \subset S_2$ of digraphs in which S_1 and S_2 are of type $(\alpha; \varnothing)$ and $(\varnothing; \lambda)$, respectively.

Now by Proposition 1.1 of [Ste2] (a description of the virtual character ϕ^α whose characteristic $\mathrm{ch}(\phi^\alpha)$ is the monomial symmetric function m_α), we have

$$(-1)^{n - \ell(\alpha)} m_\alpha = \frac{1}{n!} \sum_{w \in \mathscr{S}_n} \mathrm{sgn}(w)\, I_\alpha(w)\, p_{\rho(w)},$$

where $I_\alpha(w)$ denotes the number of subgraphs of type $(\alpha; \varnothing)$ contained in the graph of w. By applying the automorphism ω, we obtain

$$(-1)^{n - \ell(\alpha)} f_\alpha = \frac{1}{n!} \sum_{w \in \mathscr{S}_n} I_\alpha(w)\, p_{\rho(w)},$$

so the coefficient of p_λ in f_α is

$$\frac{(-1)^{n - \ell(\alpha)}}{n!} I_\alpha^\lambda.$$

The claimed identity now follows upon comparison with (3.3). ∎

3.2. THEOREM. *For any $B \subseteq [n]^2$, we have*

$$Z[B] = \sum_{\alpha, \beta} (-1)^{|\beta|} m_1(\alpha)!\, m_2(\alpha)! \cdots r_{\alpha,\beta}(\bar{B})\, f_\alpha p_\beta, \tag{3.4}$$

where $r_{\alpha,\beta}(\bar{B})$ denotes the number of subgraphs of type $(\alpha; \beta)$ in the complement $\bar{B}$.

Proof. By the Principle of Inclusion–Exclusion, we have

$$Z[B] = \sum_{S \subseteq B} (-1)^{|S|} Z_{\supseteq S},$$

summed over all non-attacking rook placements on $\bar{B}$. Apply Lemma 3.1, using the fact that $|S| = |\alpha| + |\beta| - \ell(\alpha)$ for S of type $(\alpha; \beta)$. ∎

4. THE CASE $\theta = (N)$

We now return to consideration of the symmetric functions $E^\theta_{\mu/\nu}$.

Note that the pattern of zeroes in the matrix $H_{\mu/\nu}$ (i.e., the set of positions (i, j) where $\mu_i - \nu_j + j - i < 0$) will in general be the Young diagram of some partition $\sigma = \sigma(\mu/\nu) \subseteq \delta$ in the southwest corner of the matrix. Conversely, every Young diagram that fits below the main diagonal of an $n \times n$ matrix arises as the pattern of zeroes for some $H_{\mu/\nu}$. For example, if $\mu/\nu = (3, 3, 3, 2, 2)/(2, 1, 0, 0, 0)$, then

$$H_{\mu/\nu} = \begin{bmatrix} h_1 & h_3 & h_5 & h_6 & h_7 \\ 1 & h_2 & h_4 & h_5 & h_6 \\ 0 & h_1 & h_3 & h_4 & h_5 \\ 0 & 0 & h_1 & h_2 & h_3 \\ 0 & 0 & 1 & h_1 & h_2 \end{bmatrix},$$

and so the pattern of zeroes is the Young diagram of $\sigma = (2, 2, 1)$.

Let $B_\sigma \subseteq [n]^2$ denote the board obtained by removing the diagram of σ from the southwest corner of $[n]^2$; i.e.,

$$B_\sigma = \{(i, j) \in [n]^2 : j > \sigma_{n-i+1}\}.$$

Thus, $\bar{B}_\sigma$ is a Young diagram of shape σ, and if $\sigma = \sigma(\mu/\nu)$ then B_σ is the set of positions in $H_{\mu/\nu}$ with nonzero entries. Under these circumstances, we have $\mu + \delta - w(\nu + \delta) \geqslant 0$ if and only if w is B_σ-compatible. Therefore, since $K_{(N),\alpha} = 1$ for all compositions α of N, we may deduce the following from (1.5):

4.1. PROPOSITION. *If* $\theta=(N)$ *and* $\sigma=\sigma(\mu/\nu)$, *then* $E^{\theta}_{\mu/\nu}=Z[B_\sigma]$.

Thus, in the special case $\theta=(N)$, Conjecture 1.1 amounts to the assertion that for every $\sigma\subseteq\delta$, $Z[B_\sigma]$ is a nonnegative linear combination of h_α's. This section is devoted to proving this fact whenever σ is *small*, by which we mean $\sigma_1+\ell(\sigma)\leqslant n$.

