EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 14, No. 1, 2021, 149-163
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On Independent Transversal Dominating Sets in Graphs

Daven S. Sevilleno[1,*], Ferdinand P. Jamil[2]

[1] *Department of Mathematics and Statistics, College of Liberal Arts, Sciences and Education, University of San Agustin, 5000 Iloilo City Philippines*
[2] *Department of Mathematics and Statistics, College of Science and Mathematics, Center of Graph Theory, Algebra and Analysis, Premier Research Institute of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. A set $S \subseteq V(G)$ is an independent transversal dominating set of a graph G if S is a dominating set of G and intersects every maximum independent set of G. An independent transversal dominating set which is a total dominating set is an independent transversal total dominating set. The minimum cardinality $\gamma_{it}(G)$ (resp. $\gamma_{itt}(G)$) of an independent transversal dominating set (resp. independent transversal total dominating set) of G is the independent transversal domination number (resp. independent transversal total domination number) of G. In this paper, we show that for every positive integers a and b with $5 \leq a \leq b \leq 2a-2$, there exists a connected graph G for which $\gamma_{it}(G) = a$ and $\gamma_{itt}(G) = b$. We also study these two concepts in graphs which are the join, corona or composition of graphs.

2020 Mathematics Subject Classifications: 05C22, 05C69, 05C776
Key Words and Phrases: Independent transversal dominating set, Independent transversal total dominating set, Independent transversal domination number, Independent transversal total domination number

1. Introduction

Throughout this paper, by a graph $G = (V(G), E(G))$ is meant a finite, simple and connected graph with $V(G)$ and $E(G)$ being the vertex set and edge set, respectively. For $S \subseteq V(G)$, $|S|$ is the cardinality of S. In particular, $|V(G)|$ is called the *order* of G. All basic terminologies used here are adapted from [4].

Given two graphs G and H with disjoint vertex sets,

- the *union* of G and H, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$;

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v14i1.3904

Email addresses: daven.sevilleno@g.msuiit.edu.ph (D. Sevilleno), ferdinand.jamil@g.msuiit.edu.ph (F. Jamil)

- the *join* of G and H is the graph $G+H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$;
- the *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H; and
- the *composition* $G[H]$ of G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u,v)(u',v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

For corona of graphs $G \circ H$, if $H = K_1$, we write $G \circ H = cor(G)$. It is customary to denote by H^v that copy of H whose vertices are joined with the vertex v of G. Similarly, we also write u^v to denote that copy of $u \in V(H)$ in H^v.

For vertices u and v of G, a *u-v geodesic* is any shortest u-v path. The *distance* between u and v is the length of a u-v geodesic, and is denoted by $d_G(u,v)$. The *eccentricity* of v, denoted by $e(v)$, is given by $e(v) = \max\{d_G(u,v) : u \in V(G)\}$. The *diameter* of G, denoted by $diam(G)$, is defined by $diam(G) = \max\{d_G(u,v) : u, v \in V(G)\}$. Any geodesic of length $diam(G)$ is called *diametral path* of G.

Vertices u and v of a graph G are *neighbors* if $uv \in E(G)$. The *open neighborhood* of v refers to the set $N_G(v)$ consisting of all neighbors of v. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = \cup_{v \in S} N_G[v]$. A subset $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The minimum cardinality $\gamma(G)$ of a dominating set in G is the *domination number* of G. We refer to [1, 2, 5, 7, 9, 11, 14, 16, 19] for the history, fundamental concepts and the subsequent developments of domination in graphs as well as its various applications.

Provided G has no isolated vertices, a dominating set S is a *total dominating set* of G if every vertex in S is adjacent to another vertex in S. The minimum cardinality of a total dominating set of G is the *total domination number* of G denoted by $\gamma_t(G)$. References [6, 8, 12] are excellent studies on total domination in graphs.

A subset $S \subseteq V(G)$ is an *independent set* of G if for every distinct vertices u and v in S, $uv \notin E(G)$. The maximum cardinality of an independent set is called the *independence number* of G, and is denoted by $\beta_0(G)$. Any independent set of cardinality $\beta_0(G)$ is referred to as a *β_0-set* of G. In [13], it is called a *maximum independent set*. It is worth noting that if G is complete, then $\{v\}$ is a β_0-set for all $v \in V(G)$. The symbol $XI(G)$ denotes the family of all β_0-sets of G, and $xi(G) = |XI(G)|$.

A subset $S \subseteq V(G)$ of G is said to be an *independent transversal set* of G if S intersects every β_0-set of G. The minimum cardinality of an independent transversal set of G is called the *independent transversal number*, denoted by $\beta_{0t}(G)$. In particular, $\beta_{0t}(K_n) = n$; for path P_n, $\beta_{0t}(P_n) = 1$ when n is odd and $\beta_{0t}(P_n) = 2$ otherwise; for cycle C_n on $n \geq 3$ vertices, $\beta_{0t}(C_n) = 2$ when n is even and $\beta_{0t}(C_n) = 3$ otherwise; and for the complete bipartite $K_{m,n}$, $\beta_{0t}(K_{m,n}) = 1$ if $m \neq n$, and $\beta_{0t}(K_{m,n}) = 2$ otherwise..

An independent transversal set of G is an *independent transversal dominating set* or (*ITD-set*) if S is a dominating set of G. The minimum cardinality of an ITD-set of G is called the *independent transversal domination number* of G and is denoted by $\gamma_{it}(G)$. An

ITD-set S of G with $|S| = \gamma_{it}(G)$ is called a γ_{it}-set. The study on independent transversal domination in graphs was initiated by I. Hamid [10] in 2012. It was studied further by Yero et al. [17, 18, 20] in 2016, 2017 and 2021, and by Ozeki et al. [3] in 2018.

