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ON INFINITE GROUPS
W. R. Scott

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1. Introduction. Several disconnected theorems on infinite groups will be given in this paper. In $\S 2$, a generalization of Poincare's theorem on the index of the intersection of two subgroups is proved. Other theorems on indices are given. In $\S 3$, the theorem [3, Lemma 1 and Corollary 1] that the layer of elements of infinite order in a group $G$ has order 0 or $o(G)$ is generalized to the case where the order is taken with respect to a subgroup. In $\S 4$, it is shown that the subgroup $K$ of an infinite group $G$ as defined in [3] is overcharacteristic [2]. In $\S 5$, characterizations are obtained for those Abelian groups $G$, all of whose subgroups $H$ (factor groups $G / H$ ) of order equal to $o(G)$ are isomorphic to $G$ (in this connection, compare with [7]). Again the Abelian groups, all of whose order preserving endomorphisms are onto, are found (see [6]).
2. Index theorems. If $I I$ is a subgroup of $G$, let $i(H)$ denote the index of $H$ in $G$. The cardinal of a set $S$ will be denoted by $o(S)$.

Theorem 1. Let $H_{\alpha}$ be a subgroup of $G, \alpha \in S$. Then

$$
i\left(\cap H_{\alpha}\right) \leq \prod_{i}\left(H_{\alpha}\right)
$$

Proof.

$$
g_{1} g_{2}^{-1} \in \cap H_{a}
$$

if and only if

$$
g_{1} g_{2}^{-1} \in H_{\alpha} \quad \text { for all } \alpha \in S
$$

Thus each coset of $\cap H_{\alpha}$ is the intersection of a collection of sets consisting of one coset of $H_{\alpha}$ for each $\alpha$, and the conclusion follows.

Cor ollary 1. (Poincaré) The intersection of a finite number of subgroups of finite index is again of finite index.

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Corollary 2. If $i(H)=B$, then $G$ has a normal subgroup $K$ such that $i(H) \leq B^{B}$.

Proof. Let $N(H)$ denote the normalizer of $H$, and $C l(H)$ the conjugate class of $H$. Then

$$
H \subseteq N(H), \quad o(C l(H))=i(N(H)) \leq B
$$

Thus if $K$ is the intersection of the conjugates of $H$, Theorem 1 gives $i(K) \leq B^{B}$.
Remaris. For every infinite cardinal $A$, there is a simple group $G$ of order $A$ (for example, the "alternating' group on $A$ symbols). Thus $G$ has no subgroups of index less than or equal to $B$ if $2^{B}<A$. In particular, if $A$ is such that $B<A$ implies $2^{B}<A$, then $G$ has no subgroup of index less than its order $A$. This is in sharp contrast to the behaviour of Abelian groups, which have $2^{A}$ subgroups of index $B$ for $\boldsymbol{x}_{0} \leq B \leq A, A>\boldsymbol{x}_{0}[\mathbf{4}]$. It is an unsolved problem as to whether there exists a group $G$ of order $A$ with no subgroups of order $A$, for $A>\boldsymbol{X}_{0}$.

Let $\cup$ denote the point set union, and + and $\sum$ direct sums (the lattice union of subgroups will not be used). If $T$ is a nonempty subset of a group $G$, let

$$
i_{R}(T)=\min o(S) \text { such that } U T x_{\alpha}=G, \alpha \in S
$$

Define $i_{L}(T)$ similarly, and let $i(T)$ be the smaller of $i_{R}(T)$ and $i_{L}(T)$.
Theorem 2. If $H_{i}, i=1, \cdots, n$, are subgroups of $G$ such that $i\left(H_{i}\right) \geq$ $A \geq \boldsymbol{\aleph}_{0}$, then $i\left(\cup H_{i}\right) \geq A$.

