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On infinitely extendable vector bundles on G/P

By

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Introduction

In the present paper, we shall study a structure of IE-I and IE-II vector bundle on complete homogeneous variety $X_{n,d}$. (See § 1 about the definition of $X_{n,d}$ and IE-I (or, IE-II) vector bundle on $X_{n,d}$). We have the following

Main Theorem I. *Let E be an IE-I vector bundle on the complex homogeneous variety $X_{n,d}$. Then E is isomorphic to $\bigoplus_i P_n(d; c_0^i, \cdots, c_d^i, 0, \cdots, 0)$. (see § 4 about the definition of an irreducible homogeneous vector bundle $P_n(d; c_0, \cdots c_d, 0, \cdots, 0)$ on $X_{n,d}$).*

Main Theorem II. *Let E be an IE-II vector bundle on the complex Grassmann variety $X_{n,n-e}(A)$. Then E is isomorphic to $\bigoplus_i P_n(n-e; 0, \cdots, 0, c_{n-e}^i, \cdots, c_{n-1}^i)$.*

Barth and Van de Ven showed that infinitely extendable vector bundle of rank 2 on the complex projective space is a direct sum of line bundles ([1]). And then, the author generalized this result to the case of higher rank and, in addition, any characteristic ([5] [6]). On the other hand, defining an infinitely extendable vector bundle on the infinite variety, Tjurin showed that this vector bundle is a direct sum of line bundles in characteristic 0 ([8]). The geometrical and topological meaning of the result is stated in detail in [5] and [8]. But we are not able to expect that every infinitely extendable vector bundle on an extendable variety decomposes to a direct sum of line bundles. In fact, on homogeneous varieties G/P, we have many infinitely extendable vector bundles which are indecomposable, where G is a semi-simple and simply connected algebraic group and P is a parabolic subgroup of G. Therefore our next interest is to determine the structure of infinitely extendable vector bundles on homogeneous varieties G/P where P is, in particular, a maximal parabolic subgroup. As stated in Main Theorem I and II above, we see that these vector bundles are homogeneous. Essential tools to prove these theorems are two results due to Tjurin and the author, besides we have to study the structure of parabolic subgroups of classical groups and Lie algebras corresponding to their algebraic groups and have to investigate their representation in detail (see § 3). As for IE-II vector bundle, we have not been able to determine the structure in the case of types

B, C and D because, in these cases, the structure of IE-II vector bundles on $X_{n,n-1}$ has not been known yet.

Throughout this paper, k is an algebraically closed field of characteristic 0 (All results in §1 and §2 hold in characteristic $p>0$ too.) By a scheme we understand a separated algebraic k-scheme. $\mathcal{O}_{P^n}(1)$ is the line bundle corresponding to the divisor class of hyperplanes in the n-dimensional projective space $\boldsymbol{P}^n$. When S is a quadric hypersurface in $\boldsymbol{P}^{n+1}$, $\mathcal{O}_S(1)$ denotes the line bundle corresponding to the divisor class of hyperplane sections. If E is a vector bundle on a variety, $\check{E}$ denotes the dual vector bundle of E. $\boldsymbol{Gr}(n, d)$ denotes the Grassmann variety parameterizing d-dimensional linear subspace of the n-dimensional projective space $\boldsymbol{P}^n$.

§1. Some properties of a homogeneous variety G/P and the definition of IE-I and IE-II vector bundles

Let us begin by defining complete homogeneous varieties obtained by classical group modulo its maximal parabolic subgroup and embeddings of one to another. At first let us denote the subgroup $\bar{P}_{u,v}$ of general linear group $GL(n+1, \boldsymbol{C})$ by $\{(m_{ij})_{1\leq i,j\leq u+1}\in GL(u+1, \boldsymbol{C})\,|\,m_{ij}=0$ for $w+1\leqq i\leqq u+1$ and $1\leqq j\leqq w\}$ where $1\leqq w\leqq u$. These are known to be the maximal parabolic subgroups of $GL(u+1, \boldsymbol{C})$.

A) The case of type A. Put $G_n(A)=SL(n+1, \boldsymbol{C})$ and $P_{n,d}(A)=G_n(A)\cap\bar{P}_{n,d}$. Then $P_{n,d}(A)$ is a maximal parabolic subgroup of $G_n(A)$. Hence $G_n(A)/P_{n,d}(A)$ $(=X_{n,d-1}(A))$ is a complete homogeneous variety and, indeed, is a Grassmann variety. If we define a map $j_n(A): G_n(A)\rightarrow G_{n+1}(A)$ by transforming $M=(m_{ij})_{1\leq i,j\leq n+1}$ in $G_n(A)$ to $M'=(m'_{ij})_{1\leq i,j\leq n+2}$ where $m_{ij}=m'_{ij}$ for $1\leqq i,, j\leqq n+1$, $m'_{m+2\ n+2}=1$ and $m'_{n+2\ i}=m'_{j\ n+2}=0$ for $1\leqq i, j\leqq n+1$, $j_n(A)$ provides us with a closed immersion $i_{n,d}(A)\,; X_{n,d}(A)\subsetneq X_{n+1,d}(A)$, since $j_n(A)^{-1}(P_{n+1,d}(A))=P_{n,d}(A)$. Let us call this map $i_{n,d}(A)$ a morphism of type I with respect to the type A. On the other hand, if we define a map $j'_n(A)\,; G_n(A)\subsetneq G_{n+1}(A)$ by transforming $M=(m_{ij})_{1\leq i,j\leq n+1}$ in $G_n(A)$ to $M'=(m'_{ij})_{1\leq i,j\leq n+2}$ in $G_{n+1}(A)$, where $m'_{11}=1$, $m'_{1i}=m'_{j1}=0$ for $2\leqq i, j\leqq n+2$ and $m'_{ij}=m_{i-1\ j-1}$ for $2\leqq i, j\leqq n+2$, $j'_n(A)$ provides us with the closed immersion $i'_{n,d}(A)\,; X_{n,d}(A)\subsetneq X_{n+1,d+1}(A)$. In this case let us call this map $i'_{n,d}(A)$ a morphism of type II with respect to the type A.

B) The case of type B. Let V be a vector space of rank $2n+1$ and let us choose a basis $e_1, \cdots e_n, e_{n+1}, \cdots, e_{2n+1}$ of V. Let us represent a point v of V by $v=(x_1, \cdots x_n, z, y_1, \cdots, y_n)$ with respect to this basis. If we put $Q(v)=z^2+x_1y_n+\cdots+x_ny_1$, then $G_n(B)=SO(2n+1, \boldsymbol{C})$ is defined as the subgroup of $SL(2n+1, \boldsymbol{C})$ that leaves $Q(v)$ invariant, that is, $G_n(B)=\{M\in SL(2n+1, \boldsymbol{C})\,|\,Q(Mv)=Q(v)\ v\in V\}$. Now it is well known that $\bar{P}_{2n,d}\cap G_n(B)=P_{n,d}(B)$ is a maximal parabolic subgroup of $G_n(B)$ for each $1\leqq d\leqq n$. Put $X_{n,d-1}(B)=G_n(B)/P_{n,d}(B)$. Let us define a morphism $j_n(B): G_n(B)\subsetneq G_{n+1}(B)$ as follows:

$$j_n(B):\ M=\left(\begin{array}{c|c|c} A_1 & C_1 & A_2 \\ \hline C_2 & D & C_4 \\ \hline A_3 & C_3 & A_4 \end{array}\right) \longmapsto M'=\left(\begin{array}{c|c|c|c|c} A_1 & 0 & C_1 & 0 & A_2 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline C_2 & 0 & D & 0 & C_4 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline A_3 & 0 & C_3 & 0 & A_4 \end{array}\right)$$

where A_1 is a (n, n) matrix for $1\leqq i\leqq 4$, C_i is $(n, 1)$ matrix for $i=1, 3$, C_j is a $(1, n)$ matrix for $j=2, 4$, and D is a $(1, 1)$ matrix.

The above map $j_n(B)$ yields a morphism $i_{n,d}(B): X_{n,d}(B) \subsetneq X_{n+1,d}(B)$ which is called a morphism of type I with respect to the type B. Next let us consider a morphism $j'_n(B): G_n(B) \subsetneq G_{n+1}(B)$ satisfying the condition that for $M=(m_{ij})_{1\leq i,j\leq 2+1} \in G_n(B)$, $j'_n(M)=(m'_{ij})_{1\leq i,j\leq 2n+3}$ where $m'_{ii}=1$ for $i=1$, $2n+3$, $m_{ij}=m'_{i+1\,j+1}$ for $1\leqq i$, $j\leqq 2n+1$ and $m_{ij}=0$ for others. This map yields a morphism $i'_{n,d}(B): (X_{n,d}(B) \subsetneq X_{n+1,d+1}(B)$ which is called a morphism of type II with respect to the type B.

C) The case of type C. The symplectic group $Sp(2n, \boldsymbol{C})$ $(G_n(C))$ is realised as a subgroup of $GL(2n, \boldsymbol{C})$ as follows:

Let $L=(t_{ij})_{1\leq i,j\leq 2n}$ be a skew-symmetric matrix such that $t_{i,2n-i}=1$ for $1\leqq i\leqq n$, $t_{i,2n-i}=-1$ for $n+1\leqq i\leqq 2n$, and $t_{ij}=0$ for others. Then $G_n(C)=\{M\in GL(2n, \boldsymbol{C})\,|\,ML^tM=L\}$.

It is well known that $\bar{P}_{2n-1,d} \cap G_n(C)=P_{n,d}(C)$ is a parabolic subgroup of $G_n(C)$ for $1\leqq d\leqq n$. Let us put $X_{n,d-1}(C)=G_n(C)/P_{n,d}(C)$ and let us consider a morphism $j_n(C)$: $G_n(C) \subsetneq G_{n+1}(C)$ where for $M=(m_{ij})_{1\leq i,j\leq 2n} \in G_n(C)$, $j_n(C)(M)=(m'_{ij})_{1\leq i,j\leq 2n+2}$ such that

$$j_n(C): M=\left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right) \in G_n(C) \longmapsto M'=\left(\begin{array}{c|c|c|c} A_1 & 0 & 0 & A_2 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline A_3 & 0 & 0 & A_4 \end{array}\right) \in G_{n+1}(C)$$

where for $1\leqq i\leqq 4$, A_i is a (n, n) matrix.