Provided G has no isolated vertex, a subset $S \subseteq V(G)$ is an *independent transversal total dominating set* or ($ITTD$-*set*) if S is both an ITD-set and a total dominating set of G. We denote by $\gamma_{itt}(G)$ the minimum cardinality of an $ITTD$-set of G, and is called the *independent transversal total domination number* of G. In [17], Cabrera Martinez et al. made a good introduction of this concept. The authors prove that the complexity of the decision problem associated to the computation of the value of $\gamma_{itt}(G)$ is NP-complete, under the assumption that the independence number is known.

Henceforth, all discussions of total domination or independent transversal total domination are always with respect to graphs without isolated vertices.

For graphs G of order $n \geq 2$, $\gamma_{it}(G) \leq \gamma_{itt}(G)$.

2. Preliminaries and Known Results

Observation 1. (*i*) *For the complete bipartite* $K_{m,n}$, $\gamma_{it}(K_{m,n}) = \gamma_{itt}(K_{m,n}) = 2$.

(*ii*) *For any path* P_n *of order* $n \geq 2$, *we have*

(*a*) [10] $$\gamma_{it}(P_n) = \begin{cases} 2 & \text{if } n = 2, 3, \\ 3 & \text{if } n = 6, \\ \lceil \frac{n}{3} \rceil & \text{otherwise,} \end{cases}$$

and

(*b*) $$\gamma_{itt}(P_n) = \begin{cases} 3 \text{ if } n = 4, \\ \frac{n}{2} + 1 \text{ if } n \equiv 2 (\text{mod } 4), \\ \lceil \frac{n}{2} \rceil, \text{ otherwise} \end{cases}$$

(*iii*) *For any cycle* C_n *of order* $n \geq 3$, *we have*

(*a*) [10] $$\gamma_{it}(C_n) = \begin{cases} 3 \text{ if } n = 3, 5, \\ \lceil \frac{n}{3} \rceil \text{ otherwise,} \end{cases}$$

and

(*b*) $$\gamma_{itt}(C_n) = \begin{cases} 3 \text{ if } n = 3, \\ \frac{n}{2} + 1 \text{ if } n \equiv 2 (\text{mod } 4), \\ \lceil \frac{n}{2} \rceil, \text{ otherwise.} \end{cases}$$

Theorem 1. *For any graph* G, *we have*

(*i*) [10] $\gamma(G) \leq \gamma_{it}(G) \leq \gamma(G) + \delta(G)$; *and*

(*ii*) [17] $\gamma_t(G) \leq \gamma_{itt}(G) \leq \gamma_t(G) + \delta(G)$.

Corollary 1. *Let* G *be a graph.*

(*i*) [10] *If* G *has an isolated vertex, then* $\gamma_{it}(G) = \gamma(G)$.

(*ii*) *If* G *has* K_2 *as a component, then* $\gamma_{itt}(G) = \gamma_t(G)$.

If follows from Observation 1 that if $G = P_{3n-2}$, then $\gamma_{it}(G) = \gamma(G) = n$. Thus, the following corollary is clear.

Corollary 2. *For all positive integers* n*, there exists a connected graph* G *for which* $\gamma(G) = \gamma_{it}(G) = n$.

Observation 2. *Let* G *be a connected graph of order* n*. Then*

(*i*) $\gamma_{it}(G) = n$ *if and only if* $G = K_n$*. Provided* $n \geq 2$, $\gamma_{itt}(G) = n$ *if and only if* $G = K_n$.

(*ii*) *For* $n \geq 3$,

  (*a*) $\gamma_{it}(G) = n - 1$ *if and only if* $G = P_3$.

  (*b*) $\gamma_{itt}(G) = n - 1$ *if and only if* $G = P_3$ *or* $G = P_4$.

(*iii*) $\gamma_{it}(G) = 2$ *if and only if either* $G = K_2$ *or* G *has a dominating set* $\{u, v\}$ *such that every* β_0*-set contains* u *or* v.

(*iv*) $\gamma_{itt}(G) = 2$ *if and only if either* $G = K_2$ *or* G *has a total dominating set* $\{u, v\}$ *such that every* β_0*-set contains either* u *or* v.

(*v*) $\gamma_{it}(G) = 1$ *if and only if* $n = 1$.

In Observation 2(*iii*), a β_0-set may contain both u and v. Consider, for example, the graph $G = K_{n-1} + v$ for $n \geq 2$.

3. Realization Problems

Theorem 2. [20] *For every positive integers* a, b, c *such that* $c \geq 2$ *and* $a \leq b \leq a + c$*, there exists a graph* G *such that* $\delta(G) = c$, $\gamma(G) = a$ *and* $\gamma_{it}(G) = b$.

Theorem 3. [17] *For any two positive integers* a *and* b *such that* $2 \leq a \leq \frac{2b}{3}$*, there is a graph of order* b *such that* $\gamma_{itt}(G) = a$.

Theorem 4. *For any positive integers* a *and* b *with* $3 \leq a \leq b$ *there exists a connected graph* G *for which* $\gamma_t(G) = a$ *and* $\gamma_{itt}(G) = b$.