Proof. The theorem is true for $n=1$. Induction on $n$. If, contrary to the theorem, $i\left(\cup H_{i}\right)<A$, then, say,

$$
G=\bigcup_{\alpha \in S}\left(\bigcup_{i=1}^{n} H_{i}\right) x_{\alpha}
$$

with $o(S)<A$. Since $i\left(H_{1}\right) \geq A$, there exists an $x \in G$ such that

$$
H_{1} x \cap\left(\cup_{\alpha} H_{1} x_{\alpha}\right)
$$

is empty. Hence

$$
H_{1} x \subseteq \cup_{\alpha}\left(\cup_{2}^{n} H_{i}\right) x_{\alpha}
$$

Therefore

$$
\bigcup_{i=1}^{n} H_{i} \subseteq \bigcup_{i=2}^{n} H_{i}\left(e \cup\left(\bigcup_{\alpha} x_{\alpha} x^{-1}\right)\right)=\bigcup_{\beta \in S^{\prime}}\left(\begin{array}{c}
n \\
\bigcup_{i=2}^{n}
\end{array} H_{i}\right) x_{\beta},
$$

where $o\left(S^{\prime}\right)<A$. Hence

$$
G=\underset{\alpha \in S}{\cup}\left(\bigcup_{i=1}^{n} H H_{i}\right) x_{\alpha}=\underset{\alpha \in S}{\cup} \underset{\beta \in S^{\prime}}{\cup} \bigcup_{i=2}^{n} H_{i} x_{\beta} x_{\alpha}=\underset{\gamma \in S^{\prime \prime}}{\cup} \bigcup_{i=2}^{n} H_{i} x_{\gamma}, o\left(S^{\prime \prime}\right)<A
$$

This contradicts the induction hypothesis. Hence the theorem is true.
Remark. For every infinite cardinal $A$, there is a group $G$ of order $A$, containing an increasing sequence $\left\{H_{n}\right\}$ of subgroups, each of index $A$, such that $\cup H_{n}=G$.

Let $l / A=0$ for $A \geq \boldsymbol{x}_{0}$.
THEOREM 3. If $H_{i}$ is a proper subgroup of $G,(i=1, \cdots, n)$ and $\sum 1 / i\left(H_{i}\right) \leq$ 1, then $\cup H_{i} \neq G$.

Proof. Let $H_{1}, \cdots, H_{r}$ have finite index, the others infinite index (if $r=0$, the theorem follows immediately from Theorem 2). Let

$$
D=\stackrel{r}{\bigcap_{\mathbf{i}}} H_{i}
$$

Then $U$ has finite index in $G$, and it is well known that ( $U_{i}^{r} H_{i}$ ) $\cap D x$ is empty for some $x \in G$. Hence, if $U_{1}^{n} H_{i}=G$, then $D x \subseteq U_{r+1}^{n} H_{i}$, whence $U_{r+1}^{n} H_{i}$ has finite "index" in contradiction to Theorem 2. Therefore $U_{1}^{n} H_{i} \neq G$.
3. Layers. Let $T$ be a subset of $G$, and let $n$ be a positive integer. Let

$$
\begin{aligned}
& L(n, T)=\left\{g \mid g^{n} \in T, g^{r} \notin T \text { for } 0<r<n\right\}, \\
& L(\infty, T)=\left\{g \mid g^{n} \notin T, n=1,2, \cdots\right\} .
\end{aligned}
$$

For $T=e$, the $L(n, T)$ have been called layers. The following theorem generalizes [3, Lemmal].

Theorem 4. Let $G$ be an infinite group, $H$ a subgroup, $P$ a set of primes and

$$
S=(\underset{p \in P}{\cup} \underset{\lambda}{\cup} L(\lambda p, H)) \cup L(\infty, H)
$$

Then $o(S)=0$ or $o(G)$.
Proof. Deny the theorem. Let $x \in S$. If $x \in L(\lambda p, H)$ then $x^{\lambda} \in L(p, H)$. Hence we may assume that $x \in L(\infty, H)$ or $x \in L(p, H), p \in P$.

Case 1. $o(N(x))=o(G)$, where $N(x)$ is the normalizer of $x$. Then $o(N(x)-$ $S)=o(G)$. If $y \in N(x)-S$, then $y^{r} \in H$ for some $r$ such that $(r, p)=1$ (if $p$ exists). If $x y \notin S$ then also $(x y)^{n} \in H$ for some $n$ such that ( $n, p$ ) $=1$ (if $p$ exists). Thus

$$
(x y)^{r n}=x^{r n} y^{r n} \in H,
$$

and $x^{r n} \in H$. But $(r n, p)=1$ if $p$ exists, and, in any case, we have a contradiction. Hence $x y \in S$ and

$$
o(S) \geq o(x(N(x)-S))=o(N(x)-S)=o(G)
$$

a contradiction.
Case 2. $o(N(x))<o(G)$. Then $o(C l(x))=o(G)$.
Case 2.1. $o(H)=o(G)$. Then $o(G)$ right cosets of $N(x)$ intersect $H$. Thus there are $o(G)$ elements of the form $h^{-1} x h$. But if $\left(h^{-1} x h\right)^{n} \in H$ then $x^{n} \in H$, whence $n=\lambda p$ and $h^{-1} x h \in S$. Therefore $o(S)=o(G)$, a contradiction.