By virtue of this map, we obtain the immersion $i_{n,d}(C)$: $X_{n,d}(C) \subsetneq X_{n+1,d}(C)$ which is called a morphism of type I for the type C.

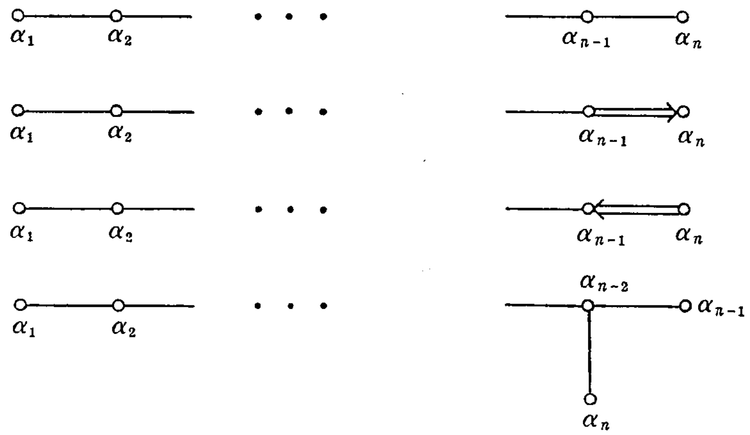
Moreover we can define $i'_{n,d}(C): X_{n,d}(C) \subsetneq X_{n+1,d+1}(C)$ in the same way as in the type B which is called a morphism of type II with respect to type B.

D) The case of type D. Let us take $G_n(D)=SO(2n, \boldsymbol{C})$. $G_n(D)$ is realised as a subgroup of $GL(2n, \boldsymbol{C})$ as follows:

Let V be a vector space of dimension $2n$. With respect to a basis $e_1, \cdots, e_{2n}$ of V, write a point $v\in V$ as $v=(x_1, \cdots, x_n, y_n, \cdots, y_1)$. Let $Q(v)$ be the quadratic form on V defined by $Q(v)=\sum_{i=1}^{n} x_i y_i$. If we take $O(2n, \boldsymbol{C})$ as the subgroup of $GL(2n, \boldsymbol{C})$ which leaves the quadratic form $Q(v)$ invariant, i.e., $O(2n, \boldsymbol{C})=\{A\in$

$GL(2n, \boldsymbol{C}) | Q(Av)=Q(v)$ for all $v \in V\}$. $SO(2n, \boldsymbol{C})$ $(=G_n(D))$ is defined as $O(2n, \boldsymbol{C}) \cap SL(2n, \boldsymbol{C})$. As for two morphisms of type I and type II, we can define $i_{n,d}(D)$: $G_n(D)/P_{n,d+1}(D)(=X_{n,d}(D)) \subsetneq G_{n+1}(D)/P_{n+1,d+1}(D)(=X_{n+1,d}(D))$ and $i'_{n,d}(D)$: $X_{n,d}(D) \subsetneq X_{n+1,d+1}(D)$ respectively in the same way as in the case of the type C. As to $P_{n,n-1}(D)$, instead of putting $\bar{P}_{2n-1,n-1} \cap G_n(D)$, we consider the maximal parabolic subgroup $P_{n,n-2}(D)$ of $G_n(D)$ obtained by omitting a simple root α_{n-1} (See Remark 1.1 below).

Remark 1.1. Let us study the structure of parabolic subgroup $P_{n,d}(*)$. Now let T_t(or, B_t) be the diagonal (or, upper triangular, resp.) matrices of $GL(t, \boldsymbol{C})$. Then it is well known that $G_n(A) \cap T_{n+1}(=T_n(A))$ (or, $G_n(A) \cap B_{n+1}(=B_n(A))$) is a maximal torus (or, Borel subgroup, resp.), $G_n(B) \cap T_{2n+1}(=T_n(B))$ (or, $G_n(B) \cap B_{2n+1}(=B_n(B))$) is a maximal torus (or, Borel subgroup, resp.) and for the type $*(=C$ or $D)$, $G_n(*) \cap T_{2n}(=T_n(*))$ (or, $G_n(*) \cap B_{2n}(=B_n(*))$, resp.) is a maximal torus (or, Borel subgroup, resp.) of $G_n(*)$. Now for the type $*(=A, B, C$ or $D)$ fix a system of roots Δ relative to $T_n(*)$ and, in addition, let Δ_+ be the set of positive roots relative to $B_n(*)$ and $S=\{\alpha_1, \cdots, \alpha_n\}$ be the system of simple roots. Then for the type $*(=A, B, C$ or $D)$, the corresponding Dynkin diagrams are as follows:



Now let us recall that for the type $*(=A, B, C$ or $D)$ the set of parabolic subgroups of $G_n(*)$ containing $B_n(*)$ is in one to one correspondence to the set of subsets of S (see 1.2 and 1.3 in [7]). Therefore it can be easily checked that $P_{n,d}(*)$ defined above is a maximal parabolic subgroup obtained by omitting α_d.

The following seems to be known (c.f. 1.9, §2 of [7] and our Remark 1.1).

Proposition 1.2. *Picard group of $X_{n,d}(*)$ is isomorphic to $\boldsymbol{Z}$. In addition, we can choose the ample line bundle as its generator. (Hereafter we shall write this generator as $L_{n,d}(*)$).*

Now for the type $*(=A, B, C$ or $D)$ put $P_{n,d,d-1}(*)=P_{n,d}(*) \cap P_{n,d-1}(*)$ and put $X_{n,d,d-1}(*)=G_n(*)/P_{n,d+1,d}(*)$. Let us study the following diagram:

(1-3*)

$$\begin{array}{ccc} & X_{n,d,d-1}(*) & \\ {\scriptstyle p_{n,d}(*)} \swarrow & & \searrow {\scriptstyle q_{n,d-1}(*)} \\ X_{n,d}(*) & & X_{n,d-1}(*) \end{array}$$

where both $p_{n,d}(*): X_{n,d,d-1}(*) \to X_{n,d}(*)$ and $q_{n,d-1}(*): X_{n,d,d-1}(*) \to X_{n,d-1}(*)$ are the canonical projections.

For the type $*=A$, we immediately have

Proposition 1.4A. *$P_{n,d}(A)$ is a $\boldsymbol{P}^d$-bundle and $q_{n,d-1}(A)$ is a $\boldsymbol{P}^{n-d}$-bundle for* $1 \leqq d \leqq n-1$.

Using this result, we obtain the following whose proof is easy and is omitted.

Proposition 1.5A. *Under the notation* 1.3A, $q_{n,d-1}(A)^*(L_{n,d-1}(A))|_{p_{n,d}(A)^{-1}(x)} \cong \mathcal{O}_{P^d}(1)$ *for all points x in $X_{n,d}(A)$ and* $p_{n,d}(A)^*(L_{n,d}(A))|_{q_{n,d-1}(A)^{-1}(y)} \cong \mathcal{O}_{P^{n-d}}(1)$ *for all points y in $X_{n,d-1}(A)$.*

In the next place, using Proposition 1.4A and Proposition 1.5A, we shall derive results corresponding to proposition 1.4A and proposition 1.5A for other types B, C and D.

For the type $*=C$, we have the following diagram for $1 \leqq d \leqq n-1$:

(1.3C)

$$\begin{array}{ccccc} & & X_{2n-1,d,d-1}(A) & & \\ & {\scriptstyle p_{2n-1,d}(A)} \swarrow & \cup\!\!\uparrow & \searrow {\scriptstyle q_{2n-1,d-1}(A)} & \\ X_{2n-1,d}(A) & & X_{n,d,d-1}(C) & & X_{2n-1,d-1}(A) \\ \cup\!\!\uparrow & {\scriptstyle p_{n,d}(C)} \swarrow & & \searrow {\scriptstyle q_{n,d-1}(C)} & \cup\!\!\uparrow \\ X_{n,d}(C) & & & & X_{n,d-1}(C) \end{array}$$

Considering the dimension of fibers of two projections $p_{n,d}(C)$ and $q_{n,d-1}(C)$, we obtain the following proposition by virtue of Proposition 1.4A.

Proposition 1.4C. *$p_{n,d}(C)$ is a $\boldsymbol{P}^d$-bundle and $q_{n,d-1}(C)$ is a $\boldsymbol{P}^{2(n-d)-1}(C)$-bundle.*

Moreover by Proposition 1.5A and Proposition 1.4C we have

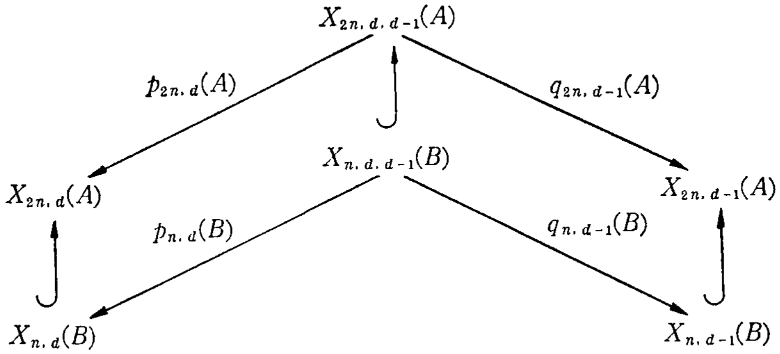
Proposition 1.5C. *Under the diagram* 1.3C, $q_{n,d-1}(C)^*(L_{n,d-1}(C))|_{p_{n,d}(C)^{-1}(x)} \cong \mathcal{O}_{P^d}(1)$ *for all points x in $X_{n,d}(C)$ and* $p_{n,d}(C)^*(L_{n,d}(C))|_{q_{n,d-1}(C)^{-1}(y)} \cong \mathcal{O}_{P^{2(n-d)-1}}(1)$ *for all points y in $X_{n,d-1}(C)$.*

To study the fiber of $p_{n,d}(*)$ and $q_{n,d-1}(*)$ for the type $*(=B, D)$, we need

Remark 1.6. We can easily show that there is a natural embedding $X_{n,0}(B) \subsetneq X_{2n,0}(A)$ $(\cong \boldsymbol{P}^{2n})$. Moreover it can be checked that $X_{n,0}(B)$ is a non-singular quadric hypersurface in $\boldsymbol{P}^{2n}$. (see B of §3 in [7]). Similarly $X_{n,0}(D)$ is a quadric hypersurface in $X_{2n-1,0}(A)$ $(\cong \boldsymbol{P}^{2n-1})$.