In order to use Theorem 3.2 in this investigation of $Z[B_\sigma]$, we need to know the h_α expansion of each term in (3.4). However, if σ is small, then the graph of $\bar{B}_\sigma$ will have no directed paths of length two or more. Therefore, the graph of any placement of non-attacking rooks on $\bar{B}_\sigma$ will be of type $(2^k1^{n-2k};\varnothing)$ for some $k\leqslant n/2$.

4.2. LEMMA. $f_{2^k1^{n-2k}}=\sum_{j=0}^{k}(-1)^{k-j}\left[\binom{n-k-j}{k-j}+\binom{n-k-j-1}{k-j-1}\right]h_{n-j}h_j$.

Proof. Let $\sigma=(r^{n-r})$ (an $r\times(n-r)$ rectangle), so that w is B_σ-compatible if and only if w belongs to the Young subgroup $\mathscr{S}_r\times\mathscr{S}_{n-r}$. Clearly, $Z[B_\sigma]=r!(n-r)!\,h_rh_{n-r}$ (cf. (3.1)). On the other hand, we may use Theorem 3.2 to compute the cycle indicator of B_σ. Since there are $k!\binom{r}{k}\binom{n-r}{k}$ ways to place k non-attacking rooks on $\bar{B}_\sigma$ (and the graph of each such placement is of type $(2^k1^{n-2k};\varnothing)$), we obtain

$$r!(n-r)!\,h_rh_{n-r}=\sum_{k=0}^{r}(k!)^2\,(n-2k)!\binom{r}{k}\binom{n-r}{k}f_{2^k1^{n-2k}},$$

or equivalently,

$$h_{n-r}h_r=\sum_{k=0}^{r}\binom{n-2k}{r-k}f_{2^k1^{n-2k}}.$$

Thus we have a triangular system of equations that uniquely determines the h_α-expansion of each forgotten symmetric function $f_{2^k1^{n-2k}}$. To complete the proof, it therefore suffices to substitute the claimed formula in the right side of the above expression and collect coefficients. This leaves us to verify

$$\sum_{k=j}^{r}(-1)^{k-j}\binom{n-2k}{r-k}\left[\binom{n-k-j}{k-j}+\binom{n-k-j-1}{k-j-1}\right]=\delta_{r,j}$$

for $j\leqslant r\leqslant n/2$. This is equivalent to

$$\sum_{k=0}^{r}(-1)^k\binom{n-2k}{r-k}\left[\binom{n-k}{k}+\binom{n-k-1}{k-1}\right]=\delta_{r,0}\qquad(4.1)$$

for $0\leqslant r\leqslant n/2$, via the change of variables $k\to k+j$, $r\to r+j$, $n\to n+2j$. If we define

$$g_r(n)=\sum_{k=0}^{r}(-1)^k\binom{n-2k}{r-k}\binom{n-k}{k},$$

then the left side of (4.1) (for $r>0$) can be identified as $g_r(n)-g_{r-1}(n-2)$. Now

$$\begin{aligned} g_r(n) &= \sum_k (-1)^k \binom{r}{k}\binom{n-k}{r} = (-1)^{n-r} \sum_k \binom{r}{k}\binom{-r-1}{n-k-r} \\ &= (-1)^{n-r}\binom{-1}{n-r} = 1 \end{aligned}$$

by a Vandermonde convolution, and so (4.1) follows. ∎

Let $r_k(\sigma)$ denote the number of ways to place k non-attacking rooks on $\bar{B}_\sigma$. The following result shows that $Z[B_\sigma]$ is indeed a nonnegative linear combination of h_α's whenever σ is small.