Case 2.2. $o(H)<o(G)$. We have, since $o(S)<o(G)$,

$$
\begin{equation*}
o(G)=o(C l(x))=\sum_{\substack{(n, p)=1 \\ n<\infty}} o(C l(x) \cap L(n, H)) \tag{1}
\end{equation*}
$$

If $o(G)=\boldsymbol{X}_{0}$, and $o(x)=\infty$, then since $H$ is finite,

$$
C l(x) \subseteq L(\infty, H) \subseteq S,
$$

a contradiction. If $o(G)=\boldsymbol{X}_{0}$, and $o(x)=m$, then $C l(x) \cap L(n, H)$ is empty for $n>m$. Hence, by (1), there exists, regardless of the size of $o(G)$, an $n$ such that $(n, p)=1$ and

$$
o(C l(x) \cap L(n, H))>o(H) o(S)
$$

Let

$$
A(n, T)=\left\{g \mid g^{n} \in T\right\}
$$

Then $A(n, H) \supseteq L(n, H)$, hence

$$
o(C l(x) \cap A(n, H))=\sum_{h} o(C l(x) \cap A(n, h))>o(H) o(S) .
$$

Hence there exists an $h_{0} \in H$ such that

$$
o\left(C l(x) \cap A\left(n, h_{0}\right)>o(S)\right.
$$

There is then a $b \in G$ such that $\left(b^{-1} x b\right)^{n}=h_{0}$, whence

$$
x \in C l(x) \cap A\left(n, b h_{0} b^{-1}\right)
$$

If

$$
q \in C l(x) \cap A\left(n, b h_{0} b^{-1}\right)
$$

then

$$
q^{n}=b h_{0} b^{-1}=x^{n} .
$$

Hence if $q^{r} \in H$, then

$$
x^{n r}=q^{n r} \in H
$$

and $p \mid n r$, whence $p \mid r$. Thus $q \in S$ in any case. We have

$$
\begin{aligned}
o(S) \geq o\left(C l(x) \cap A\left(n, b h_{0} b^{-1}\right)\right) & =o\left(b\left(C l(x) \cap A\left(n, h_{0}\right)\right) b^{-1}\right) \\
& =o\left(C l(x) \cap A\left(n, h_{0}\right)\right)>o(S) .
\end{aligned}
$$

This contradiction shows that the theorem is true.
Corollary. If $H$ is a subgroup of the group $G$, then $o(L(\infty, H))=o(G)$ or 0 .

Proof. In Theorem 4, let $P$ be the empty set.
4. An over-characteristic subgroup. Neumann and Neumann [2] have defined a subgroup $K$ of $G$ to be over-characteristic in $G$ if and only if (i) $K$ is normal, and (ii) $G / K \cong G / H$ implies $K \subseteq H$.

Define (see [3]) a subgroup $K$ of an infinite group $G$ as follows. Let $E(x)$ be the set of $g \in G$ such that $x$ is not in the subgroup generated by $g$, and let $K$ be the set of $x \in G$ such that $o(E(x))<o(G)$.

Theorem 5. If $G$ is infinite, and $K$ is defined as above, then $K$ is an overcharacteristic subgroup of $G$.

Proof. (i) $K$ is normal since it is fully characteristic [3, Theorem 6].
(ii) Let $G / K \cong G / H$.

Case 1. $K$ is finite. Then [3, Corollary 3 to Theorem 8]

$$
K_{2}=K(G / K)=e .
$$

Hence $K(G / H)=e$. Now

$$
o(G / H)=o(G / K)=o(G) .
$$

If there exists a $k \in K-H$, then

$$
o(E(k H)) \leq o(E(k))<o(G)=o(G / H) .
$$

Hence $k H \in K(G / H)$. This is a contradiction. Hence $K \subseteq H$, and $K$ is overcharacteristic.