Firstly, let us consider the case of type B. There is the following diagram for $0 \leqq d \leqq n-1$:

(1.3B)



Then we have

Proposition 1.4B. *$p_{n,d}(B)$ is a $\boldsymbol{P}^d$-bundle and every fiber of $q_{n,d-1}(B)$ is isomosphic to a non-singular quadric hypersurface in the projective space $\boldsymbol{P}^{2(n-d)}$.*

Proof. The former is obvious. Each fiber of $q_{n,d-1}(B)$ is isomorphic to $P_{n,d-1}(B)/P_{n,d,d-1}(B)$. On the other hand, we can easily check that there is an embedding $i: G_{n-d}(B) \subsetneq G_n(B)$ such that for $M=(m_{ij})_{1\leq i,j\leq 2(n-d)+1}$ in $G_{n-d}(B)$, $i(M)=(m'_{ij})_{1\leq i,j\leq 2n+1}$, where for $1 \leqq i, j \leqq 2(n-d)+1$, $m'_{i+d,j+d}=m_{ij}$, $m'_{ii}=1$ for $1 \leqq i \leqq d$ or $2n-d+1 \leqq i \leqq 2n+1$ and $m'_{ij}=0$ for others. Hence, since $P_{n,d}(B)/P_{n,d,d-1}(B) \cong G_{n-d}(B)/P_{n-d,0}(B)$, each fiber of $q_{n,d-1}(B)$ is a quadric hypersurface S of $\boldsymbol{P}^{2(n-d)}$ by virtue of Remark 1.6. q. e. d.

Moreover, combining Proposition 1.5A and Proposition 1.4B, we obtain

Proposition 1.5B. *Under the diagram* 1.3B, *$q_{n,d-1}(B)^*(L_{n,d-1}(B))|_{p_{n,d}(B)^{-1}(x)} \cong \mathcal{O}_{\boldsymbol{P}^d}(1)$ for all points x in $X_{n,d}(B)$. $p_{n,d}(B)^*(L_{n,d}(B))|_{q_{n,d-1}(B)^{-1}(y)}$ is isomorphic to $\mathcal{O}_S(1)$ (see the last part of introduction about $\mathcal{O}_S(1)$) for all points y in $X_{n,d-1}(B)$ in the case $1 \leqq d \leqq n-2$ and it is isomorphic to $\mathcal{O}_{\boldsymbol{P}^1}(1)$ in the case $d=n-1$.*

Secondly, we obtain the results for type D in the same way as above.

Proposition 1.4D. *$p_{n,d}(D)$ is a $\boldsymbol{P}^d$-bundle and every fiber of $q_{n,d-1}(D)$ is isomorphic to a non-singular quadric hypersurface ie $\boldsymbol{P}^{2(n-d)-1}$ for $0 \leqq d \leqq n-3$. Both the fibers of $p_{n-1}(D)$ and $q_{n-2}(D)$ are $\boldsymbol{P}^{n-1}$, the fibers of $p_{n-2}(D)$ are $\boldsymbol{Gr}(n-1, 1)$ and fibers of $q_{n-3}(D)$ are $\boldsymbol{P}^1$,*

Proposition 1.5D. *Under the diagram* 1.3D, *for a point x in $X_{n,d}(D)$, set $\bar{L}_{n,d-1}(x)=q_{n,d-1}(D)^*(L_{n,d-1}(D))|_{p_{n,d}(D)^{-1}(x)}$. Then if $1 \leqq d \leqq n-3$, $\bar{L}_{n,d-1}(x) \cong \mathcal{O}_{\boldsymbol{P}^d}(1)$. Moreover $\bar{L}_{n,n-2}(x) \cong \mathcal{O}_{\boldsymbol{P}^{n-1}}(1)$ and $\bar{L}_{n,n-3}(x) \cong \mathcal{O}_{Gr(n-1,1)}(1)$ where $\mathcal{O}_{Gr(n-1,1)}(1)$ is the* *ample generator of* Pic $Gr(n-1, 1)$. *On the other hand, for a point y in* $X_{n,d-1}(D)$, *set* $\tilde{L}_{n,d}(y)=p_{n,d}(D)^*L_{n,d}(D)|_{q_{n,d-1}(D)^{-1}(y)}$. *Then,* $\tilde{L}_{n,n-2}(y)\cong\mathcal{O}_{P^1}(1)$, $\tilde{L}_{n,n-1}(y)\cong\mathcal{O}_{P^{n-1}}(1)$ *and* $\tilde{L}_{n,d}(y)\cong\mathcal{O}_S(1)$ *for* $1\leqq d\leqq n-3$.

Under these preliminaries, we can define infinitely extendable vector bundles on $X_{n,d}(*)$ for $*=A, B, C$ or D.

Definition 1.6. For $*(=A, B, C$ or $D)$, let E be a vector bundle on $X_{n,d}(*)$. Then E is said to be infinitely extendable of type I (or, type II), if for all integers $m(\geqq n)$, there is a vector bundle E_m on $X_{m,d}(*)$ and there is a morphism of type I $(=i_{m,d}(*): X_{m,d}(*)\subsetneq X_{m+1,d}(*))$ (or, a morphism of type II $(=i'_{m,d}(*): X_{m,d}(*)\subsetneq X_{m+1,d+1}(*))$, resp.) such that $i_{m,d}(*)^*(E_{m+1})\cong E_m$ (or, $i'_{m,d}(*)^*(E_{m+1})\cong E_m$, resp.) (Hereafter we abbreviate an infinitely extendable vector bundle of type I (or, II) to an IE-I vector bundle (or, an IE-II vector bundle, resp.).

§ 2. Fundamental properties of IE-I or IE-II vector bundles on $X_{n,d}$

In this section since we consider some properties of IE-I (or, IE-II) vector bundles on $X_{n,d}(*)$ which are common in type A, B, C and D $(=*)$, we sometimes abbreviate $X_{n,d}(*)$ (or, $X_{n,d,d-1}(*)$) for $*(=A, B, C$ and $D)$ to $X_{n,d}$ (or, $X_{n,d,d-1}$, resp.). Similarly $L_{n,d}(*)^{\otimes c}$ is abbreviated to $\mathcal{O}_{X_{n,d}}(c)$. Therefore we shall use the following diagram instead of (1-3*):

(2-1)
$$\begin{array}{ccccc} & & X_{n,d,d-1} & & \\ & \swarrow^{p_{n,d}} & & \searrow^{q_{n,d-1}} & \\ X_{n,d} & & & & X_{n,d-1} \end{array}$$

In order to study properties of IE-I (or, IE-II) vector bundles, let us recall two theorems.

Theorem 2.2. *Let E be an infinitely extendable vector bundle on the projective space* $\boldsymbol{P}^n$ *over an algebraically closed field of any characteristic. Then E is a direct sum of line bundle. (See* [5] *and* [6]).

Theorem 2.3. *Let* $X_\infty \overset{s}{\subsetneq} \boldsymbol{P}_\infty$ *be a non-singular infinite projective variety and let E be a vector bundle on* X_∞ *of rank n. Then E is a direct sum of line bundles. (See* [8])

The following is well-known.

Proposition 2.4. *Let S be a non-singualr quadric hypersurface in* $\boldsymbol{P}^n$, *then* $H^i(\boldsymbol{P}^n, \boldsymbol{Z})\cong H^i(S, \boldsymbol{Z})$ *for* $0\leqq i\leqq n-2$. *Moreover* Pic $S\cong\boldsymbol{Z}$, *generated by* $\mathcal{O}_S(1)$, *if* $n\geqq 4$.

Using Proposition 2.4, we have

Proposition 2.5. *Let X and Y be algebraic k-schemes and* $f: X\to Y$ *a proper* *flat morphism, where either for all points y in Y,* $f^{-1}(y)\cong \boldsymbol{P}^n$, *or, for all points y in Y,* $f^{-1}(y)$ *is a non-singular quadric hypersurface in* $\boldsymbol{P}^{n+1}$, *let E be a vector bundle of rank r on X with* $n\geqq 2(r+1)$. *Assume that all points y in Y,* $E|_{f^{-1}(y)}$ *is a direct sum of line bundles, say,* $\bigoplus_{i=1}^{r}\mathcal{O}_{f^{-1}(y)}(a_i)$, $(a_1\geqq\cdots,\cdots,\geqq a_r)$.

In addition, assume that there is a line bundle L on X such that $L|_{f^{-1}(y)}\cong \mathcal{O}_{f^{-1}(y)}(1)$ *for all points y in Y.*

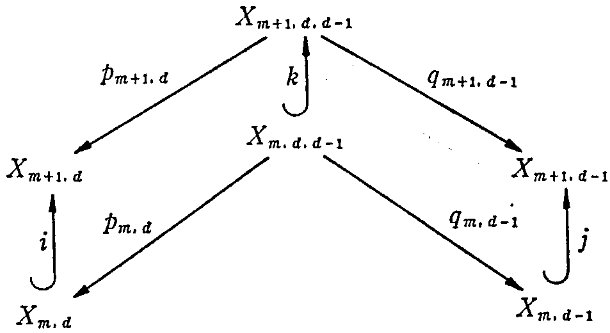
Then the sequence $(a_1,\cdots,a_r)$ *is independent of y.*

Proof. Since the Euler-Poincare characteristic of $E\otimes L^m|_{f^{-1}(y)}$ is independent of y, this proposition is obvious by proposition 2.4. q.e.d.