4.3. THEOREM. *Assume σ is small, and let $m=\min(\ell(\sigma), \sigma_1)$. We have*

$$Z[B_\sigma] = \sum_{j<m} j!(n-2j)\, R_j(\sigma)\, h_{n-j}h_j + m!(n-2m)!\, r_m(\sigma)\, h_{n-m}h_m,$$

where $R_j(\sigma)$ denotes the number of ways to place $n-j-1$ non-attacking rooks on $[n-j-1]^2$ so that exactly j rooks occupy positions belonging to the Young diagram of shape σ in the southwest corner of $[n-j-1]^2$.

Proof. If σ is small, then as we remarked earlier, the graph of any placement of non-attacking rooks on $\bar{B}_\sigma$ will be of type $(2^k1^{n-2k}; \varnothing)$ for some $k \leqslant m$, and so Theorem 3.2 implies

$$Z[B_\sigma] = \sum_{k=0}^{m} k!(n-2k)!\, r_k(\sigma)\, f_{2^k1^{n-2k}}.$$

If we apply Lemma 4.2, using the fact that

$$\binom{n-k-j}{k-j} + \binom{n-k-j-1}{k-j-1} = \frac{(n-2j)(n-k-j-1)!}{(k-j)!\,(n-2k)!}$$

except when $j=k=m$, we obtain

$$\begin{aligned} Z[B_\sigma] &= m!\,(n-2m)!\, r_m(\sigma)\, h_{n-m}h_m \\ &\quad + \sum_{j<m} j!\,(n-2j)\, h_{n-j}h_j \sum_{k\geqslant j} (-1)^{k-j}\binom{k}{j}(n-k-j-1)!\, r_k(\sigma). \end{aligned}$$

Therefore, to complete the proof, we need only verify that

$$R_j(\sigma) = \sum_{k\geqslant j} (-1)^{k-j}\binom{k}{j}(n-k-j-1)!\, r_k(\sigma)$$

for $j < m$. However, this is a straightforward application of the Principle of Inclusion–Exclusion (cf. (3.2)). We should point out that since σ is small, we have $\max(\ell(\sigma), \sigma_1) \leqslant n-m$, so the Young diagram of σ will indeed fit inside a square board of order $n-j-1$ whenever $j<m$. ∎

Thus Conjecture 1.1 is valid when $\theta = (N)$ and $\sigma(\mu/\nu)$ is small.

4.4. *Remark.* It is interesting to note that in the above proof, no use is made of the fact that the cells of $\bar{B}_\sigma$ form a Young diagram. Indeed, the proof applies equally well to any board B obtained by deleting from $[n]^2$ any subset of the southwestern $m \times (n-m)$ rectangle. These boards are distinguished by the fact that the graph of $\bar{B}$ is bipartite. By Theorem 3.2, it follows that for such B, we have

$$Z[B] = \sum_k k!\,(n-2k)!\,r_k(\bar{B})\,f_{2^k 1^{n-2k}},$$

where in this case (cf. (3.2)), $r_k(\bar{B})$ equals the number of k-edge matchings in the graph of $\bar{B}$. In this context, the fact that $Z[B]$ has a nonnegative h_α-expansion could be equivalently viewed as the assertion that matching polynomials of bipartite graphs satisfy a certain system of linear inequalities. Although there is a considerable amount known about matching polynomials (e.g., see [GG], [Go]), these inequalities seem to be new.

5. Some Conjectures

In this section, we use Z_σ as an abbreviation for the cycle-indicator $Z[B_\sigma]$, or equivalently (Prop. 4.1), for $E^\theta_{\mu/\nu}$ in the case $\theta = (N)$, where $\sigma = \sigma(\mu/\nu)$.

The following conjecture, if true, would reduce Conjecture 1.1 to the case $\theta = (N)$.

5.1. *Conjecture.* $E^\theta_{\mu/\nu}$ is a nonnegative integer linear combination of Z_σ's.