Proof. We consider the following cases:

Case 1: Suppose that $a = b$. If $a = b = 3$, then we choose $G = C_5$. Suppose that $a = b \geq 4$. If $a \equiv 0, 2 (\text{mod } 4)$, then choose $G = P_{2a}$. If $a \equiv 1, 3 (\text{mod } 4)$, then choose $G = P_{2a-1}$. In any case, $\gamma_t(G) = \gamma_{itt}(G) = a$.

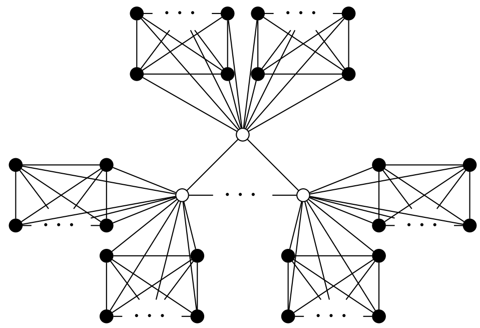

Figure 1: Graph $G = C_a \circ \{K_n \cup K_n\}$ with $a \geq 3$

Case 2: Suppose that $a < b$. The $b = a + n$ for some positive integer n. Consider the corona $G = C_a \circ \{K_n \cup K_n\}$ as shown in Figure 1. For convenience, write $(K_n \cup K_n)^v = K_n^v \cup K_n^v$. Clearly, $V(C_a)$ is a γ_t-set of G so that $\gamma_t(G) = a$. Let $v \in V(C_a)$ and define $S = V(C_a) \cup V(K_n^v)$. Then, S is a total dominating set of G. Now a β_0-set M of G is of the form $M = \cup_{v \in V(C_a)} \{a^v, b^v\}$, where a^v and b^v belong to different components of $K_n^v \cup K_n^v$. Then $M \cap S \neq \varnothing$. Thus, S is an $ITTD$-set of G. Thus, $\gamma_{itt}(G) \leq |S| = a + n = b$. Let $T \subseteq V(G)$ be a total dominating set with $|T| < b$. Then $V(K_n^v) \setminus T \neq \varnothing$ for all $v \in V(C_a)$. For each $v \in V(C_a)$, pick $a^v, b^v \in V(K_n^v \cup K_n^v) \setminus T$ such tha a^v and b^v are from different components. Then $M = \{a^v, b^v : v \in V(G)\}$ is a maximum independent set of G. Since $M \cap T = \varnothing$, T is not an $ITTD$-set of G. Since T is arbitrary, $\gamma_{itt}(G) = b$. □

Theorem 5. *For every positive integers n, a and b with $5 \leq a \leq b \leq 2a - 2$, there exists a connected graph G for which $\gamma_{it}(G) = a$ and $\gamma_{itt}(G) = b$.*

Proof. For $k \geq 2$, denote by $G(k)$ the graph given in Figure 2.

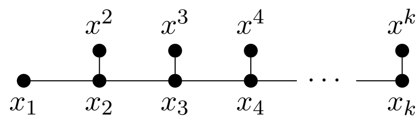

Figure 2: Graph $G(k)$

Suppose that $a = b$. Take the graph G as in Figure 3 obtained from the graph $G(a-1)$ by adding the path $[x_{a-1}, x_a, x_{a+1}, x_{a+2}, x^{a+2}]$. Observe that $\{x_2, x_3, ..., x_{a-1}, x_{a+2}\}$ is a

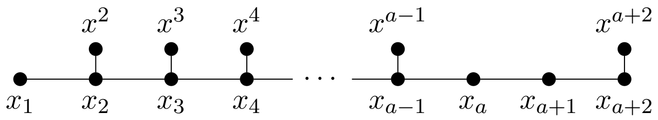

Figure 3: Graph G with $\gamma_{it}(G) = \gamma_{itt}(G)$

γ-set of G and every maximum independent set of G contains either x_{a+2} or x^{a+2}. Thus, the set $D = \{x_2, x_3, ..., x_{a-1}, x_{a+2}, x^{a+2}\}$ is a γ_{it}-set of G. Note that $[x_2, x_3, ..., x_{a-1}]$ is a path of length at least 2. Hence D is also a γ_{itt}-set of G. Therefore $\gamma_{it}(G) = \gamma_{itt}(G) = (a-2)+2 = a$.

Suppose now that $a < b$. Then $b = a + k$ for some positive integer $1 \leq k \leq a-2$. For $k = 1$, consider the graph G in Figure 4 obtained from $G(a-2)$ by adding the paths $[x_{a-2+3j}, x_{a-1+3j}, x_{a+3j}, x_{a+1+3j}, x^{a+1+3j}]$ where $j = 0, 1$. By a similar argument

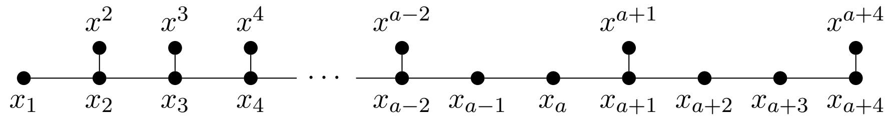

Figure 4: Graph G with $\gamma_{itt}(G) = \gamma_{it}(G) + 1$

$D = \{x_2, x_3, ..., x_{a-2}, x_{a+1}, x_{a+4}, x^{a+4}\}$ is a γ_{it}-set of G. This implies that $\gamma_{it}(G) = (a-3)+3 = a$. It also follows that $D \cup \{x^{a+1}\}$ is a γ_{itt}-set of G. Thus, $\gamma_{itt}(G) = a + 1 = b$.