Using Theorem 2.2 and Proposition 2.5, we have the following

Proposition 2.5, AC I. *Let * be either type A or type C and let E be an IE-I vector bundle of rank r on* $X_{n,d}(*)$. *Then we have the following: For all points* $y\in X_{n,d-1}(*)$, $p_{n,d}{}^*E|_{q_{n,d-1}^{-1}(y)}$ *is isomorphic to* $\bigoplus_{i=1}^{\alpha}\mathcal{O}_{P^{n(*)}}(a_i)^{\oplus r_i}$ *such that* $a_1>\cdots>a_\alpha$ *and* $r_i>0$. $(n(A)=n-d$ *and* $n(C)=2(n-d)-1)$. *In addition* $a_1,\cdots,a_\alpha$ *and* $r_1,\cdots,r_\alpha$ *are independent of y.*

Proof. Let us consider the following diagram:



Then we see that for a point $\bar{y}$ in $X_{m,d-1}$, the morphism $k|_{q_{m,d-1}^{-1}(\bar{y})}: q_{m,d-1}^{-1}(\bar{y})\hookrightarrow q_{m+1,d-1}^{-1}(j(\bar{y}))$ is a morphism of type I.

Since for all points y in $X_{n,d-1}$, $p_{n,d}^*E|_{q_{n,d-1}^{-1}(y)}$ is an IE-I vector bundle, we can show the former of this proposition by Theorem 2.2. On the other hand, the latter is obvious by Proposition 2.5. q.e.d.

In the same way, we obtain the following

Proposition 2.6, ABCD II. *Let * be one of A, B, C and D. Let E be an IE-II vector bundle of rank r on* $X_{n,n-e}(*)$. *Then we have the following: For all points* $x\in X_{n,n-e+1}(*)$, $q_{n,n-e}^*E|_{p_{n,n-e+1}^{-1}(x)}$ *is isomorphic to* $\bigoplus_{i=1}^{\alpha}\mathcal{O}_{P^{n-e+1}}(a_i)^{\oplus r_i}$ *such that* $a_1>\cdots>a_\alpha$ *and* $r_i>0$ *except type D and* $e=3$. *In addition* $a_1,\cdots,a_\alpha$ *and* $r_1,\cdots,r_\alpha$ *are independent of x.*

Proof. It is obvious by virtue of Theorem 2.2 and Proposition 1.4*.
q. e. d.

Moreover, by Theorem 2.3, we have

Proposition 2.5. BD I. *Let * be either B or D and let E be an IE-I vector bundle of rank r on* $X_{n,d}(*)$. *Then for all points* $y \in X_{n,d-1}(*), p_{n,d}(*)^* E|_{q_{n,d-1}(*)^{-1}(y)}$ *is isomorphic to* $\bigoplus_{i=1}^{\alpha} \mathcal{O}_S(a_i)^{\oplus r_i}$ *such that* $a_1 > \cdots > a_\alpha$ *and* $r_i > 0$, *where in the case of* $*=B$ (*or, D*), S $(=q_{n,d-1}(*)^{-1}(y))$ *is a non-singular quadric hypersurface in* $\boldsymbol{P}^{2(n-d)}$ (*or,* $\boldsymbol{P}^{2(n-d)-1}$, *resp.*). *In addition* $a_1, \cdots, a_\alpha$ *and* $r_1, \cdots, a_\alpha$ *are independent of y.*

Corollary 2.7. *Let * be one of A, B, C and D and let E be an IE-I* (*or, IE-II*) *vector bundle on* $X_{n,d}(*)$ $(X_{n,n-e}(*)$, *resp.*). *Then* $a_1, \cdots, a_\alpha$ *and* $r_1, \cdots, r_\alpha$ *obtained for* E_m *are independent of* $m (m \geqq n)$. (*About* E_m, *see the definition* 1.6)

The following proposition plays an important role in the proof of the main result (Theorem 2.11) in this section.

Proposition 2.8. *Let X and Y be an algebraic k-schemes and let* $q: X \to Y$ *be a proper flat morphism where either for all points y in Y,* $q^{-1}(y)$ *is a d-dimensional projective space, or for all points y in Y,* $q^{-1}(y)$ *is a non-singular quadric hypersurface in* $\boldsymbol{P}^{d+1}$. *Assume that for all points y in Y,* $F|_{q^{-1}(y)}$ *is isomorphic to* $\bigoplus_{i=1}^{\alpha} \mathcal{O}_{q^{-1}(y)}(a_i)^{\oplus r_i}$ *and assume that* $(a_1, \cdots, a_\alpha)$ *and* $(r_1, \cdots, r_\alpha)$ *are independent of y, where* $a_1 = 0 > a_2 > \cdots > a_\alpha$, $r_i > 0$. *Then we have the following:*

1) $q_* F$ *is a vector bundle of rank* r_1 *on Y and* $q^* q_* F$ *is a subbundle of F.*
2) *If* $i=1$, $F \cong q^* q_* F$.

If $i \geqq 2$, $(F|q^* q_* F)|_{q^{-1}(y)}$ *is isomorphic to* $\bigoplus_{i \geqq 2}^{\alpha} \mathcal{O}_{q^{-1}(y)}(a_i)^{\oplus r_i}$.

Proof. This proposition is easily shown by the base change theorem (Theorem (7.7.6) in [3]). For more detail see Corollary 3.3 and Remark 3.4 in [5].
q. e. d.

Corollary 2.9. *Under the diagrem 2.1, let E be a vector bundle on* $X_{n,d,d-1}$ *and let us put* $p = p_{n,d}$ *and* $q = q_{n,d-1}$. *Assume that for all points y in* $X_{n,d-1}$ $p^* E|_{q^{-1}(y)}$ *is isomorphic to* $\bigoplus_{i=1}^{\alpha} \mathcal{O}_{q^{-1}(y)}(a_i)^{\oplus r_i}$ *with* $a_1 > a_2 > \cdots > a_\alpha$ *and* $r_i > 0$. *Then there are vector bundles on* $X_{n,d-1}$: $E_1, \cdots, E_\alpha$ *which are fitted in following exact sequences:*

$$0 \longrightarrow q^* E_1 \otimes p^* \mathcal{O}(a_1) \longrightarrow p^* E \longrightarrow F_2 \longrightarrow 0$$

$$0 \longrightarrow q^* E_2 \otimes p^* \mathcal{O}(a_2) \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

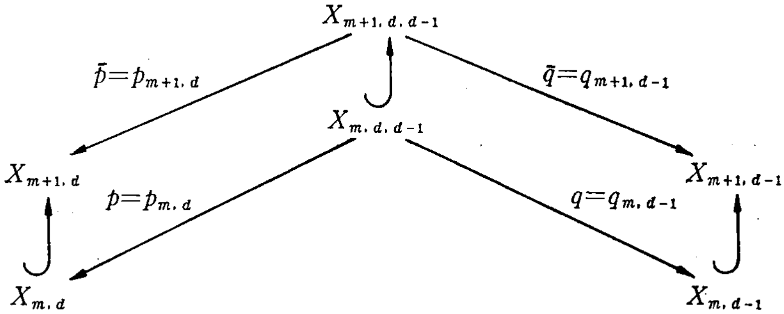
$$0 \longrightarrow q^* E_{\alpha-1} \otimes p^* \mathcal{O}(a_{\alpha-1}) \longrightarrow F_{\alpha-1} \longrightarrow q^* E \otimes p^* \mathcal{O}(a_\alpha) \longrightarrow 0,$$

where $F_2, \cdots, F_{\alpha-1}$ *are vector bundles on* $X_{n,d,d-1}$ *and where* $\mathcal{O}(a)$ *means* $\mathcal{O}_{X_{n,d}}(a)$.

Proof. Using Proposition 1.5* and Proposition 2.8 repeatedly, this proposition is proved easily.
q. e. d.

For the case of $d \geqq 1$, we shall prove that if E is an IE-I vector bundle on $X_{n,d}$, so is E_i for $1 \leqq i \leqq \alpha$ obtained in Corollary 2.9. For this purpose we prove

Proposition 2.10. *Let F (or, F') be a vector bundle on $X_{m,d,d-1}$ (or, $X_{m+1,d,d-1}$, resp.) and let $j: X_{m,d,d-1} \hookrightarrow X_{m+1,d,d-1}$ be a canoical morphism induced by $j_m(*)$: $G_m(*) \hookrightarrow G_{m+1}(*)$. Moreover let us consider the following diagram:*



Assume that

1) *j^*F' is isomorphic to F.*

2) *$F'|_{\bar{q}^{-1}(y)}$ is isomorphic to $\bigoplus_{i=1}^{\alpha} \mathcal{O}_{\bar{q}^{-1}(y)}(a_i)^{\oplus r_i}$ with $a_1 >, \cdots, > a_\alpha$ and $r_i > 0$ for all points $y \in X_{m+1,d-1}$.*

Then we have

I) *$q_*(F \otimes p^*\mathcal{O}_{X_{m,d}}(-a_1))$ is isomorphic to $k^*(\bar{q}_*(F' \otimes \bar{p}^*\mathcal{O}_{X_{m+1,d}}(-a_1))$.*

II) *F_1 is isomorphic to j^*F_1' where F_1 and F_1' are fitted in the following exact secqence:*

$$0 \longrightarrow q^*q_*(F \otimes p^*\mathcal{O}_{X_{m,d}}(-a_1)) \otimes p^*\mathcal{O}_{X_{m,d}}(a_1) \longrightarrow F \longrightarrow F_1 \longrightarrow 0$$

$$0 \longrightarrow \bar{q}^*\bar{q}_*(F' \otimes \bar{p}^*\mathcal{O}_{X_{m+1,d}}(-a_1)) \otimes \bar{p}^*\mathcal{O}_{X_{m+1}}(a_1) \longrightarrow F' \longrightarrow F_1' \longrightarrow 0.$$

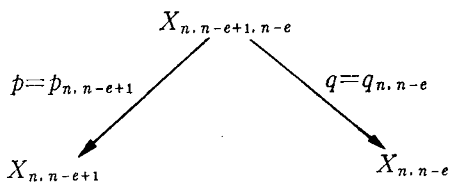
Proof. One can easily check this proposition by virtue of Proposition 2.8 and the base change theorem (Theorem (7.7.6) and Remark (7.7.9) in [3]).

q.e.d.