We remark that the integer constraint is necessary for this conjecture to be interesting. Indeed, for any $\alpha \vdash n$, there exist boards B_σ with the property that w is B_σ-compatible if and only if w belongs to the Young subgroup $\mathscr{S}_\alpha$. For such σ, one has

$$Z_\sigma = \alpha_1! \cdots \alpha_l!\,h_\alpha,$$

and so Conjecture 1.1 would trivially imply that any $E^\theta_{\mu/\nu}$ is a nonnegative *rational* linear combination of Z_σ's.

5.2. EXAMPLE. For $n=2$, there are two partitions σ such that $\sigma \subseteq \delta = (1, 0)$; namely, $\varnothing$ and (1). For these partitions, we have $Z_{\varnothing} = p_1^2 + p_2$ and $Z_{(1)} = p_1^2$. It follows that $F_{\mu/\nu}(x, y)$ has the decomposition (cf. (1.1), (1.2))

$$\begin{aligned} F_{\mu/\nu}(x, y) &= p_1^2(y)\, h_{\mu_1-\nu_1}(x)\, h_{\mu_2-\nu_2}(x) + p_2(y)\, h_{\mu_1-\nu_2+1}(x)\, h_{\mu_2-\nu_1-1}(x) \\ &= Z_{(1)}(y)\, s_{\mu/\nu}(x) + Z_{\varnothing}(y)\, h_{\mu_1-\nu_2+1}(x)\, h_{\mu_2-\nu_1-1}(x). \end{aligned}$$

The coefficient of $s_\theta(x)$ in the above expression can be easily shown to be a nonnegative integer combination of the Z_σ's via (1.4) and the Littlewood–Richardson rule [M, I.9].

5.3. EXAMPLE. For $n=3$, there are five partitions σ such that $\sigma \subseteq \delta = (2, 1, 0)$; the corresponding cycle indicators Z_σ are

$$\begin{aligned} Z_{\varnothing} &= p_1^3 + 3p_2 p_1 + 2p_3, & Z_{(2)} = Z_{(1,1)} &= p_1^3 + p_2 p_1 \\ Z_{(1)} &= p_1^3 + 2p_2 p_1 + p_3, & Z_{(2,1)} &= p_1^3. \end{aligned}$$

In this case, judicious applications of (1.1) can be used to show that

$$\begin{aligned} F_{\mu/\nu}(x, y) = {} & Z_{(2,1)}(y)\, s_{\mu/\nu}(x) + Z_{(2)}(y)[h_{\mu_3-\nu_3} s_{(\mu_1,\mu_2)/(\nu_1,\nu_2)} - s_{\mu/\nu}](x) \\ & + Z_{(1,1)}(y)[h_{\mu_1-\nu_1} s_{(\mu_2,\mu_3)/(\nu_2,\nu_3)} - s_{\mu/\nu}](x) \\ & + Z_{(1)}(y)[h_{\mu_3-\nu_1-2} s_{(\mu_1+1,\mu_2+1)/(\nu_2,\nu_3)} \\ & + h_{\mu_1-\nu_3+2} s_{(\mu_2,\mu_3)/(\nu_1+1,\nu_2+1)}](x) \\ & + Z_{\varnothing}(y)\, h_{\mu_1-\nu_3+2}(x)\, h_{\mu_2-\nu_2}(x)\, h_{\mu_3-\nu_1-2}(x). \end{aligned}$$

Again, the Littlewood–Richardson rule can be used to verify that for each θ, the coefficient of $s_\theta(x)$ in the above decomposition is a nonnegative integer combination of Z_σ's.

Recall that the Kostka number $K_{\theta,\alpha}$ can be interpreted as the number of *tableaux* (also known as reverse column-strict plane partitions) of *shape* θ and *content* α [M, (6.4)], [Sta1, (27)]; i.e., the number of integer arrays of shape θ in which: (a) the rows are weakly increasing, (b) the columns are strictly increasing, and (c) the number i occurs α_i times. For instance,

$$\begin{matrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 5 & \\ 3 & 5 & & & \end{matrix}$$

is a tableau of shape (5, 4, 2) and content (3, 4, 2, 0, 2).