Case 2. $K$ is infinite. Then [3, Theorem 5] $K$ is a $p^{\infty}$ group, and [3, Theorem 8] $G / K$ is finite. If there exists a $k \in K-H$ then

$$
k^{\prime P^{n}}=k
$$

implies $k^{\prime} \in K-H$, and

$$
o\left(k^{\prime} H\right) \geq p^{n+1} .
$$

This contradicts the finiteness of $G / H$. Therefore $K \subseteq H$, and since $G / K$ is finite, $K=H$. Hence $K$ is over-characteristic.
5. Abelian groups with special properties. ${ }^{1}$ If $G$ is an Abelian group such that $0 \subset H \subset G$ implies $G \cong H$ for subgroups $H$, then it is trivial that $G$ is 0 or cyclic of prime or infinite order, and conversely. This naturally leads to the problem of finding those groups which possess the following property:
${ }^{1}$ For the facts used without proof in this section, see [1].
$\left(P_{1}\right) G$ is Abelian, and if $H$ is a subgroup of $G$ such that $o(H)=o(G)$ then $G \cong H$.

Theorem 6. G has property ( $P_{1}$ ) if and only if (i) $G$ is finite Abelian, (ii) $G$ is a $p^{\infty}$ group, (iii) $G$ is a direct sum of cyclic groups of order $p, p a$ fixed prime, (iv) $G$ is infinite cyclic, or (v) $G$ is the direct sum of a nondenumerable number of infinite cyclic groups.

Proof. If $G$ is of one of the above five types, then it is either trivial or well-known that $G$ has property ( $P_{1}$ ).

Conversely, suppose that $G$ is infinite and has property $\left(P_{1}\right)$. Let $T$ be the torsion subgroup of $G$.

Case 1. $o(T)<o(G)$. Then (see, for example, [3, proof of Theorem 9, Case 1]) there is a free Abelian subgroup $H$ of $G$ such that $o(H)=o(G)$. Hence $G \cong H$. If the rank of $G$ is non-denumerable, we are done. If the rank of $G$ is countable, then $G$ is countable and contains an infinite cyclic subgroup. $\operatorname{By}\left(P_{1}\right), G$ is infinite cyclic.

Case 2. $o(T)=o(G)$. Then $G \cong T$, that is, $G$ is periodic. If $G_{p}$ is a nonzero $p$-component of $G$, then $G=G_{p}+H_{p}$, hence $G \cong G_{p}$ or $G \cong H_{p}$, a contradiction unless $H_{p}=0$. Hence $G$ is a $p$-group. Thus $G=D+R$, where $D$ is a divisible (that is, $n D=D$ ) and $R$ a reduced (no divisible non-zero subgroups) $p$-group. Hence $G \cong R$ or $G \cong D$, that is $G$ is reduced or divisible.

Case 2.1. $G$ is a divisible p-group. Then $G=\sum C_{\alpha}$ where $C_{a}$ is a $p^{\infty}$ group. If there is more than one summand, then there is a subgroup

$$
H=C^{*}+\sum C_{a},
$$

$\alpha \neq \alpha_{0}$, where $C^{*}$ is a proper subgroup of $C_{\alpha_{0}}$. Hence $o(H)=o(G)$, but $H$ is not divisible, a contradiction. Therefore $G$ is a $p^{\infty}$ group in this case.

Case 2.2. $G$ is a reduced $p$-group. Then $G$ has a cyclic direct summand $C$ of order, say, $p^{n}$. Zorn's lemma may be applied to sets $S$ of cyclic groups $C_{\alpha}$ of order $p^{n}$ such that $\sum C_{a}, C_{\alpha} \in S$, exists and is pure in $G$ (that is, a servant subgroup of $G$ ). There is then a maximal such set $S^{*}$, and if $K=\sum C_{a}$, $C_{\alpha} \in S^{*}$, then $K$ is a pure subgroup of bounded order. Hence $K$ is a direct summand, $G=K+A$. It is clear that $A$ has no cyclic direct summands of order $p^{n}$. This implies, by property $\left(P_{1}\right)$, that $o(A)<o(G)$, hence $G \cong K$. If, now, $n>1$, there is a subgroup $H$ of $K$ of order $o(G)$ such that $H \not \equiv K$. Therefore $n=1$.

Theorem 6 has a dual.
$\left(P_{2}\right) G$ is Abelian, and $o(G / H)=o(G)$ implies $G \cong G / H$.
Theorem 7. $G$ has property $\left(P_{2}\right)$ if and only if (i) $G$ is finite Abelian, (ii) $G$ is infinite cyclic, (iii) $G$ is a direct sum of cyclic groups of order $p$, (iv) $G$ is a $p^{\infty}$ group, or ( v ) $G$ is the direct sum of a non-denumerable number of $p^{\infty}$ groups.