The following theorem is an immediate consequence of Corollary 2.9 and Proposition 2.10 which is very important for the proof of Main Theorem.

Theorem 2.11 I. *Let E be an IE-I vector bundle on $X_{n,d}$. Then under the notations of Corollary 2.9, $E_1, E_2, \cdots E_\alpha$ are also IE-I vector bundles on $X_{n,d-1}$.*

In the same way we can get the result of IE-II vector bundle corresponding to Theorem 2.11 I. Let us consider the diagram as follows:



Theorem 2.11 II. *Let E be an IE-II vector bundle on $X_{n,n-e}$. Then except type D and $e=3$, we obtain the following sequences of vector bundles on $X_{n,n-e+1,n-e}$:*

$$0 \longrightarrow p^*E_1 \otimes q^*\mathcal{O}(a_1) \longrightarrow q^*E \longrightarrow F_2 \longrightarrow 0$$

$$0 \longrightarrow p^*E_2 \otimes q^*\mathcal{O}(a_2) \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

$$0 \longrightarrow p^*E_{\alpha-1} \otimes q^*\mathcal{O}(a_{\alpha-1}) \longrightarrow F_{\alpha-1} \longrightarrow p^*E \otimes q^*\mathcal{O}(a_\alpha) \longrightarrow 0$$

where $a_1, \cdots, a_\alpha$ are integers with $a_1>, \cdots, >a_\alpha$ where $E_1, \cdots, E_\alpha$ are vector bundles on $X_{n,n-e+1}$, and $F_2, \cdots, F_{\alpha-1}$ are vector bnndles on $X_{n,n-e+1,n-e}$ and $\mathcal{O}(a)$ means $\mathcal{O}_{X_{n\cdot n-e+1}}(a)$. In addition $E_1, \cdots, E_\alpha$ are IE-II vector bundles on $X_{n,n-e+1}$.

Proof. It is obvious by virtue of Proposition 2.6 ABCD II and Corollary 2.7. q. e. d.

§3. Representations of parabolic subalgebra $\mathfrak{p}_{n,d}$

Let G be a semi-simple and simply connected linear algebraic group and let P, B and H be a parabolic subgroup, a Borel subgroup and a maximal torus of G, respectively, where $P \supset B \supset H$. Moreover let $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{b}$ and $\mathfrak{h}$ be the semi-simple Lie algebra, the parabolic subalgebra, the Borel subalgebra and the Cartan subalgebra corresponding to G, P, B and H. Hereafter we maintain this notation.

In this section, homogeneous vector bundles on G/P are considered. We know that giving a representation ϕ of P is equivalent to doing a representation $\bar{\phi}$ ef $\mathfrak{p}$. Hence we shall study homogeneous vector bundles in terms of Lie algebra.

In the first place, let us recall fundamental results about semi-simple Lie algebra which are found in details in [4].

Let $\mathfrak{g}=\mathfrak{h}+\sum\mathfrak{g}_\alpha$ be the decomposition of $\mathfrak{g}$ into $\mathfrak{h}$-invariant spaces through the ajoint representation, where $\mathfrak{h}$ acts on $\mathfrak{g}_\alpha$ through the character α. In particular let Δ be the set of characters α of $\mathfrak{h}$ with $\mathfrak{g}_\alpha \neq 0$. Now if $\mathfrak{r}$ is a subspace of $\mathfrak{g}$ invariant under the ajoint representation of $\mathfrak{h}$, we denote $\Delta(\mathfrak{r})$ by the subset of Δ consisting of all the roots α with $\mathfrak{g}_\alpha \subset \mathfrak{r}$. Moreover, let us define $\mathfrak{r}^0$ to be the set $\{z \in \mathfrak{g} \mid (z, y)=0 \text{ for all } y \in \mathfrak{r}\}$ where $(*, *)$ is the Cartan-Killing form on $\mathfrak{g}$. On the other hand, we know that the restriction $(*, *)$ of $\mathfrak{g}$ to $\mathfrak{h}$ is non-singular. Hence one can define a map $\mu \rightarrow x_\mu$ of $\mathfrak{h}'=\mathrm{Hom}_{\boldsymbol{Z}}(\mathfrak{h}, \boldsymbol{Z})$ onto $\mathfrak{h}$ by the relation $(x, x_\mu)=\langle x, \mu\rangle$ for all $x \in \mathfrak{h}$. This implies that the mapping defines a non-singular bilinear form on $\mathfrak{h}$ given by $(\mu, \lambda)=\langle x_\mu, \lambda\rangle$.

Then we have the following proposition with respect to the parabolic subalgebra.

Proposition 3.1. (*Proposition* 5.3. *in* [4]) *If $\mathfrak{n}=\mathfrak{p}^0$, $\mathfrak{p}=\mathfrak{g}_1+\mathfrak{n}$ is an orthogonal direct sum where $\mathfrak{g}_1$ is reductive in $\mathfrak{p}$ and $\mathfrak{n}$ is both the maximal nilpotent ideal in $\mathfrak{p}$ and the set of all nilpotent elements in the radical of $\mathfrak{p}$.*

Now let $Z(\subset \mathfrak{h}')$ be the set of all integral linear forms on $\mathfrak{h}$ and let us put $D_1=\{\mu \in Z \mid (\mu, \phi) \geqq 0 \text{ for all } \phi \in \Delta(\mathfrak{m}_1)\}$, where $\mathfrak{m}_1=\mathfrak{m} \cap \mathfrak{g}_1$ such that $\mathfrak{m}=\mathfrak{b}^0$. Then we have two propositions:

Proposition 3.2.1. *Let G_1 be a reductive group. Then every irreducible representation of G_1 is equivalent to ν_1^ξ for one and only one $\xi \in D_1$ where ν_1^ξ is the unique irreducible representation of G_1 having ξ as the highest weight.*

Proposition 3.2.2. *Let φ be an irreducible representation of P and let $P = G_1N$ be the Levi decomposition for P where G_1 (or, N) is the reductive part (or, the unipotent radical part, resp.) of P. Then φ is trivial on N and is equivalent to ν_1^ξ for some $\xi \in D_1$ on G_1. And conversely, given $\xi \in D_1$, the representation ν_1^ξ of G_1 on V_1^ξ extends to an irreducible representation*

$$\nu_1^\xi : P \longrightarrow \mathrm{End}\, V_1^\xi$$

of P on V_1^ξ by making it trivial on N.

For these two propositions, see 5.5 and 6.1 in [4].

Now let us employ some of notations in [4]. Put $\Delta_+ = \Delta(\mathfrak{m})$ and let $\Pi(\subseteqq \Delta_+)$ be the set of simple roots corresponding to Δ_+. Then, for any $\phi \in \Delta$, we have $\phi = \sum_{\alpha \in \Pi} n_\alpha(\phi)\alpha$. Let U be the set of parabolic subalgebra $\mathfrak{p}(\supseteqq \mathfrak{b})$ of $\mathfrak{g}$. Then $\mathfrak{p} \to \Pi(\mathfrak{p})$ defines a bijective map of the set of parabolic subalgebras containing $\mathfrak{b}$ to the set of subsets of Π such that

$$\Delta(\mathfrak{p}) \cap \Delta_- = \{\phi \in \Delta_- \mid n_\alpha(\phi) = 0 \text{ for all } \alpha \in \Pi(\mathfrak{p})\} .$$

Moreover we have the following

Proposition 3.3. (*see Proposition* 5.4 *in* [4]). *Under the above notation, let $\mathfrak{p}$ be a parabolic subalgebra contaning $\mathfrak{b}$. Then $\Delta(\mathfrak{p}) = \Delta(\mathfrak{g}_1) \cup \Delta(\mathfrak{n})$ is a disjoint union* (*see proposition* 3.1). *Moreover,*

$$\Delta(\mathfrak{g}_1) = \{\phi \in \Delta_+ \mid n_\alpha(\phi) = 0 \textit{ for all } \alpha \in \Pi(\mathfrak{p})\} \quad .$$

$$\Delta(n) = \{\phi \in \Delta_+ \mid n_\alpha(\phi) > 0 \textit{ for all } \alpha \in \Pi(\mathfrak{p})\} .$$

Now we will return our attention to the semi-simple algebraic groups $G_n(*)$, parabolic subgroups $P_{n,d}(*)$, Borel subgroups $B_n(*)$ and maximal tori $H_n(*)$ defined in § 1 (abbreiviated to G_n, $P_{n,d}$, B_n and H_n, respectively). Then let $\mathfrak{g}_n$, $\mathfrak{p}_{n,d}$, $\mathfrak{b}_n$ and $\mathfrak{h}_n$ be Lie algebras corresponding to G_n, $P_{n,d}$, B_n and H_n respectively. These notation will be maintained hereafter.

At first according to Dynkin diagram in § 1, let us consider $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ as the set of simple roots corresponding to Δ_+.

Then we have the following

Remark 3.4. $\Pi(\mathfrak{p}_{n,d}) = \alpha_d$.