For $2 \leq k \leq a-4$, consider the graph G in Figure 5 obtained from $G(a-k-1)$ by adding $(k+1)$ paths $[x_{a-k-1+3j}, x_{a-k+3j}, x_{a-k+1+3j}, x_{a-k+2+3j}, x^{a-k+2+3j}]$ $(j = 0, 1, \cdots, k-1)$ and $[x_{a+2k}, x_{a+2k+1}, x_{a+2k+2}, x_{a+2k+3}, x_{a+2k+4}, x_{a+2k+5}]$. Then $D = \{x_2, x_3, \ldots, x_{a-k-1}\} \cup$

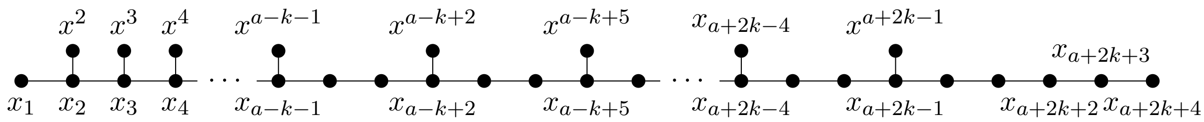

Figure 5: Graph G with $\gamma_{itt}(G) = \gamma_{it} + k$, where $2 \leq k \leq a-4$

$\{x_{a-k+2+3j} : j \in \{0, 1, 2, \ldots, k-1\}\} \cup \{x_{a+2k+2}, x_{a+2k+4}\}$ is a γ_{it}-set of G. On the other hand, $D \cup \{x^{a-k+2+3j} : j \in \{0, 1, 2, \ldots, k-1\}\} \cup \{x_{a+2k+3}\}$ is a γ_{itt}-set of G. Thus, $\gamma_{it}(G) = (a-k-2)+k+2 = a$ and $\gamma_{itt}(G) = a + k = b$.

For $k = a-3$, consider the graph G in Figure 6 obtained from $G(2)$, by adding $(a-2)$ paths $[x_2, x_3, x_4, x^4]$, $[x_{4+3j}, x_{5+3j}, x_{6+3j}, x_{7+3j}, x^{7+3j}]$, $j = 0, 1, 2, ..., a-5$, and $[x_{a+2k-2}, x_{a+2k-1}, x_{a+2k}, x_{a+2k+1}, x_{a+2k+2}, x_{a+2k+3}]$. Then $D = \{x_2\} \cup \{x_{4+3j} | j = 0, 1, 2, ..., a-4\} \cup \{x_{a+2k+1}, x_{a+2k+2}\}$ is a γ_{it}-set of G. On the other hand, $D \cup \{x_3\} \cup \{x^{4+3j} | j = 0, 1, 2, ..., a-4\}$ is a γ_{itt}-set of G. Thus, $\gamma_{it}(G) = 1 + a - 3 + 2 = a$, and $\gamma_{itt}(G) = a + (a-3) = a + k$.

Now, for $k = a-2$, consider the graph G as in Figure 7 obtained by adding $(a-2)$ P_5 paths $[x_{2+3j}, x_{3+3j}, x_{4+3j}, x_{5+3j}, x^{5+3j}]$, $j = 0, 1, \ldots, a-3$, to the path $G(2)$. Then $D = \{x_{3a-4}, x^{3a-4}, x_{2+3j} : j \in \{0, 1, 2, \ldots, a-3\}\}$ is a γ_{it}-set of G, and $D \cup \{x^{2+3j} : j \in \{0, 1, 2, \ldots, a-3\}\}$ forms a γ_{itt}-set of G. Therefore, $\gamma_{it}(G) = 2 + (a-2)$ and $\gamma_{itt}(G) = a + (a-2) = b$. □

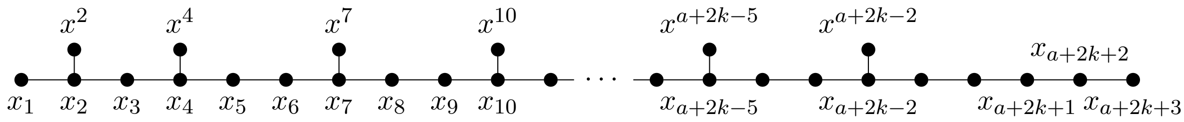

Figure 6: Graph G with $\gamma_{itt}(G) = \gamma_{it}(G) + a - 3$

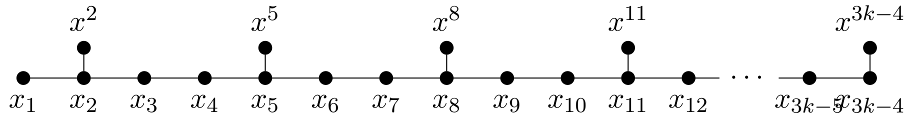

Figure 7: Graph G with $\gamma_{itt}(G) = \gamma_{it}(G) + a - 2$

4. On join of graphs

For an independent subset $D \subseteq V(G+H)$, either $D \subseteq V(G)$ or $D \subseteq V(H)$. In particular, if $\beta_0(H) < \beta_0(G)$, then D is a β_0-set of $G+H$ if and only if $D \subseteq V(G)$ and is a β_0-set of G.