Proof. If $G$ is of one of the above five types, then it is clear that $G$ has property $\left(P_{2}\right)$.

Conversely suppose that $G$ is infinite and has property $\left(P_{2}\right)$.
Case 1. $o(G / T)=o(G)$. Then, by $\left(P_{2}\right) G$ is torsionsfree. Let $C$ be a cyclic subgroup of $G$. Then $2 C$ is cyclic, and $G / 2 C$ has an element of order 2 , hence $o(G / 2 C)<o(G)$. Therefore $o(G)=\boldsymbol{\aleph}_{0}$, and $o(G / C)$ is finite, hence $G$ is cyclic.

Case 2. $o(G / T)<o(G)$. Hence $o(T)=o(G)$. Let $S$ be a maximal linearly independent set of elements, $B$ the subgroup generated by $S$ (set $B=0$ if $S$ is empty). Then $T \cap B=0$, hence $T$ is isomorphic to a subgroup of $G / B$, and therefore $o(G / B)=o(G)$. But $G / B$ is periodic, hence $G$ is periodic. It follows, just as in the proof of Theorem 6 , that $G$ is either a divisible or a reduced p-group.

Case 2.1. $G$ is a divisible $p$-group. Then $G=\sum C_{\alpha}$, where $C_{a}$ is a $p^{\infty}$ group. If the number of summands is non-denumerable, we are done. If not, then $G$ is homomorphic to a $p^{\infty}$ group, and $o(G)=\boldsymbol{K}_{0}$. Therefore by $\left(P_{2}\right), G$ is a $p^{\infty}$ group.

Case 2.2. $G$ is a reduced p-group. Then, almost exactly as in Case 2.2 of Theorem 6, it follows that $G$ is the direct sum of cyclic groups of order $p$.

Remark. Szélpál [7] has shown that if $G$ is an Abelian group which is isomorphic to all proper quotient groups, then $G$ is a cyclic group of order $p$ or a $p^{\infty}$ group. Theorem 7 may be considered as a generalization of this theorem.

Szele and Szélpál [6] have shown that if $G$ is an Abelian group such that every non-zero endomorphism is onto, then $G$ is a cyclic group of order $p$, a $p^{\infty}$ group, or the rationals. The following theorem may be considered as a generalization.
$\left(P_{3}\right) G$ is Abelian, and if $\sigma$ is an endomorphism of $G$ such that $o(G \sigma)=o(G)$ then $G \sigma=G$.

Theorem 8. $G$ has property $\left(P_{3}\right)$ if and only if (i) $G$ is finite Abelian, (ii) $G$ is a $p^{\infty}$ group, or (iii) $G$ is the group of rationals.

Proof. If $G$ is of one of the above three types, then it is clear that $\left(P_{3}\right)$ is satisfied.

Conversely, suppose that $G$ is an infinite group satisfying $\left(P_{3}\right)$.

Case 1. $G$ is torsion-free. Then if $p G \neq G$ for some $p$, the transformation $g \sigma=p g$ is an isomorphism of $G$ into itself, so that $o(G \sigma)=o(G), G \sigma \neq G$, a contradiction. Hence $p G=G$ for all $p$, and therefore $G=\sum R_{\alpha}$, where $R_{\alpha}$ is is isomorphic to the group of rationals. If there is more than one summand, then there is a projection $\sigma$ of $G$ onto $\sum R_{\alpha}, \alpha \neq \alpha_{0}$, a contradiction. Hence $G$ is the group of rationals.

Case 2. $G$ is not torsion-free. Then $G=A+B$ where $A$ is finite (and nonzero ) or a $p^{\infty}$ group. Thus the projection $\sigma$ of $G$ onto the larger of $A$ and $B$ yields a contradiction unless $B=0$. But in this case, since $G$ is infinite, $G=A$ is a $p^{\infty}$ group.

Finally ( compare with Szele [5]) consider the following property.
$\left(P_{4}\right) \quad G$ is Abelian, and if $\sigma$ is an endomorphism of $G$ such that $o(G \sigma)=o(G)$ then $\sigma$ is an automorphism of $G$

Corollary. $G$ has property $\left(P_{4}\right)$ if and only if $(\mathrm{i}) G$ is finite Abelian, or (ii) $G$ is the group of rationals.

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