Moreover let us recall a well-known result;

Proposition 3.5. *Let $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \mathfrak{h}'} \mathfrak{g}_\alpha$ be as in the first part of this section. Assume that α and α' are roots and, moreover, $\alpha + \alpha'$ is also a root. Then we have $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}] = \mathfrak{g}_{\alpha+\alpha'}$.*

Now let E_{α_i} be the root vector corresponding to a root α_i. Then we can check easily that

$$E_{\alpha_d}=[E_{\alpha_d+\alpha_{d+1}}, E_{-\alpha_{d+1}}] \quad \text{for } d\leqq n-1,$$

$$E_{\alpha_d}=[E_{\alpha_{d-1}+\alpha_d}, E_{-\alpha_{d-1}}] \quad \text{for } d\geqq 1$$

where $E_{\alpha_d+\alpha_{d+1}}$ and $E_{\alpha_{d-1}+\alpha_d}$ are elements in the nilpotent radical of $\mathfrak{p}_{n,d}$. Therefore, combining Proposition 3.3, Remark 3.4 and Proposition 3.5, we obtain

Proposition 3.6. *Let ϕ be a representation of $\mathfrak{p}_{n,d}$. Assume that either $\phi(E_{\alpha_{d-1}})=0$ or $\phi(E_{\alpha_{d+1}})=0$. Then $\phi(E_{\alpha_d})=0$. In addition to this, we have $\phi(\mathfrak{n})=0$ where $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{p}_{n,d}$.*

Remark 3.7. It is well known that if a Lie algebra $\mathfrak{g}$ is reductive, then $\mathfrak{g}$ decomposes into a sum of the center of $\mathfrak{g}$ and semi-simple algebra $D\mathfrak{g}(=[\mathfrak{g}, \mathfrak{g}])$. Let us determine the structure of $D\mathfrak{g}_1$ of the reductive subalgebra $\mathfrak{g}_1$ in $\mathfrak{p}_{n,d}(*)$ for $*=A$, B, C and D. Since we already have known Dynkin diagram for the type A, B, C and D, and the relations among their roots respectively, we immediately obtain the following

Type	$D\mathfrak{g}_1$	
A	$\mathfrak{sl}(d)+\mathfrak{sl}(n-d+1)$	
B	$\mathfrak{sl}(d)+\mathfrak{o}(2(n-d)+1)$	
C	$\mathfrak{sl}(d)+\mathfrak{sp}(n-d)$	
D	$\mathfrak{sl}(d)+\mathfrak{o}(2(n-d))$	for $n-d\geqq 2$
	$\mathfrak{sl}(n)$	for $n-d=0$ or 1

Remark 3.8. Let $\mathfrak{g}$ be one of simple Lie algebras $\mathfrak{sl}(n+1)$, $\mathfrak{o}(2n+1)$, $\mathfrak{sp}(n)$ and $\mathfrak{o}(2n)$ and let ϕ be a representation: $\mathfrak{g}\rightarrow\mathfrak{gl}(V)$ such that $\dim V<n$. Then it is easy to check that ϕ is trivial. (Later $\mathfrak{g}_n(A)$, $\mathfrak{g}_n(B)$, $\mathfrak{g}_n(C)$ and $\mathfrak{g}_n(D)$ denotes $\mathfrak{sl}(n+1)$, $\mathfrak{o}(2n+1)$, $\mathfrak{sp}(n)$ and $\mathfrak{o}(2n)$ respectively.)

Now let us recall that $\mathfrak{p}_{n,d}(*)=\mathfrak{g}_1+\mathfrak{n}$ where $\mathfrak{g}_1$ is reductive and $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{p}_{n,d}(*)$ and that $D(\mathfrak{g}_1)=\mathfrak{sl}(d)+\mathfrak{g}_{n-d}(*)$.

Therefore using Proposition 3.6, Remark 3.7 and Remark 3.8, we have

Proposition 3.9.1. *Under above notations, let $\phi:\mathfrak{p}_{n,d}(*)\rightarrow\mathfrak{gl}(V)$ be a representation such that* $\dim V<n-d$. *Then ϕ is trivial on $\mathfrak{g}_{n-d}(*)$ and $\mathfrak{n}$.*

Moreover we have following

Proposition 3.9.2. *Under the same notation in Proposition* 3.9.1, *assume that* $\dim V<d$. *Then ϕ is trivial on $\mathfrak{sl}(d)$ and n.*

§ 4. Proof of Main Theorem

In this section, by using the results obtained in § 3, let us determine the structure of homogeneous vector bundles on $X_{n,d}$ satisfying some conditions. At first we shall study D_1 defined in § 3 in detail. (Hereafter in this section we maintain the notation in § 3.)

Recall that $\mu \in Z$ if and only if $2(\mu, \phi)/(\mu, \mu)$ is an integer for any $\phi \in \Delta$. (see 5.5 in [4]). Moreover we can characterize Z in easier way as follows: Z is isomorphic to $\boldsymbol{Z}^{\oplus n}$ by the map σ:

$$\mu \longrightarrow (2(\mu, \alpha_1)/(\alpha_1, \alpha_1), \cdots, 2(\mu, \alpha_i)/(\alpha_i, \alpha_i), \cdots, 2(\mu, \alpha_n)/(\alpha_n, \alpha_n))$$

for any $\alpha_i \in \Pi$.

Furthermore we have the following

Proposition 4. 1. *D_1 is isomorphic to the subset of $\boldsymbol{Z}^{\oplus n}(=\boldsymbol{N}_0^{d-1} \times \boldsymbol{Z} \times \boldsymbol{N}_0^{n-d})$ by the above map σ where $\boldsymbol{N}_0$ is the set of non-nagative integers.*

Proof. It is obvious by the definition of D_1 in § 3. q. e. d.

In the sequel, for any element $\mu \in Z$, $\sigma(\mu)$ is written in the form $\sigma(\mu) = (c_0, \cdots, c_i, \cdots, c_{n-1})$ according to Proposition 4. 1.

Before studying irreducible homogeneous vector bundles on $X_{n,d}$, let us state the following remark.

Remark 4. 2. $SO(2n+1, \boldsymbol{C})(=G_n(B))$ and $SO(2n, \boldsymbol{C})(=G_n(D))$ are not simply connected. Therefore for $*=B, D$, let $\bar{G}_n(*)$ be a simply connected covering of $G_n(*)$ and $\bar{P}_{n,d}(*)$ be a parabolic subgroup of $\bar{G}_n(*)$ corresponding to a parabolic subgroup $P_{n,d}(*)$ of $G_n(*)$. Then $X_{n,d}(*)$ is isomorphic to $\bar{G}_n(*)/\bar{P}_{n,d+1}(*)$. Moreover let $\bar{T}_n(*)$ be the maximal torus corresponding to $T_n(*)$ in $\bar{G}_n(*)$. Then it is important to recall that the system of roots for $\bar{G}_n(*)$ (relative to $\bar{T}_n(*)$) is the same as the one for $G_n(*)$ (relative to $T_n(*)$). (see 1.9 in [7])

(4. 3) By virtue of Proposition 3. 2. 2, Proposition 4. 1 and Remark 4. 2 we can write an irreducible homogeneous vector bundle corresponding to $\sigma(\mu)=(c_0, \cdots, c_{n-1})$ as $P_n(d: c_0, \cdots, c_d, \cdots, c_{n-1})$ with non-negative integers c_i for $i \neq d$and an integer c_d.

We shall study the sufficient condition for homogeneous vector bundles on $X_{n,d}$ to be isomorphic to a direct sum of irreducible homogenous vector bundles which are isomorphic to $P_n(d; c_0, \cdots, c_d, 0, \cdots, 0)$.

Proposition 4. 4. *Let E be an irreducible homogeneous vector bundle of rank r on $X_{n,d}$. Assume $n-d>r$. Then E is isomorphic to one of $P_n(d; c_0, \cdots, c_d, 0, \cdots, 0)$ where for $0 \leqq i \leqq d-1$, c_i is a non-negative integer.*

Proof. Let E be an irreducible vector bundle obtained by an irreducible representation $\phi: P_{n,d+1} \rightarrow GL(V)$, which induces a representation of Lie algebra $\tilde{\phi}: \mathfrak{p}_{n,d+1} \rightarrow \mathfrak{gl}(V)$. Therefore the assumption yields this proposition by virtue of Remark 3. 8, Theorem 3. 9. 1 and Proposition 4. 1. q. e. d.

Before continuing our argument, we shall show a result which is necessary for some proposition stated later. Let G_0 be a connected linear algebraic group and $G_0=G_1N$ be a Levi decomposition where G_1 (or, N) is a reductive part (or, a unipotent radical, resp.) of G_0. Now let ϕ be a representation of G_0 and $\tilde{\phi}$ be the corresponding representation of $\mathfrak{g}_0$ which is a Lie algebra of G_0. Moreover let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{g}_0$ corresponding to N. Then we have

Proposition 4.5. *Under the above notations, assume that $\tilde{\phi}$ is trivial on $\mathfrak{n}$. Then ϕ is completely reducible.*

Proof. By virtue of the well-known relation between Lie group and the corresponding Lie algebra, $\mathrm{Ker}\,\tilde{\phi}$ is equal to the subalgebra of $\mathfrak{g}_0$ corresponding to $\mathrm{Ker}\,\phi$. Hence we see easily that $\mathrm{Ker}\,\phi\supset N$, which implies that ϕ is completely reducible by virtue of Weyl's Theorem. q.e.d.

Corollary 4.6. *Let E be a homogeneous vector bundle of rank r on $X_{n,d}$. Assume that $n-d>r$. Then E is isomorphic to $\bigoplus_{i=1}^{t} P_n(d, c_0^i, \cdots c_{d-1}^i, c_d^i, 0, \cdots 0)$ where c_j^i is a non-negative integer for $1\leqq i\leqq t$ and $0\leqq j\leqq d-1$.*

Proof. Let E be a vector bundle obtained by a representation $\phi: P_{n,d+1}\to GL(V)$. By proposition 3.9.1, the corresponding representation of Lie algebra $\mathfrak{p}_{n,d+1}$; $(=\tilde{\phi})$ is trivial on $\mathfrak{n}$ which is the nilpotent radical of $\mathfrak{p}_{n,d+1}$. Hence by Proposition 4.5, we see ϕ is completely reducible. q.e.d.

The next proposition is necessary for the main theorem.