Proposition 1. *Let G and H be connected graphs with $\beta_0(H) < \beta_0(G)$. Then $D \subseteq V(G+H)$ is an independent transversal (total) dominating set of $G+H$ if and only if one of the following holds:*

(*i*) *$D \subseteq V(G)$ and D is an independent transversal (total) dominating set of G;*

(*ii*) *$D \cap V(G)$ is an independent transversal set of G and $D \cap V(H) \neq \varnothing$.*

Proof. Let $D \subseteq V(G+H)$. Assume that D is an independent transversal (total) dominating set of $G+H$. In view of the preceding remark, $D \cap V(G) \neq \varnothing$. Suppose that $D \subseteq V(G)$. Then D is a (total) dominating set of G. Let $M \subseteq V(G)$ be a β_0-set of G. Then M is a β_0-set of $G+H$. Thus, $D \cap M \neq \varnothing$. This shows that D is an independent transversal (total) dominating set of G. Suppose that D intersects both $V(G)$ and $V(H)$, and let $M \subseteq V(G)$ be a β_0-set of G. Since M is a β_0-set of $G+H$, $M \cap (D \cap V(G)) = M \cap D \neq \varnothing$. Thus, $D \cap V(G)$ is an independent transversal set of G.

Conversely, since (total) dominating sets of G are (total) dominating sets of $G+H$, in view of the preceding remark, if (*i*) holds for $D \subseteq V(G)$, then D is an independent transversal (total) dominating set of $G+H$. Suppose that (*ii*) holds for D. Then D is a (total) dominating set of $G+H$. Let $M \subseteq V(G+H)$ be a β_0-set of $G+H$. Then $M \subseteq V(G)$ and is a β_0-set of G. Thus, $D \cap M = (D \cap V(G)) \cap M \neq \varnothing$. This shows that D is an independent transversal (total) dominating of $G+H$. □

Proposition 2. *Let G and H be connected graphs with $\beta_0(G) = \beta_0(H)$. Then $D \subseteq V(G+H)$ is an independent transversal (total) dominating set of $G+H$ if and only if $D \cap V(G)$ and $D \cap V(H)$ are independent transversal sets of G and H, respectively.*

Proof. Let $D \subseteq V(G+H)$. Suppose that D is an ITD-set of $G+H$. We claim that D intersects both $V(G)$ and $V(H)$ so that D is in fact an $ITTD$-set of $G+H$. Suppose not, say $D \subseteq V(G)$. Pick a β_0-set $M \subseteq V(H)$ of H. Since M is a β_0-set of $G+H$, $D \cap M \neq \varnothing$, which is impossible.

Now, for all β_0-sets $M \subseteq V(G)$ of G, M is a β_0-set of $G+H$ so that $(D \cap V(G)) \cap M = D \cap M \neq \varnothing$. This means that $D \cap V(G)$ is an independent transversal set of G. Similarly, $D \cap V(H)$ is an independent transversal set of H.

Conversely, suppose that $D \cap V(G)$ and $D \cap V(H)$ are independent transversal sets of G and H, respectively. Then D is a total dominating set of $G+H$. Let $M \subseteq V(G+H)$ be a β_0-set of $G+H$. Either $M \subseteq V(G)$ and is a β_0-set of G or $M \subseteq V(H)$ and is a β_0-set of H. Either case yields $D \cap M \neq \varnothing$. Thus, D is an independent transversal (total) dominating set of $G+H$. □

Corollary 3. *Let G and H be connected graphs with $\beta_0(H) \leq \beta_0(G)$. Then the following hold:*

(i) *If $\beta_0(H) = \beta_0(G)$, then $\gamma_{it}(G+H) = \gamma_{itt}(G+H) = \beta_{0t}(G) + \beta_{0t}(H)$.*

(ii) *If $\beta_0(H) < \beta_0(G)$ and $\beta_{0t}(G) < \gamma_{it}(G)$, then $\gamma_{it}(G+H) = \gamma_{itt}(G+H) = 1+\beta_{0t}(G)$.*

(iii) *If $\beta_0(H) < \beta_0(G)$ and $\beta_{0t}(G) = \gamma_{it}(G) = \gamma_{itt}(G)$, then $\gamma_{it}(G+H) = \gamma_{itt}(G+H) = \gamma_{it}(G)$.*

(iv) *If $\beta_0(H) < \beta_0(G)$ and $\beta_{0t}(G) = \gamma_{it}(G) < \gamma_{itt}(G)$, then $\gamma_{it}(G+H) = \gamma_{it}(G)$ and $\gamma_{itt}(G+H) = 1 + \beta_{0t}(G)$.*

Proof. The case where $\beta_0(G) = \beta_0(H)$ is immediate from Proposition 2. Assume that $\beta_0(G) > \beta_0(H)$. It follows from Proposition 1 that $\gamma_{it}(G+H) = \min\{\gamma_{it}(G), 1+\beta_{0t}(G)\}$. Suppose that $\gamma_{it}(G) > \beta_{0t}(G)$. Then $\gamma_{it}(G+H) = 1+\beta_{0t}(G)$ and any γ_{it}-set of $G+H$ is a total dominating set of $G+H$. Thus, $\gamma_{itt}(G+H) = 1+\beta_{0t}(G)$. Suppose that $\gamma_{it}(G) = \beta_{0t}(G)$. If $\gamma_{it}(G) = \gamma_{itt}(G)$, then $\gamma_{itt}(G+H) = \gamma_{it}(G+H) = \gamma_{it}(G)$. Suppose that $\gamma_{it}(G) < \gamma_{itt}(G)$. Then $1+\beta_{0t}(G) \leq \gamma_{itt}(G)$. In view of Proposition 1, $\gamma_{itt}(G+H) = 1+\beta_{0t}(G)$. □

