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On inequalities for the gamma function†

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Abstract

We present various inequalities for the classical gamma function of Euler. Among others, we prove the following result:

Let α be a real number. For all $x, y \in (0,1]$ and for all $x, y \in [1, \infty)$ we have

$$x^{\alpha}\Gamma(x) + y^{\alpha}\Gamma(y) \leq 1 + (xy)^{\alpha}\Gamma(xy).$$

The constant 1 *is sharp.*

Keywords: Gamma function, inequalities, completely monotonic, convex, concave, mean values, diophantine equation, Fibonacci and Lucas numbers.

MSC: 26A48, 26D07, 33B15, 11B39, 11D41.

§1. Introduction

The classical gamma function, also known as Eulerian integral of the second kind, is defined for positive real numbers x by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt = \frac{1}{x} \prod_{k=1}^{\infty} \Big\{ \Big(1 + \frac{1}{k}\Big)^{x} \Big(1 + \frac{x}{k}\Big)^{-1} \Big\}.$$

Since the Γ-function has remarkable applications in various mathematical fields as well as in physics and other branches, it has been investigated thoroughly by many researchers.

†Dedicated to Professor Bent Fuglede on the occasion of his 90th birthday