Proposition 4.7. *If $E_i=P_n(d; c_0^i, \cdots c_d^i, 0, \cdots, 0)$ for $i=1, 2$, then $E_1\otimes E_2$ is isomorphic to $\bigoplus_{j=1}^{t} P_n(d; b_0^j, \cdots, b_d^j, 0, \cdots, 0)$.*

Proof. For $i=1, 2$, let ϕ_i be the irreducible representation of $P_{n,d+1}$ which yields the vector bundle E_i. Since ϕ_i is trivial on N where N is the unipotent radical of $P_{n,d+1}$, we see that representation $\phi_1\otimes\phi_2$ is also trivial on N and therefore it is a representation of a reductive group. Hence $\phi_1\otimes\phi_2$ is completely reducible. It is easy to check that any direct summand of $E_1\otimes E_2$ is of type $P_n(d; b_0, \cdots, b_d, 0, \cdots, 0)$. q.e.d.

Remark 4.8. $P_n(d; c_0, \cdots c_d, 0, \cdots, 0)$ is an IE-I vector bundle.

Proof. Since we already know that $\mathfrak{p}_{n,d+1}(*)=(\mathfrak{sl}(d+1)+\mathfrak{g}_{n-d-1}(*)+\mathfrak{r})+\mathfrak{n}$ where $\mathfrak{n}$ is a nilpotent radical of $\mathfrak{p}_{n,d+1}(*)$ and $\mathfrak{r}$ is a center of the reductive part of $\mathfrak{p}_{n,d+1}(*)$, we see that the representation ϕ corresponding to the vector bundle $P_n(d; c_0, \cdots c_d, 0, \cdots 0)$ is trivial on $\mathfrak{g}_{n-d-1}(*)+\mathfrak{n}$. Hence we have only to take $P_m(d; c_0, \cdots c_d, 0, \cdots, 0)$ as E_m for $m\geqq n$. q.e.d.

From now on let ϕ be an irreducible representation of $\mathfrak{g}_n(A)$, V be a representation space of ϕ and let us put $\phi|_{\mathfrak{p}_{n,n}(A)}=\psi$. Moreover let V_0 be an $(n+1)$-dimensinal vector space and $e_1, \cdots e_{n+1}$ be a basis of V_0. Then it is known that V is a $\mathfrak{g}_n(A)$-module generated by the vector of highest weight of ϕ: $v=e_1{}^{c_1}\otimes$

$(e_1\wedge e_2)^{c_2}\otimes\cdots\otimes(e_1\wedge\cdots\wedge e_n)^{c_n}$ with a sequence of non-negative integers; $c_1, \cdots, c_n$. Moreover we can consider $\mathfrak{p}_{n,n}(A)$-invariant submodule of $V(=W)$ generated by v, which yields a irreducible representation of $\mathfrak{p}_{n,n}(A)$ $(=\xi: \mathfrak{p}_{n,n}(A)\to\mathfrak{gl}(W))$. In the same way for an irreducible representation ϕ of $\mathfrak{g}_n(*)$, we can consider an irreducible subrepresentation ψ of $\phi|_{\mathfrak{p}_{n,n}(*)}$ of $\mathfrak{p}_{n,n}(*)$ for all $*$. Therefore we have the following

Proposition 4.9. *Let ϕ be an irreducible representation of $\mathfrak{g}_n(*)$ corresponding to an element $(c_0, \cdots c_{n-1})$ in $D_1(n+1)$. Then the above irreducible subresentation of $\phi|_{\mathfrak{p}_{n,n}(\cdot)}$ corresponds to $(c_0, \cdots c_{n-1})$ in $D_1(n)$. Here $D_1(d)$ denotes D_1 in Proposition 4.1 with respect to $\mathfrak{p}_{n,d}(*)$.* $(\mathfrak{p}_{n,n+1}(*)=\mathfrak{g}_n(*))$.

Remark 4.10. Let ϕ be an irreducible representation of $\mathfrak{p}_{n,d}(*)$ $(=\mathfrak{sl}(d)+\mathfrak{g}_{n-d}(*)+\mathfrak{r}+\mathfrak{n})$ where $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{p}_{n,d}(*)$ and $\mathfrak{r}$ is a center of the reductive part of $\mathfrak{p}_{n,d}(*)$. Assume that ϕ is trivial on $\mathfrak{n}$ and $\mathfrak{g}_{n-d}(*)$. Since it is easy to check that $\mathfrak{p}_{n,d+1,d}(*)$ $(=\mathfrak{p}_{n,d+1}(*)\cap\mathfrak{p}_{n,d}(*))$ contains $\mathfrak{sl}(d)+\mathfrak{r}$, we see that $\phi|_{\mathfrak{p}_{n,d+1,d}(\cdot)}$ is an irreducible representation of $\mathfrak{p}_{n,d+1,d}(*)$.

Let us study an irreducible homogeneous vector bundle $P_n(d; a_0, \cdots a_d, 0\cdots 0)$ from a different point of view. In the first place the following proposition is very important for Main theorem.

Proposition 4.11. *Under the diagrom 2.1, $p^*_{n,d}P_n(d; c_0, \cdots, c_{d-1}, 0, \cdots, 0)$ has a quotient vector bundle $q^*_{n,d-1}P_n(d-1; c_0, \cdots c_{d-1}, 0, \cdots, 0)$ on $X_{n,d,d-1}$.*

Proof. Let ϕ be the representation of $P_{n,d+1}$ which yields the vector bundle $E=P_n(d; c_0, \cdots, c_{d-1}, 0, \cdots, 0)$. In order to prove this proposition, it suffices to show that there is a representation ψ of $P_{n,d}$ which yields the vector bundle $P_n(d-1; c_0, \cdots c_{d-1}, 0, \cdots, 0)$ and which satisfies the following:

$\psi|_{P_{n,d+1,d}}$ is an irreducible subrepresentation of $\phi|_{P_{n,d+1,d}}$. Therefore instead of Lie group we shall consider the above statement in terms of Lie algebra. For the corresponding representations of Lie algebra, we shall use the same notation ϕ, ψ. We know that $\mathfrak{p}_{n,d+1}(*)$ $(=\mathfrak{sl}(d+1)+\mathfrak{g}_{n-d}(*)+\mathfrak{r}+\mathfrak{n})$. Moreover by the assumption, we see that ϕ is trivial on $\mathfrak{g}_{n-d}(*)+\mathfrak{r}+\mathfrak{n}$. Hence by virtue of Proposition 4.9 and Remark 4.10, this proposition is obvious. q.e.d.

Next assume that $c_0, \cdots c_d$ are integers where c_i is non-negative for $0\leq i\leq d-1$. Let us consider a coherent sheaf $P_n(c_0, \cdots c_d)$ on $X_{n,d}$ defined inductively as follows: Under the diagram 2-1, for $d=0$, put $P_n(c_0)=\mathcal{O}_{X_{n,0}}(c_0)$. If $P_n(c_0, \cdots c_j)$ is defined, put $P_n(c_0, \cdots, c_{j+1})=p_{n,j*}q^*_{n,j-1}P_n(c_0, \cdots, c_j)\otimes\mathcal{O}_{X_{n,j+1}}(c_{j+1})$.

Our next task is to show that $P_n(c_0, \cdots, c_d)$ defined in the above is isomorphic to $P_n(d; c_0, \cdots, c_d, \cdots, 0)$ in (4.3).

Now let us consider the exact sequence obtained in Proposition 4.9:

$$(4.12)\qquad 0\longrightarrow F\longrightarrow p^*_{n,d}P_n(d; c_0, \cdots, c_{d-1}, 0, \cdots, 0)$$
$$\longrightarrow q^*_{n,d-1}P_n(d-1; c_0, \cdots, c_{d-1}, 0, \cdots, 0)\longrightarrow 0.$$

Particularly let us study the structure of the above sequence (4.12) on the fiber of $p_{n,d}$. For this purpose, it suffices to investigate $P_{n,d+1}\times_{P_{n,d+1,d}}V$ and its subbundle $P_{n,d+1}\times_{P_{n,d+1,d}}W$ where W is a representation space of the subrepresenttion $\psi|_{P_{n,d+1,d}}$ of $\varphi|_{P_{n,d+1,d}}$. On the other hand, we can easily show that $P_{n,d+1}/P_{n,d+1,d}\cong G_d(A)/P_{d,d}(A)$. Moreover as was seen in the proof in Proposition 4.9, ϕ can be regarded as representation of $P_{n,d+1}$ extending a representation τ of $G_d(A)$ $(=SL(d+1, \boldsymbol{C}))$. Therefore the above two facts mean that $P_{n,d+1}\times_{P_{n,d+1,d}}V$ is isomorphic to $SL(d+1, \boldsymbol{C})\times_{P_{d,d}(A)}V$ and, in the same way, $P_{n,d}\times_{P_{n,d,d-1}}W$ is isomorphic to $SL(d+1, \boldsymbol{C})\times_{P_{d,d}(A)}W$. Then τ yields an irreducible subrepresentation of τ $(=\tau' : P_{d,d}(A)\rightarrow GL(W)$ induced by $\psi|_{P_{n,d+1,d}})$. It is obvious that τ' is an irreducible representation by Remark 4.12.