Example 1. (1) *For positive integers m, n and p with $p > \max\{m, n\}$,*

$$\gamma_{it}(K_{m,p} + K_{n,p}) = \gamma_{itt}(K_{m,p} + K_{n,p}) = 2.$$

(2) *For the fan F_n on $n+1 \geq 3$ vertices,*

$$\gamma_{it}(F_n) = \begin{cases} 2, & \text{if } n = 4 \text{ or } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even and } n \neq 4, \end{cases} \quad \text{and} \quad \gamma_{itt}(F_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

(3) *For the wheel W_n on $n+1$ vertices,*

$$\gamma_{it}(W_n) = \begin{cases} 2, & \text{if } n = 4; \\ 3, & \text{if } n \in \{3,5,7,9\} \text{ or } n \text{ is even and } n \neq 4; \\ 4, & \text{otherwise,} \end{cases} \quad \text{and}$$

$$\gamma_{itt}(W_n) = \begin{cases} 2, & \text{if } n = 4; \\ 3, & \text{if } n \in \{3,5\} \text{ or } n \text{ is even and } n \neq 4; \\ 4, & \text{otherwise.} \end{cases}$$

(4) *For all positive integers $n \geq 2$ and $p \geq 2$, $\gamma_{it}(K_{n,n} + K_p) = \gamma_{itt}(K_{n,n} + K_p) = 2$.*

5. On corona of graphs

It is worth noting that $G \circ H$ is composed of the joins $H^v + v = H^v + \langle v \rangle$, $v \in V(G)$, joined together by the edges of G. Thus,

$$V(G \circ H) = V(G) \cup \left(\cup_{v \in V(G)} V(H^v)\right) = \cup_{v \in V(G)} V(H^v + v).$$

Theorem 6. [5] *Let G be a connected graph and H any graph. Then $S \subseteq V(G \circ H)$ is a dominating set of $G \circ H$ if and only if $S \cap V(H^v + v)$ is a dominating set of $H^v + v$ for each $v \in V(G)$.*

Observation 3. *Let G be a nontrivial connected graph of order n, and let p be a positive integer. Then*

$$\gamma_{it}(G \circ K_p) = \begin{cases} n, & \text{if } p = 1; \\ n + p, & \text{if } p \geq 2, \end{cases} \quad \text{and} \quad \gamma_{itt}(G \circ K_p) = n + p.$$

A sharp bound for $\gamma_{it}(G \circ H)$ is provided in [20]

Theorem 7. [20] *Let G be a graph of order $n \geq 2$. Then for any graph H such that $\beta_0(H) \geq 2$,*

$$n - 1 + d_\beta(H) \leq \gamma_{it}(G \circ H) \leq n + d_\beta(H),$$

where $d_\beta(H)$ is the largest number of pairwise disjoint β_0-sets of H. Moreover, if there is a $d_\beta(H)$-set which is a dominating set in H, then $\gamma_{it}(G \circ H) = n - 1 + d_\beta(H)$.

Observe that $d_\beta(H) \leq \beta_{0t}(H)$ for any graph H. In what follows, we determine $\gamma_{it}(G \circ H)$ in terms of $\beta_{0t}(H)$.

Lemma 1. *Let G be a connected graph and H a noncomplete graph. Then a subset $S \subseteq V(G \circ H)$ is a β_0-set of $G \circ H$ if and only if $S = \cup_{v \in V(G)} S_v$, where $S_v \subseteq V(H^v)$ is a β_0-set of H^v for each $v \in V(G)$.*

Proposition 3. *Let G and H be nontrivial connected graphs with H noncomplete. Then $S \subseteq V(G \circ H)$ is an ITD-set of $G \circ H$ if and only if $S = A \cup \big(\cup_{v \in V(G)} S_v\big)$, where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for all $v \in V(G)$ satisfying the following:*

(i) *For each $v \in V(G) \setminus A$, S_v is a dominating set of H^v.*

(ii) *There exists $v \in V(G)$ for which S_v is an independent transversal set of H^v.*

Proof. Assume that $S \subseteq V(G \circ H)$ is an ITD-set of $G \circ H$. Then $S = A \cup \big(\cup_{v \in V(G)} S_v\big)$, where $A = S \cap V(G)$ and $S_v = S \cap V(H^v)$ for all $v \in V(G)$. Statement (i) follows immediately from Theorem 6. To prove (ii), suppose that for each $v \in V(G)$ there exists a β_0-set M_v of H^v for which $S_v \cap M_v = \varnothing$. By Lemma 1, $M = \cup_{v \in V(G)} M_v$ is a β_0-set of $G \circ H$. Since S is an independent transversal set of $G \circ H$, $\cup_{v \in V(G)} (S_v \cap M_v) = S \cap M \neq \varnothing$, which is impossible. This proves (ii).

Conversely, assume that (i) and (ii) hold for S. By condition (i) and the fact that $S \cap V(H^v + v)$ is a dominating set in $H^v + v$ whenever $v \in A$, S is a dominating set of $G \circ H$. Let $M \subseteq V(G \circ H)$ be a β_0-set of $G \circ H$. By (ii), there exists $v \in V(G)$ for which S_v is an independent transversal set of H^v. Since $M_v = M \cap V(H^v)$ is a β_0-set of H^v, $S_v \cap M_v \neq \varnothing$. Thus, $S \cap M \neq \varnothing$. Therefore, S is an ITD-set of $G \circ H$. □

In view of Proposition 3, the following assertion is clear.