Note that (1.5) shows that the coefficient of p_1^n in $E_{\mu/\nu}^\theta$ is $K_{\theta,\mu-\nu}$, whereas the coefficient of p_1^n in Z_σ is always 1. Thus, whenever $E_{\mu/\nu}^\theta$ is expressed as a linear combination of Z_σ's, the sum of the coefficients must be $K_{\theta,\mu-\nu}$. We have a (not completely precise) conjecture that there should exist a "natural" decomposition of $E^\theta_{\mu/\nu}$ proving Conjecture 5.1, in which the terms of the decomposition are indexed by the tableaux of shape θ with content $\mu-\nu$.

To describe this conjecture, suppose we have a pair of partitions σ, τ such that $\sigma \subseteq \delta(r)$ and $\tau \subseteq \delta(n-r)$, where $\delta(r)=(r-1, r-2, ..., 0)$ denotes the rth staircase. Let $\sigma \oplus \tau \subseteq \delta(n)$ denote the partition whose diagram is obtained by adjoining σ and τ to the sides of an $r \times (n-r)$ rectangle. In these terms $w \in \mathscr{S}_n$ will be $B_{\sigma\oplus\tau}$-compatible if and only if $w = w_1 w_2 \in \mathscr{S}_r \times \mathscr{S}_{n-r}$, where w_1 (resp., w_2) is compatible with B_σ (resp., B_τ). We therefore have

$$Z_{\sigma\oplus\tau} = Z^{(r)}_\sigma Z^{(n-r)}_\tau,$$

where the superscripts r and $n-r$ remind us that the cycle indicators are taken with respect to boards of order r and $n-r$, respectively. We will refer to $\sigma \oplus \tau$ as a *factorization* of *type* $(r, n-r)$. More generally, given any composition $\alpha = (r_1, ..., r_l)$ of n, together with partitions $\sigma^j \subseteq \delta(r_j)$, we will refer to $\sigma^1 \oplus \cdots \oplus \sigma^l$ as a factorization of type α.

5.4. *Conjecture.* If $\theta \vdash N = |\mu| - |\nu|$ and $\sigma = \sigma(\mu/\nu)$, then

$$E^\theta_{\mu/\nu} = \sum_T Z_{\sigma(T)},$$

summed over all tableaux T of shape θ and content $\mu-\nu$, where $\sigma(T)$ is some partition satisfying $\sigma \subseteq \sigma(T) \subseteq \delta$. Moreover, for each T there exists some sort of "descent set" $S = \{j_1, ..., j_l\} \subseteq [n-1]$ satisfying (but not necessarily defined by) the following properties: (1) If every $j+1$ lies in a lower row of T than every j, then $j \in S$; (2) if no $j+1$ lies in a lower row of T than any j, then $j \notin S$; (3) $\sigma(T)$ has a factorization $\sigma^0(T) \oplus \cdots \oplus \sigma^l(T)$ of type $(j_1, j_2 - j_1, ..., n - j_l)$, for suitable partitions $\sigma^i(T)$.

This conjecture is imprecise on two counts: we do not know how to define the descent set S, and given S, we do not know how to define the component partitions $\sigma^i(T)$. When μ/ν is a border-strip, it is easy to see that the summands of (2.3) are in one-to-one correspondence with the tableaux of shape θ and content $\mu-\nu$ (cf. Lemma 2.3). Furthermore, in view of the fact that

$$J_{(a_1)} \cdots J_{(a_l)} = Z_\sigma, \qquad \text{where} \quad \sigma = \delta(a_1 - 1) \oplus \cdots \oplus \delta(a_l - 1),$$

one may verify that Theorem 2.4 does agree with the terms of our conjecture, provided that we define the descent set of a tableau T to consist of those numbers j with the property that there exists some j in a higher row of T than some $j+1$.

Our final conjecture is based on the observation that for any partition $\sigma \subseteq \delta$, the positions of $[n]^2$ indexed by $\bar{B}_\sigma$ can be used to define a partial order P_σ of $[n]$ in which $i > j$ if and only if $(i, j) \in \bar{B}_\sigma$.