Next let us restrict the exact sequence (4.12) on the fiber $p_{n,d}^{-1}(x)$ $(\cong \boldsymbol{P}^d)$ for $x\in X_{n,d}$ as follows:

$$(4.13)\qquad 0\longrightarrow F|_{p_{n,d}^{-1}(x)}\longrightarrow \mathcal{O}_{p_{n,d}^{-1}(x)}^{\oplus r}\longrightarrow q_{n,d-1}^{*}P_n(d-1\,;c_0,\cdots c_{d-1},0,\cdots,0)|_{p_{n,d}^{-1}(x)}\longrightarrow 0,\quad \text{where}\quad r=\text{rank}\ P_n(d\,;c_0,\cdots c_{d-1},0,\cdots,0).$$

Further let us consider the sequence of cohomologies obtained from (4.13):

$$(4.14)\qquad 0\longrightarrow H^0(\boldsymbol{P}^d, F|_{\boldsymbol{P}^d})\xrightarrow{s} H^0(\boldsymbol{P}^d, \mathcal{O}_{\boldsymbol{P}^d}{}^{\oplus r})\xrightarrow{t} H^0(\boldsymbol{P}^d, E)\longrightarrow$$
$$\text{where}\quad E=q_{n,d-1}^{*}P_n(d-1\,;c_0,\cdots,c_{d-1},0,\cdots 0)|_{p_{n,d}^{-1}(x)}.$$

Then we have

Lemma 4.13. *t is an isomorphism.*

Proof. We can regard the exact sequence (4.14) as an exact sequence of $SL(d+1, \boldsymbol{C})$-modules. Since E is an irreducible homogeneous vector bundle, $H^0(\boldsymbol{P}^d, E)$ is an irreducible $SL(d+1, \boldsymbol{C})$-module. Hence we see that t is surjective or t is a zero map. If t is a zero map, s is an isomorphism, which implies that $F|_{p_{n,d}^{-1}(x)}\cong \mathcal{O}_{p_{n,d}^{-1}(x)}{}^{\oplus r}$. This is a contradiction. Hence t is surjective. On the other hand, it is obvious that t is injective. q. e. d.

Therefore we have the following.

Theorem. 4.14. *$P_n(d\,;c_0,\cdots,c_d,0,\cdots 0)$ is isomorphic to $P_n(c_0,\cdots,c_d)$.*

Proof. We prove this theorem by induction on d. It is easy to see that $P_n(0\,;c_0)\cong P_n(c_0)$. For $d\geqq 1$, if we take the direct image $p_{n,d*}$ of the exact sequence (4.12), our assertion is obvious by Proposition 4.13. q. e. d.

Let us show that $H^1(X_{n,d}, P_n(d\,;c_0,\cdots c_d,0,\cdots 0))=0$ for $d\leqq n-2$. In the first place, assume that $c_d\geqq 0$. Then by virtue of Bott's Theorem $H^1(X_{n,d}, P_n(d, c_0,\cdots c_d, 0,\cdots,0))=0$ for $0\leqq d\leqq n-1$. In the second place, assume that $c_d<0$. Let us consider the following vector bundle on $X_{n,d,d-1}$;

$$M=q_{n,d-1}^{*}P_n(d-1, c_0,\cdots,c_{d-1},0,\cdots,0)\otimes p_{n,d}^{*}\mathcal{O}_{X_{n,d}}(c_d).$$

We want to show $H^1(X_{n,d,d-1}, M)=0$ for $d\leqq n-2$. For this purpose, the following proposition is necessary.

Proposition 4.15. *Let S be a hypersurface in $\boldsymbol{P}^n$ for $n\geqq 3$ and $\mathcal{O}_S(1)$ is the line bundle corresponding to the hyperplane section of S. Then $H^i(S, \mathcal{O}_S(a))=0$ for any negative integer a and $i=0, 1$.*

Proof. Well-known.

Now let us consider the spectral sequence $E_2^{i,j}=H^i(X_{n,d-1}, R^j q_{n,d-1*}(M)) \Rightarrow E^{i+j}=H^{i+j}(X_{n,d,d-1}, M)$. Combining Proposition 1-4*, Proposition 1-5* and Proposition 4.15, we have $E_2^{1,0}=E_2^{0,1}=0$, which implies that $H^1(X_{n,d,d-1}, M)=0$ for $d\leqq n-2$. On the other hand, since $M|_{p_{n,d}^{-1}(x)}$ is an irreducible homogeneous vector bundle $P_d(d-1; c_0, \cdots, c_{d-1})$ on $\boldsymbol{P}^d$ for a point x in $X_{n,d}$, we see that $H^0(X_{n,d}, R^1 p_{n,d*}M)=0$ by the result. Hence we have $H^1(X_{n,d}, P_n(d; c_0, \cdots, c_d, 0\cdots, 0))=0$ by Theorem 4.14 and Leray's Spectral sequence.

Summing up the above results, we have

Proposition 4.16. *For $d\leqq n-2$, $H^1(X_{n,d}, P_n(d; c_0, \cdots, c_d, 0, \cdots, 0))=0$ and if $c_{n-1}\geqq 0$, $H^1(X_{n,n-1}, P_n(n-1; c_0, \cdots, c_{n-1}))=0$.*

Proposition 4.17. *Let E be a vector bundle on $X_{n,d}$ for $n-1>d\geqq 1$ satisfying the following exact sequence;*

$$0 \longrightarrow \bigoplus_i P_n(d; a_0^i, \cdots a_d^i, 0, \cdots, 0) \longrightarrow E \longrightarrow \bigoplus_j P_n(d; b_0^j, \cdots, b_d^j, 0, \cdots, 0) \longrightarrow 0 .$$

Then E is isomorphic to $(\bigoplus_i P_n(d; a_0^i, \cdots, a_d^i, 0, \cdots, 0) \oplus (\bigoplus_j P_n(d; b_0^j, \cdots b_d^j, 0, \cdots, 0))$.

Proof. It is obvious by Proposition 4.7 and Proposition 4.16. q.e.d.

The next proposition plays an important role in the proof of Main Theorem.

Proposition 4.18. *Let E be a vector bundle on $X_{n,d}$ such that $E\otimes(\bigoplus P_n(d; a_0, \cdots, a_d, 0, \cdots 0))\cong(\bigoplus P_n(d; b_0, \cdots, b_d, 0, \cdots 0))$. Then E is isomorphic to $\bigoplus P_n(d; c_0, \cdots c_d, 0, \cdots, 0)$.*

Proof. Since we see $(\bigoplus P_n(d; a_0^i, \cdots a_d^i, 0, \cdots 0)\otimes(\bigoplus P_n(d; a_0^i, \cdots, a_d^i, 0, \cdots, 0\ v))$ has a trivial line bundle as a direct summand by virtue of a trace map, $E\otimes(\bigoplus P_n(d; a_0^i, \cdots, a_d^i, 0, \cdots, 0))\otimes(\bigoplus P_n(d; a_0^i, \cdots a_d^i, 0, \cdots 0\ v))$ has E as a direct summand. Therefore Proposition 4.7 and Krull-Schmitt Theorem of vector bundles provide us with this proposition. q.e.d.

Finally we shall prove our Main Theorems.

Proof of Main Theorem I.

We shall prove theorem by induction on d. In the case of $d=0$, our assertion is easily shown by virtue of Theorem 2.2 and Theorem 2.3.

Assume that $d\geqq 1$. By induction assumption and Theorem 2.11. I, we see that E_i is isomorphic to $\bigoplus_{j_i} P_n(d-1; b_0^{i,j_i}, \cdots, b_{d-1}^{i,j_i}, 0, \cdots, 0)$. Therefore putting $-b=\min_{ij_i}\{0, b_{d-1}^{i,j_i}|i, j_i\}$ and tensoring exact sequence of vector bundle by $q_{n,d-1}^*\mathcal{O}_{X_{n,d-1}}(b)$, we get the following sequences:

$$0 \longrightarrow q^*_{n,d-1}E'_1 \otimes p^*_{n,d}\mathcal{O}_{X_{n,d}}(a_1) \longrightarrow p^*_{n,d}E \otimes q^*_{n,d-1}\mathcal{O}_{X_{n\cdot d-1}}(b)$$
$$\longrightarrow F_2 \otimes q^*_{n,d-1}\mathcal{O}_{X_{n\cdot d-1}}(b) \longrightarrow 0$$
$$0 \longrightarrow q^*_{n,d-1}E'_{\alpha-1} \otimes p^*_{n,d}\mathcal{O}_{X_{n\cdot d}}(a_{\alpha-1}) \longrightarrow F_{\alpha-1} \otimes q^*_{n,d-1}\mathcal{O}_{X_{n\cdot d-1}}(b)$$
$$\longrightarrow q^*_{n,d-1}E'_\alpha \otimes p^*_{n,d}\mathcal{O}_{X_{n\cdot d}}(a_\alpha) \longrightarrow 0$$

where $E'_i = \bigoplus_{j_i} P_n(d-1\,;b_0^{i,j_i}, \cdots, b_{d-1}^{i,j_i}+b, 0, \cdots, 0)$. Take the direct image $R^i p_{n,d*}$ of these sequences. Then since we can easily check that $R^1 p_{n,d*} q^*_{n,d-1}E'_i=0$ by proposition 4.16, we have the following exact sequence;

$$0 \longrightarrow \bar{E}_1 \longrightarrow E \otimes P_n(d\,;0, \cdots, 0, b, 0, \cdots 0) \longrightarrow p_{n,d*}(F_2 \otimes q^*_{n,d-1}\mathcal{O}_{X_{n\cdot d-1}}(b)) \longrightarrow 0$$
$$0 \longrightarrow \bar{E}_{\alpha-2} \longrightarrow p_{n,d*}(F_{\alpha-2} \otimes q^*_{n,d-1}\mathcal{O}_{X_{n\cdot d-1}}(b))$$
$$\longrightarrow p_{n,d*}(F_{\alpha-1} \otimes q^*_{n,d-1}\mathcal{O}_{X_{n,d-1}}(b)) \longrightarrow 0$$
$$0 \longrightarrow \bar{E}_{\alpha-1} \longrightarrow p_{n,d*}(F_{\alpha-1} \otimes q^*_{n,d-1}\mathcal{O}_{X_{n\cdot d-1}}(b)) \longrightarrow \bar{E}_\alpha \longrightarrow 0$$

where $\bar{E}_i = \bigoplus P_n(d\,;b_0^{i,j_i}, \cdots, b_{d-1}^{i,j_i}+b, a_i, 0, \cdots 0)$. By virtue of Proposition 4.17 and Proposition 4.18, we complete our proof. q. e. d.

Proof of Main theorem II. We can show this theorem in the same way as above using theorem 2.2. q. e. d.

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