Proposition 4. *Let G and H be nontrivial connected graphs with H noncomplete. Then $S \subseteq V(G \circ H)$ is an independent transversal total dominating set of $G \circ H$ if and only if $S = A \cup_{v \in V(G)} S_v$, where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for all $v \in V(G)$ satisfying the following:*

(i) *For each $v \in V(G) \setminus A$, S_v is a total dominating set of H^v.*

(ii) *For each $v \in A$, $S_v \neq \varnothing$ or $N_G(v) \cap S \neq \varnothing$.*

(iii) *There exists $v \in V(G)$ for which S_v is an independent transversal set of H^v.*

Corollary 4. *Let G and H be nontrivial connected graphs, where G is of order n and H noncomplete. Then*

$$\gamma_{it}(G \circ H) = n - 1 + \min\{\gamma_{it}(H), 1 + \beta_{0t}(H)\}, \tag{1}$$

and

$$\gamma_{itt}(G \circ H) = n - 1 + \min\{\gamma_{itt}(H), 1 + \beta_{0t}(H)\}. \tag{2}$$

Proof. Let $v \in V(G)$ and let $B_1, B_2 \subseteq V(H^v)$ be a γ_{it}-set and a β_{0t}-set of H^v, respectively. Put $S_1 = (V(G) \setminus \{v\}) \cup B_1$ and $S_2 = V(G) \cup B_2$. By Proposition 3, both S_1 and S_2 are ITD-sets of $G \circ H$. Thus, $\gamma_{it}(G \circ H) \leq \min\{|S_1|, |S_2|\} = n - 1 + \min\{\gamma_{it}(H), 1 + \beta_{0t}(H)\}$.

Let $S = A \cup \left(\cup_{v \in V(G)} S_v\right) \subseteq V(G \circ H)$ where $A \subseteq V(G)$ be a γ_{it}-set of $G \circ H$. By Proposition 3, $|S_w| \geq 1$ for all $w \in V(G) \setminus A$ and there exists $v \in V(G)$ for which S_v is an independent transversal set of H^v. We consider two cases:

Case 1: Suppose that $v \in A$. Then

$$\begin{aligned} \gamma_{it}(G \circ H) = |S| &= |A| + \sum_{w \in V(G)} |S_w| \\ &\geq n + |S_v| \\ &\geq n - 1 + \min\{\gamma_{it}(H), 1 + \beta_{0t}(H)\}. \end{aligned}$$

Case 2: Suppose that $v \notin A$. Then S_v is an ITD-set of H^v. Thus,

$$\begin{aligned} \gamma_{it}(G \circ H) = |S| &= |A| + \sum_{w \in V(G)} |S_w| \\ &\geq n - 1 + |S_v| \\ &\geq n - 1 + \min\{\gamma_{it}(H), 1 + \beta_{0t}(H)\}. \end{aligned}$$

This proves Equation 1.

Similar arguments will prove Equation 2. □

6. On composition of graphs

For a subset $C \subseteq V(G[H])$, we can always write $C = \cup_{x \in A}(\{x\} \times A_x)$ for some $A \subseteq V(G)$ and $A_x = \{y \in V(H) : (x, y) \in A\}$.

Theorem 8. [15] *Let G and H be connected graphs, and $C = \cup_{x \in S}(\{x\} \times S_x) \subseteq V(G[H])$. Then C is a dominating set of $G[H]$ if and only if one of the following holds:*

(i) S is a total dominating set of G;

(ii) S is a dominating set of G and S_x is a dominating set of H for each $x \in S \setminus N_G(S)$.

Lemma 2. *Let G and H be nontrivial connected graphs, and $C = \cup_{x \in S}(\{x\} \times S_x) \subseteq V(G[H])$. Then C is an (maximum) independent set of $G[H]$ if and only if S is an (maximum) independent set of G and S_x is an (maximum) independent set of H for each $x \in S$.*

Proposition 5. *Let G and H be nontrivial connected graphs, and $C = \cup_{x \in S}(\{x\} \times S_x) \subseteq V(G[H])$. Then C is an independent transversal dominating set of $G[H]$ if and only if each of the following holds:*

(i) One of the following holds:

(a) S is an $ITTD$-set of G.

(*b*) *S is an ITD-set of G and S_x is a dominating set of H for each $x \in S \setminus N_G(S)$.*

(*ii*) *For every pair of β_0-sets A and B of G and H, respectively, there exists $x \in S \cap A$ for which $B \cap S_x \neq \varnothing$.*

Proof. Suppose that conditions (i) and (ii) hold for S. Condition (i) implies that C is a dominating set of $G[H]$ by Theorem 8. Let $D = \cup_{x\in A} (\{x\} \times A_x) \subseteq V(G[H])$ be a β_0-set of $G[H]$. By Lemma 2, A is a β_0-set of G and A_x is a β_0-set of H for each $x \in A$. Since S is an ITD-set of G, $S \cap A \neq \varnothing$. By condition($ii$), there exists $x \in S \cap A$ for which $S_x \cap A_x \neq \varnothing$. Let $y \in S_x \cap A_x$. Then $(x, y) \in C \cap D$. Since D is arbitrary, C is an ITD-set of $G[H]$.