For any partition α of n, let us define $c_\alpha(P)$ to be the number of ways to partition an n-element poset P into (unordered) chains of cardinality $\alpha_1, \alpha_2, ...$, and let $\bar{c}_\alpha(P) := m_1(\alpha)!\, m_2(\alpha)! \cdots c_\alpha(P)$ denote the number of partitions of P into *ordered* chains of cardinality $\alpha_1, \alpha_2, ...$. When $B = B_\sigma$, the quantity $m_1(\alpha)!\, m_2(\alpha)! \cdots r_{\alpha,\varnothing}(\bar{B})$ appearing in Theorem 3.2 can be identified as $\bar{c}_\alpha(P_\sigma)$. Thus by Proposition 4.1 and Theorem 3.2, the special case $\theta = (N)$ of Conjecture 1.1 is equivalent to the assertion that

$$\sum_{\alpha \vdash n} \bar{c}_\alpha(P_\sigma) \cdot f_\alpha$$

is a nonnegative linear combination of the h_λ's. Since the automorphism ω maps f_α to m_α and h_λ to the product of elementary symmetric functions e_λ [M, I.2], this conjecture could also be expressed as the assertion that if $P = P_\sigma$, then

$$\sum_{\alpha \vdash n} \bar{c}_\alpha(P) \cdot m_\alpha \tag{5.1}$$

is a nonnegative linear combination of the e_λ's.

Since (5.1) is well-defined for any n-element poset P, it is natural to consider the problem of determining the class of posets that share with the P_σ's the (conjectured) property that the coefficient of e_λ in (5.1) is nonnegative for all partitions λ of n. We will refer to this condition as the *Poset–Chain* (PC) property.

Recall that a poset P is an induced subposet of Q if there is an order-preserving injection of P into Q whose inverse is also order-preserving. By a theorem of Dean and Keller [DK], the posets P_σ are characterized by the fact that they do not contain $[3]+[1]$ or $[2]+[2]$ as induced subposets, where "+" denotes disjoint union of posets, and by abuse of notation, $[k]$ denotes the k-element chain $\{1 < \cdots < k\}$. This suggests the possibility that the presence of $[3]+[1]$ and/or $[2]+[2]$ as an induced subposet destroys the Poset–Chain property. However, Remark 4.4 shows that any poset P that contains no three-element chain (a class which includes many examples containing induced copies of $[2]+[2]$) does satisfy the Poset–Chain property. Thus we are left with $[3]+[1]$ as a possible culprit.

We have used the computer algebra package *Maple* to classify all posets on $\leqslant 7$ vertices according to whether or not they contain $[3]+[1]$ as an induced subposet, and whether or not they satisfy the Poset–Chain property. The following table lists the number in each category.

Number of vertices	4	5	6	7
PC true, no induced [3] + [1]	15	49	173	639
PC false, no induced [3] + [1]	0	0	0	0
PC true, induced [3] + [1]	0	5	39	468
PC false, induced [3] + [1]	1	9	106	938

5.5. *Conjecture.* Any partial order that does not contain [3] + [1] as an induced subposet satisfies the Poset–Chain property.

5.6. *Remark.* For any $B \subseteq [n]^2$, let $g_\lambda(B)$ denote the coefficient of h_λ in $Z[B]$. There is a standard inner product $\langle \cdot, \cdot \rangle$ of symmetric functions [M, I.4] for which the h_λ's and m_λ's are dual bases. We therefore have $g_\lambda(B) = \langle Z[B], m_\lambda \rangle$. Since $h_n = \sum_{\lambda \vdash n} m_\lambda$ and $\langle p_\lambda, h_n \rangle = 1$ for all $\lambda \vdash n$, it follows that

$$\eta(B) = \langle Z[B], h_n \rangle = \sum_{\lambda \vdash n} g_\lambda(B),$$

where, as in Section 3, $\eta(B)$ denotes the number of B-compatible permutations. This suggests that for $B = B_\sigma$ (or any board arising from a poset satisfying the Poset–Chain property), it should be possible to assign a partition λ to each B-compatible permutation so that $g_\lambda(B)$ counts the number of these assigned to λ.

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