Conversely, assume that C is an ITD-set of $G[H]$. By Theorem 8, since C is a dominating set of $G[H]$, S is a dominating set of G and S_x a dominating set of H for each $x \in S \setminus N_G(S)$ or S is a total dominating set of G. Let $A \subseteq V(G)$ be a β_0-set of G and $B \subseteq V(H)$ a β_0-set of H. Since $C^* = \cup_{x\in A} (\{x\} \times B)$ is a β_0-set of $G[H]$, $C \cap C^* \neq \varnothing$. Let $(x, y) \in C \cap C^*$. Then $x \in S \cap A$. So far, we have shown that S is an ITD-set of G, thus (i) holds. Moreover, $y \in S_x \cap B$ so that $S_x \cap B \neq \varnothing$. This proves ($ii$). □

It should be noted that if $S \cap A = \{x\}$ (singleton) in Theorem 5(ii), then S_x is an independent transversal set of H.

Consider $G = P_5[P_4]$. Write $P_5 = [x_1, x_2, x_3, x_4, x_5]$ and $P_4 = [y_1, y_2, y_3, y_4]$. Then $C = \{(x_2, y_2), (x_3, y_1), (x_3, y_2), (x_4, y_2)\}$ is a γ_{it}-set of G. Put $S = \{x_2, x_3, x_4\}$ and $A = \{x_1, x_3, x_5\}$. Then S is a γ_{itt}-set of P_5 and A is the unique β_0-set of P_5. Further, $S \cap A = \{x_3\}$ and $S_{x_3} = \{y_1, y_2\}$ is an independent transversal set of P_4.

The following is immediate from Proposition 5.

Proposition 6. *Let G and H be nontrivial connected graphs, and $C = \cup_{x\in S} (\{x\} \times S_x) \subseteq V(G[H])$. Then C is an independent transversal total dominating set of $G[H]$ if and only if each of the following holds:*

(i) *S is an independent transversal total dominating set of G.*

(ii) *For every pair of maximum independent sets A and B of G and H, respectively, there exists $x \in S \cap A$ for which $B \cap S_x \neq \varnothing$.*

Given an ITD-set (resp. $ITTD$-set) $S \subseteq V(G)$ of G, define $M_G(S)$ (resp. $M^t_G(S)$) to be any subset of S of minimum cardinality such that for each $x \in M_G(S)$ (resp. $x \in M^t_G(S)$), $x \in M$ for some $M \in XI(G)$, and put

$$\eta(G) = \min\{|M_G(S)| : S \text{ is an } \gamma_{it}\text{-set of } G\}, and$$

$$\eta_t(G) = \min\{|M^t_G(S)| : S \text{ is a } \gamma_{itt}\text{-set of } G\}.$$

Corollary 5. *For all nontrivial connected graphs G and positive integers $p \geq 2$,*

$$\gamma_{it}(G[K_p]) \leq (p-1)\eta(G) + \gamma_{it}(G),$$

and

$$\gamma_{itt}(G[K_p]) \leq (p-1)\eta_t(G) + \gamma_{itt}(G).$$

Equality in each is attained if $XI(G)$ consists of pairwise disjoint β_0-sets of G.

Proof. Let $S \subseteq V(G)$ be a γ_{it}-set of G for which $|M_G(S)| = \eta(G)$, and let $y \in V(K_p)$. Define

$$C = [\cup_{x \in M_G(S)} (\{x\} \times V(K_p))] \cup [\cup_{x \in S \setminus M_G(S)} \{(x, y)\}].$$

By Proposition 5, C is an ITD-set of $G[K_p]$. Consequently,

$$\gamma_{it}(G[K_p]) \leq |C| = p \cdot \eta(G) + (\gamma_{it}(G) - \eta(G)) = (p-1)\eta(G) + \gamma_{it}(G).$$

Now suppose that $XI(G)$ consists of pairwise disjoint β_0-sets of G. Then $|M_G(S)| = \eta(G) = xi(G)$ for all ITD-sets S of G. Let $C = \cup_{x \in S} (\{x\} \times S_x\})$ be a γ_{it}-set of $G[K_p]$. In view of Proposition 5, S is an ITD-set of G. Write

$$C = \left(\cup_{x \in M_G(S)} (\{x\} \times S_x)\right) \cup \left(\cup_{x \in S \setminus M_G(S)} (\{x\} \times S_x)\right),$$

and we claim that $S_x = V(K_p)$ for all $x \in M_G(S)$. Let $x \in M_G(S)$, and let $y \in V(K_p)$. Pick a β_0-set $A \subseteq V(G)$ of G for which $x \in S \cap A$. By Proposition 5(ii), since $\{y\}$ is a β_0-set of K_p, there exists $u \in S \cap A$, consequently $u \in M_G(S)$, such that $S_u = \{y\}$. By the minimality of the cardinality of $M_G(S)$, $x = u$ so that $y \in S_x$. Accordingly, $S_x = V(K_p)$. Thus,

$$\gamma_{it}(G \circ K_p) = |C| \geq p \cdot \eta(G) + (|S| - \eta(G)) \geq (p-1)\eta(G) + \gamma_{it}(G).$$

Similar arguments will prove the desired results for $\gamma_{itt}(G \circ K_p)$. □

If, in particular, $G = K_{1,n}$ on $n+1$ vertices, then $\eta(G) = 1$, $\gamma_{it}(G) = \gamma_{itt}(G) = 2$, and $\gamma_{it}(G[K_p]) = \gamma_{itt}(G[K_p]) = p + 1$.

Acknowledgements

The authors would like to thank the referees for reviewing the initial paper and for the invaluable comments and suggestions that eventually led to this much improved version of the work. This research is funded by the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP), Philippines.

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